People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Higher Normal School of Technological Education-Skikda

Department of Mathematics and computer science

Dissertation

Presented to obtain a degree in Mathematics as a teacher of secondary school

Entitled

Analytic study of some Timoshenko systems

Presented by:

MENAIL Chaima

Board of Examiners:

Chairwoman: SEGHIER Fatma Zohra	MCA	ENSET SKIKDA
Supervisor: BOUCENNA Djalal	MCA	ENSET SKIKDA
Examiner: MANSOURI Bouzid	MCA	ENSET SKIKDA
Examiner: KENEF Esma	MCA	ENSET SKIKDA

June’s session 2024

Acknowledgment

First and foremost, I thank Allah and praise him for granting me success and giving me the ability and effort to complete this humble work.

I would like to express my deepest gratitude to my supervisor Dr. BOUCENNA Djalal, for his invaluable guidance, support, and encouragement throughout my research. His immense knowledge, patience, mastery and sincerity were instrumental in shaping this thesis. I learned from him not only in terms of science and knowledge, but also in terms of his treatment, principls and values. Asking Allah to reward him with great goodness and bless him.

I am also extremely thankful to the board of examiners, Dr. MANSOURI Bouzid and Dr. KENEF Esma, for their evaluating, constructive comments and suggestions for this modest dissertation.

Finally, I extend my sincere thanks to everyone who contributed to the success of this work.

Dedicate

To my virtuous lady, my mother, who sacrificed her life for me and my family.

To my noble man, my father, who did the impossible for me since I was born.

To my siblings Asma and Abderrahman, for whom I fight always to be a good model.

To my closest and favorite person, my all, Hadeel, who taught me the meaning of friendship.

To my uncle, my second father, to my aunt, her children and husband.

To all my uncles, their wives and children, specially Hadjer and Abir.

To my long-time best friends, Sara, Ahlem and Marwa.

To all of my close pals from my time in univesity, Amani, Amel, Hadil, Aziza, Dounia, Anfel, Salwa, Manel, Ines...

To my present and compensation for this year, Nour-Elhouda, the best I knew.

To companions of the Quran,"Halakat Elnour", each with her own name.

To my first pupils, who have trust and faith in me.

To everyone who taught me Quran, religion, and throughout my academic journey, and finally by my suprevisor Dr. BOUCENNA.

To everyone who has ever helped me or done me a favor, may Allah reward them all with goodness.

To myself, for all hard work and patience. It was entirely owing to Allah's success and assistance. Praise be to him first and last.

Abstract

In this dissertation, we are concerned with a nonlinear Timoshenko system modeling clamped thin elastic beams with time delay and a memory-type Timoshenko system with Dirichlet boundary conditions and a very general class of relaxation functions. For the nonlinear Timoshenko system with delay, under suitable assumptions on the data, we establish the well-posedness of the problem with respect to weak solutions and establish the exponential stability of the system under the usual equal wave speeds assumption, and we state the well-posedness of the system and establish its general stability considering the case of equal-speeds and the case of non-equal-speeds of propagation for the memory-type Timoshenko system.

Keywords: Timoshenko system, time delay, exponential stability, general decay, equal and non-equal speeds of the wave propagation.

Contents

Introduction

The Timoshenko system is a mathematical model that describes shear deformation and rotational bending effects on the beam.

The Timoshenko beam model was first derived in [22] to describe the dynamics of a thick beam and it consists of a system of two coupled hyperbolic equations of the form:

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0 & in\,(0,L)\times(0,+\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) = 0 & in\,(0,L)\times(0,+\infty)\,, \end{cases} \tag{0.1}$$

where φ is the transverse displacement, ψ is the rotational angle of the filament of the beam and ρ_1, ρ_2, b and K are fixed positive physical constants. Many results dealing with global existence and stability of the system have been proved. In order to study the large-time (asymptotic) behaviour of system (0.1), various types of dissipation, such as boundary and/or internal feedback, heat or thermoelasticity, infinite memory, and Kelvin-Voigt damping, have been utilized. In this regard, see for example [2, 5, 10, 13, 15]. The system above is exponentially stable due to some damping terms on both equations of (0.1) without imposing any condition on the speeds of wave propagation. But if the damping effect is acting, the system is exponentially stable if and only if it has equal speeds of wave propagation, that is,

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}.$$

First, concentrating on the stabilisation of a nonlinear Timoshenko system with delay, the insert of delay terms in the feedback of these systems adds an extra layer of complexity to the analysis of their stability properties. Delay terms can arise naturally in applications where there is a time lag between the measurement of the system state and the application of the control force. Several authors have studied the stabilization of Timoshenko systems with delay terms, like Nicaise and Pignotti [18, 19], Said-Houari and Laskri [21], Datko et al [8]. Munoz Rivera and Racke [3] treated a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1 \varphi_{tt} + \sigma_1 (\varphi_x, \psi)_x = 0, \\ \rho_2 \psi_{tt} - \chi (\psi_x)_x + \sigma_2 (\varphi_x, \psi) + d\psi_t = 0, \end{cases} \tag{0.2}$$

in a bounded interval. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general.

On the other side, the stabilisation of a viscoelastic-type Timoshenko system had received a considerable attention since the work of Ammar-Khodja et al. [4], in which the authors studied the following problem

$$\begin{cases} \rho_1 \varphi_{tt} - K (\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K (\varphi_x + \psi) + \int_0^t g (t-s) \psi_{xx} (s) \, ds = 0, \\ \varphi (0,t) = \varphi (L,t) = \psi (0,t) = \psi (L,t) = 0, \quad for \quad t \geq 0, \end{cases} \tag{0.3}$$

in $(0, L) \times (0, +\infty)$, where g is a positive non-increasing differentiable L^1-function defined on $[0, +\infty)$. They showed that the system is stable if and only if $\dfrac{K}{\rho_1} = \dfrac{b}{\rho_2}$. Guesmia and Messaoudi [11] established the same decay result of [4] by weakening some of the assumptions on g.

This dissertation consists of three chapters, the first one contains some preliminary concepts about Sobolev space, some inequalities, operator basics and Hille-Yoshida Theorem.

The second chapter contains some preliminaries, poof of well-posedness and exponential stability for a nonlinear Timoshenko system with delay in case of equal speeds.

In the third chapter, we present also some preliminaries, well-posedness theorem and establishing the general stability of a viscoelastic Timoshenko system for both equal and non-equal speeds cases.

CHAPTER 1

Preliminary and basic concepts

1.1 Definitions and preliminary notions

1.1.1 Lebesgue and Sobolev space

Definition 1.1. [1] For $1 \leq p < +\infty$, $L^p(\Omega)$ is the space of measurable fonctions whose p-th powers are integrable over Ω. Equipped with the norm:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}.$$

$L^p(\Omega)$ is a Banach space.

For $p = +\infty, L^\infty(\Omega)$ is the space of measurable functions f which are essentially bounded over Ω, that is, there exists a constant $C > 0$ such that $|f(x)| \leq C$ almost everywhere in Ω. Equipped with the norm:

$$\|f\|_{L^\infty(\Omega)} = inf\left\{C \in \mathbb{R}^+ \quad \text{such that} \quad |f(x)| \leq C \quad \text{a.e in} \quad \Omega\right\}.$$

$L^\infty(\Omega)$ is a Banach space. Recall that, if Ω is a bounded open set, then $L^p(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq p \leq \infty$.

Definition 1.2. [1] Let Ω be an open set of $\mathbb{R}^n$ equipped with the Lebesgue measure. We define by $L^2(\Omega)$ the space of measurable functions which are of square integrable in Ω. Under the scalar product:

$$\langle f, g \rangle = \int_\Omega f(x)g(x)dx.$$

$L^2(\Omega)$ is a Hilbert space. We denote the corresponding norm by

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2 \, dx\right)^{1/2}.$$

Notation: [1] Let $\alpha = (\alpha_k)_{i=1}^n \in \mathbb{N}^n$ be a multi-index, that is, a vector of n components which are non-negative integres $\alpha_i \geq 0$. We denote by $|\alpha| = \sum_{i=1}^n \alpha_i$ and, for a function v,

$$D^\alpha v\,(x) = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}\,(x)\,.$$

Definition 1.3. [1] For every enteger $m \geq 0$, the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) \text{ such that } , \forall \alpha \text{ with } |\alpha| \leq m, \partial^\alpha v \in L^p(\Omega)\}\,,$$

where the partial derivative $\partial^\alpha v$ is taken in the weak sense. Equipped with the norm:

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|^p\right)^{1/p}.$$

The space $W^{m,p}(\Omega)$ is a Banach space.

Definition 1.4. [1] For an enteger $m \geq 0$, the Sobolev space $H^m(\Omega)$ is defined by:

$$H^m(\Omega) = \left\{v \in L^2(\Omega) \text{ such that, } \forall \alpha \text{ with } |\alpha| \leq m, \partial^\alpha v \in L^2(\Omega)\right\},$$

where the partial derivative $\partial^\alpha v$ is to be taken in the weak sens, Equipped with the scalar product:

$$\langle u, v \rangle = \int_\Omega \sum_{|\alpha| \leq m} \partial^\alpha u(x) \partial^\alpha v(x) dx$$

and the norm $\|u\|_{H^m(\Omega)} = \sqrt{\langle u, u \rangle}$. $H^m(\Omega)$ is a Hilbert space.

Definition 1.5. [1] Let Ω be an open set of $\mathbb{R}^n$. The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \left\{v \in L^2(\Omega) \text{ such that, } \forall i \in \{1, ..., n\}, \frac{\partial v}{\partial x_i} \in L^2(\Omega)\right\},$$

where $\frac{\partial v}{\partial x_i}$ is the weak partial derivative of v. Equipped with the scalar product:

$$\langle u, v \rangle = \int_\Omega (u(x)v(x) + \nabla u(x).\nabla v(x))\, dx$$

and with the norm:

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega \left(|u(x)|^2 + |\nabla u(x)|^2\right) dx\right)^{1/2}.$$

The Sobolev space $H^1(\Omega)$ is a Hilbert space.

Let us now define another Sobolev space which is a subspace of $H^1(\Omega)$ and which will be very useful for problems with Dirichlet boundary conditions.

Definition 1.6. [1] Let $C_c^\infty(\Omega)$ be the space of functions of class C^∞ with compact support in Ω. The Sobolev spase $H_0^1(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Equipped with the scalar product of $H^1(\Omega)$. The Sobolev space $H_0^1(\Omega)$ is a Hilbert space.

Corollary 1.1. *[1] Let Ω be an open bounded regular set of class C^1. The space $H_0^1(\Omega)$ coincides with the subspace of $H^1(\Omega)$ composed of functions which are zero on the boundary $\partial\Omega$.*

1.1.2 Some important inequalities

Theorem 1.1. *(Young's inequality) Let $a, b \geq 0$. For any $\epsilon > 0$ we have*

$$ab \leq \frac{a^2}{4\epsilon} + \epsilon b^2.$$

Theorem 1.2. *[7](Holder's inequality) Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p, q \leq +\infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$f, g \in L^1(\Omega) \quad and \quad \int_\Omega |fg|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Theorem 1.3. *[7](Cauchy-Schwarz's inequality) We put $p = q = 2$ in Holder's inequality, we obtain the Cauchy-Schwarz inequality:*

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

Proposition 1.1. *[7](Poincaré's inequality) Suppose that $1 \leq p < +\infty$ and Ω is a bounded open set. Then there exists a constant C such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1.$$

Corollary 1.2. *[1] Let Ω be an open set of $\mathbb{R}^n$ bounded in at least one space direction. Then the seminorm*

$$|v|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla v(x)|^2\, dx \right)^{1/2}$$

is a norm over $H_0^1(\Omega)$ which is equivalent to the usal norm induced by that of $H^1(\Omega)$.

Theorem 1.4. *(Jensen's inequality) Let $G : [a, b] \to \mathbb{R}$ be a convex function. Assume that the fuctions $f : \Omega \to [a, b]$ and $h : \Omega \to \mathbb{R}$ are integrable such that $h(x) \geq 0$, for any $x \in \Omega$ and $\int_\Omega h(x)\, dx = k > 0$. Then,*

$$G\left(\frac{1}{k}\int_\Omega f(x)h(x)dx\right) \leq \frac{1}{k}\int_\Omega G(f(x))\, h(x)dx.$$

1.1.3 Green's Formula

Theorem 1.5. *[1] (Green's formula) Let Ω be an open bounded regular set of class C^1. If u and v are functions of $H^1(\Omega)$, they satisfy*

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx + \int_{\partial\Omega} u(x)v(x)\eta_i(x)ds,$$

where $\eta = (\eta_i)_{1\leq i\leq n}$ is the outward unit normal to $\partial\Omega$.

Theorem 1.6. *[1](Green's formula) Let Ω be an open bounded regular set of class C^2. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \Delta u(x)\, v(x)\, dx = -\int_\Omega \nabla u(x)\, .\nabla v(x)\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v(x)\, ds.$$

1.1.4 Lax-Milgram Theorem

Definition 1.7. [7] A bilinear form $a : H \times H \to \mathbb{R}$ is said to be:

1. Continuous if there is a constant C such that:

$$|a(u, v)| \leq C\, \|u\|_H\, \|v\|_H\,, \quad \forall u, v \in H.$$

2. Coercive if there is a constant $\alpha > 0$ such that:

$$a(v, v) \geq \alpha\, \|v\|_H^2\,, \quad \forall v \in H.$$

Theorem 1.7. *[7] (Lax-Milgram) Assume that $a(u, v)$ is a continuous coercive bilinear form on H. Then, given any $\phi \in H^*$ (H^* is the topological duality of H), there exists a unique element $u \in H$ such that*

$$a(u, v) = \langle \phi, v\rangle\, .$$

1.1.5 Operator basics

Definition 1.8. [9] A linear operator A with domain $D(A)$ in a Banach space X, i.e., $A : D(A) \subset X \to X$, is closed if the graph $G(A) = \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$.

Lemma 1.1. *[9] A linear operator A with domain $D(A)$ in a Banach space X, i.e., $A : D(A) \subset X \to X$, is closed if for the sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ the limits $x_n \to x \in X$ and $Ax_n \to y \in X$ exist, then $x \in D(A)$ and $Ax = y$.*

1.1.6 M-dissipative operator

Definition 1.9. [9] A linear operator $(A, D(A))$ on a Banach space X is called dissipative if $\forall\lambda > 0, \forall x \in D(A)$:

$$\|(\lambda I - A)\, x\| \geq \lambda \|x\| .$$

Proposition 1.2. *[9] For a dissipative operator $(A, D(A))$ the following property holds:*

$$\lambda I - A \quad \textit{is injective}, \quad \forall\lambda > 0.$$

Definition 1.10. [7] A unbounded linear operator $A : D(A) \subset H \to H$ is called m-dissipative if:

1. A is dissipative,
2. $R(\lambda I - A) = H$ for all $\lambda > 0$. i.e.,

$$\forall\lambda > 0, \forall f \in H, \exists u \in D(A), \quad \text{such that} \quad \lambda u - Au = f.$$

Proposition 1.3. *[7] Let $A : D(A) \subset H \to H$ (H denotes a Hilbert space) be a m-dissipative operator, then*

1. *$D(A)$ is dense in H,*
2. *A is a closed operator.*

1.1.7 Spectrum of operator

Definition 1.11. [9] Let $A : D(A) \subset X \to X$ be a closed linear operator on a Banach space X. We call

$$\rho(A) = \left\{\lambda \in \mathbb{C} : (\lambda I - A) : D(A) \to X \quad is \quad bijective\right\}$$

the resolvent set and its complement $\sigma(A) = \mathbb{C}/\rho(A)$ the spectrum of A. For $\lambda \in \rho(A)$, the inverse

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

is, by the closed graph theorem, a bounded operator on X and is called the resolvent of A at the point λ.

Definition 1.12. [20, 23] Let A be a linear operator from a Banach space X into X. Let λ be a complex number. The spectrum of A consists of three mutually exclusive parts:

1. The point spectrum $\sigma_p(A)$ defined as: $\lambda \in \sigma_p(A)$ if $(\lambda I - A)$ does not admit of an inverse,
2. The residual spectrum $\sigma_r(A)$ defined as: $\lambda \in \sigma_r(A)$ if $D\left((\lambda I - A)^{-1}\right)$ is not dense in X,
3. The continuous spectrum $\sigma_c(A)$ defined as: $\lambda \in \sigma_c(A)$ if $D\left((\lambda I - A)^{-1}\right)$ is dense in X but $(\lambda I - A)^{-1}$ is not continuous.

1.2 Semigroups of bounded linear operators

Definition 1.13. [20] Let X be a Banach space. A one parameter family $\{T(t)\}_{t\geq 0}$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if:

$$T(0) = I,$$

$$T(t+s) = T(t)T(s), \quad \forall t, s \geq 0.$$

Remark 1.1. When the second property is true for all $t, s \in \mathbb{R}$, then we say that $\{T(t)\}_{t\in\mathbb{R}}$ is a group.

1.2.1 Uniformly continuous semigroups

Definition 1.14. [20] A semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ is uniformly continuous if:

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

1.2.2 Strongly continuous semigroups

Definition 1.15. [20] A semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t\to 0} T(t)\, x = x, \quad \forall x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

Definition 1.16. [20] The linear operator A defined by:

$$D(A) = \left\{ x \in X, \quad \lim_{t\to 0} \frac{T(t)x - x}{t} \quad \text{exists} \right\}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)\, x - x}{t}, \quad \forall x \in D(A)$$

is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$.

Theorem 1.8. *[20] Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup. There exist constants $\omega > 0$ and $M > 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall 0 \leq t < \infty.$$

Notation: We denote by $\mathcal{SG}(M, \omega)$ the set of C_0 semigroup $\{T(t)\}_{t\geq 0}$ for which there exists $\omega > 0$ and $M > 1$ such that

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall 0 \leq t < \infty.$$

In this case, we say that $\{T(t)\}_{t\geq 0}$ is an exponentially bounded C_0 semigroup.

Definition 1.17. [20] Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup. From Theorem 1.8 it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$.

1. If $\omega = 0$, $\{T(t)\}_{t\geq 0}$ is called uniformly bounded.

2. If $\omega = 0$ and $M = 1$, $\{T(t)\}_{t\geq 0}$ called a C_0 semigroup of contractions.

Theorem 1.9. *[20] A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded operator.*

1.3 Hille-Yosida Theorem

Theorem 1.10. *[20](Hille-Yosida) A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of C_0 semigroup $\{T(t)\}_{t\geq 0} \in \mathcal{SG}(M, \omega)$ if and only if:*

1. *A is closed,*
2. *$\overline{D(A)} = X$,*
3. *There exists $\omega \geq 0$ and $M \geq 1$ such that $\Lambda_\omega \subset \rho(A)$ and for $\lambda \in \Lambda_\omega$, we have:*

$$\|R(\lambda; A)^n\| \leq \frac{M}{(Re(\lambda) - \omega)^n}, \quad \forall n \in \mathbb{N}^*.$$

Theorem 1.11. *[20](Lumer phillips) Let $(A, D(A))$ be an unbounded linear operator of dense domain in X:*

1. *If A is dissipative and there exists a $\lambda_0 > 0$ such that $Im\ (\lambda_0 I - A) = X$ then, A is the infinitesimal generator of a C_0 contraction semigroup on X.*
2. *If A is the generator of a C_0 contraction semigroup on X then, A is dissipative and $Im\ (\lambda I - A) = X$ for all $\lambda > 0$.*

Theorem 1.12. *[7] Let A be a maximal monotone operator (-A is m-dissipative). Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty), H) \cap C([0, +\infty), D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad on\, [0, +\infty), \\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0| \quad and \quad \left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0|, \quad \forall t \geq 0.$$

Theorem 1.13. *[7] In a Banach space X, let the following problem*

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = f(t) \quad on\ [0,T], \\ u(0) = u_0. \end{cases}$$

Assume that A is m-accretive (-A is m-dissipative). Then for every $u_0 \in D(A)$ and every $f \in C^1([0,T]; X)$ there exists a unique solution u of the above problem with

$$u \in C^1([0,T]; X) \cap C([0,T]; D(A)).$$

Moreover, u is given by the formula

$$u(t) = S_A(t)u_0 + \int_0^t S_A(t-s) f(s)\, ds,$$

where A is the generator of $S(t)$.

CHAPTER 2

Global existence and exponential stability for a nonlinear Timoshenko system with delay

2.1 Presentation of the problem

In this chapter, we consider the following nonlinear Timoshenko System with delay:

$$\begin{cases} \rho_1 \varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \mu_1 \psi_t + \mu_2 \psi_t\left(x, t-\tau\right) + f\left(\psi\right) = 0, \end{cases} \tag{S}$$

where $(x,t) \in (0,1) \times \mathbb{R}^+$. Here $\varphi = \varphi(x,t)$ denotes the transverse diplacement of the beam, $\psi = \psi(x,t)$ denotes the rotation angle of the beam's filament and ρ_1, ρ_2, K, b are positive constants related to physical properties of the beam.

In this system, $\mu_1 \psi_t$ represents a frictional damping and $f(\psi)$ is a forcing term. The time delay is given by $\mu_2 \psi_t(x, t-\tau)$, where μ_1, μ_2, τ are positive constants.

The system is added by initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0, \quad \varphi_t(x,0) = \varphi_1, \quad \psi(x,0) = \psi_0, \quad \psi_t(x,0) = \psi_1, \\ \psi_t(x, t-\tau) = f_0(x, t-\tau), \quad t \in (0,\tau), \end{cases} \tag{2.1}$$

where f_0 is prescribed, and the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad \forall t \geq 0. \tag{2.2}$$

Our objective along this chapter is to extend the results of Said-Houari and Laskri [21] to a nonlinear framework by adding a force term $f(\psi)$. In this chapter, we present some preliminary results, we prove the well-posedness of the system by using semigroup theorem and some technical lemmas that are necessary to establish the exponential stability of our system by using energy methods.

2.2 Preliminaries

Hypothese: We assume $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\left|f\left(\psi^1\right) - f\left(\psi^2\right)\right| \leq K_0\left(\left|\psi^1\right|^\theta + \left|\psi^2\right|^\theta\right)\left|\psi^1 - \psi^2\right|, \quad \forall \psi^1, \psi^2 \in \mathbb{R}, \tag{2.3}$$

where $K_0, \theta > 0$, and we assume for $\hat{f}(z) = \int_0^z f(s)\, ds$ that

$$0 \leq \hat{f}(\psi) \leq f(\psi)\psi, \quad \psi \in \mathbb{R}. \tag{2.4}$$

In order to deal with the delay feedback term, we define the following new dependent variable:

$$z(x,\rho,t) = \psi_t(x, t - \tau\rho), \quad x \in (0,1), \rho \in (0,1), t > 0. \tag{2.5}$$

Differentiating z with respect to t and ρ, we get

$$z_t(x,\rho,t) = \psi_{tt}(x, t - \tau\rho)$$

and

$$z_\rho(x,\rho,t) = -\tau\psi_{tt}(x, t - \tau\rho).$$

Implies that

$$z_\rho(x,\rho,t) + \tau z_t(x,\rho,t) = 0.$$

Then, equations of (S) are transformed to

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) \\ \quad +\mu_1\psi_t(x,t) + \mu_2 z(x,1,t) + f(\psi(x,t)) = 0, \\ z_\rho(x,\rho,t) + \tau z_t(x,\rho,t) = 0, \quad x \in (0,1), \rho \in (0,1), \quad t > 0, \end{cases} \tag{2.6}$$

with the initial and boundary conditions:

$$\begin{cases} \varphi(x,0)=\varphi_0, \quad \varphi_t(x,0)=\varphi_1, \quad \psi(x,0)=\psi_0, \quad \psi_t(x,0)=\psi_1, \quad x\in(0,1) \\ z(x,\rho,0)=f_0(x,-\tau\rho), (x,t)\in(0,1)\times(0,\tau), \\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0, \quad t\geq 0, \\ z(x,0,t)=\psi_t(x,t), \quad x\in(0,1), \quad t\geq 0. \end{cases} \tag{2.7}$$

We introduce two new dependent variables $u=\varphi_t$ and $v=\psi_t$, then the problem (2.6)-(2.7) is reduced to the following problem as a first-order evolutionary equation:

$$\begin{cases} \dfrac{dU}{dt}(t)=AU+F, \quad t>0, \\ U(0)=U_0=(\varphi_0,\varphi_1,\psi_0,\psi_1,f_0(.,-\tau))^T, \end{cases} \tag{2.8}$$

such that $U=(\varphi,u,\psi,v,z)^T$, and

$$AU=\begin{pmatrix} u \\ \dfrac{K}{\rho_1}(\varphi_{xx}+\psi_x) \\ v \\ \dfrac{b}{\rho_2}\psi_{xx}-\dfrac{K}{\rho_2}(\varphi_x+\psi)-\dfrac{\mu_1}{\rho_2}v-\dfrac{\mu_2}{\rho_2}z(.,1) \\ -\dfrac{1}{\tau}z_\rho \end{pmatrix}, F=\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{1}{\rho_2}f(\psi) \\ 0 \end{pmatrix},$$

with the domain

$$D(A)=\left\{(\varphi,u,\psi,v,z)^T\in H; v=z(.,0), \text{ in } (0,1)\right\}, \tag{2.9}$$

where

$$\begin{aligned} H &= \left(H^2(0,1)\cap H_0^1(0,1)\right)\times H_0^1(0,1)\times\left(H^2(0,1)\cap H_0^1(0,1)\right) \\ &\quad \times H_0^1(0,1)\times L^2\left(0,1;H_0^1(0,1)\right). \end{aligned}$$

We define the energy space $\mathcal{H}$ by

$$\mathcal{H}=H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L^2((0,1)\times(0,1)). \tag{2.10}$$

We equip $\mathcal{H}$ with the following scalar product

$$\begin{aligned} \left\langle U,\overline{U}\right\rangle_{\mathcal{H}} &= \int_0^1\left(\rho_1 u\overline{u}+\rho_2 v\overline{v}+K(\varphi_x+\psi)\left(\overline{\varphi}_x+\overline{\psi}\right)+b\psi_x\overline{\psi_x}\right)dx \\ &\quad \xi\int_0^1\int_0^1 z(x,\rho)\overline{z}(x,\rho)\,d\rho dx, \end{aligned} \tag{2.11}$$

such that $U=(\varphi,u,\psi,v,z)^T$, $\overline{U}=\left(\overline{\varphi},\overline{u},\overline{\psi},\overline{v},\overline{z}\right)^T$ and ξ is a positive constant which satisfies

$$\tau\mu_2\leq\xi\leq\tau\left(2\mu_1-\mu_2\right). \tag{2.12}$$

Lemma 2.1. *Let $(\varphi,\varphi_t,\psi,\psi_t,z)$ be the solution of problem (2.6)-(2.7). The energy functional is defined by*

$$\begin{aligned}E(t)=\ &\frac{1}{2}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2+K\left(\varphi_x+\psi\right)^2+b\psi_x^2\right)dx\\&+\frac{\xi}{2}\int_0^1\int_0^1 z^2\left(x,\rho,t\right)d\rho dx+\int_0^1\hat{f}\left(\psi\left(t\right)\right)dx,\end{aligned}$$

satisfies for a positive constant C and for any $t\geq 0$ that

$$E'(t)\leq -C\int_0^1\psi_t^2\left(x,t\right)dx-C\int_0^1 z^2\left(x,1,t\right)dx\leq 0 \tag{2.13}$$

and there exist two positive constants γ_0 and C_1, independent of initial data in $\mathcal{H}$, such that for any $t\geq 0$

$$\begin{aligned}E(t)\geq\ &\gamma_0\left(\int_0^1\varphi_t^2dx+\int_0^1\psi_t^2dx+\int_0^1\left(\varphi_x+\psi\right)^2dx+\int_0^1\psi_x^2dx\right.\\&\left.+\int_0^1\int_0^1 z^2\left(x,\rho\right)d\rho dx\right)-C_1.\end{aligned} \tag{2.14}$$

Proof. Multiplying $(2.6)_1$ and $(2.6)_2$ by φ_t and ψ_t respectively and integrating over $(0,1)$ respect to x, and multiplying $(2.6)_3$ by $\frac{\xi}{\tau}z$ and integrating twice over $(0,1)$ respect to x and τ, we get

$$\begin{cases}\rho_1\int_0^1\varphi_{tt}\varphi_t dx-K\int_0^1\left(\varphi_x+\psi\right)_x\varphi_t dx=0,\\ \rho_2\int_0^1\psi_{tt}\psi_t dx-b\int_0^1\psi_{xx}\psi_t dx+K\int_0^1\left(\varphi_x+\psi\right)\psi_t dx\\ \quad+\mu_1\int_0^1\psi_t^2dx+\mu_2\int_0^1 z\left(x,1,t\right)\psi_t dx+\int_0^1\psi_t f\left(\psi\right)dx=0,\\ \frac{\xi}{\tau}\int_0^1\int_0^1 z_\rho\left(x,\rho,t\right)zd\rho dx+\xi\int_0^1\int_0^1 z_t\left(x,\rho,t\right)zd\rho dx=0.\end{cases} \tag{2.15}$$

By summing $(2.15)_1$ and $(2.15)_2$ and using integration by parts and boundary conditions, we get

$$\begin{aligned}&\frac{\rho_2}{2}\frac{d}{dt}\int_0^1\psi_t^2dx+\frac{b}{2}\frac{d}{dt}\int_0^1\psi_x^2dx+\frac{K}{2}\frac{d}{dt}\int_0^1\left(\varphi_x+\psi\right)^2dx+\frac{\rho_1}{2}\frac{d}{dt}\int_0^1\varphi_t^2dx+\frac{d}{dt}\hat{f}\left(\psi\right)\\&=-\mu_1\int_0^1\psi_t^2dx-\mu_2\int_0^1 z\left(x,1,t\right)\psi_t dx.\end{aligned} \tag{2.16}$$

Using integration by parts and (2.5), we get

$$\begin{aligned}
\xi \int_0^1 \int_0^1 z_t(x,\rho,t)\, z d\rho dx &= \frac{\xi}{2}\frac{d}{dt}\int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho dx \\
&= -\frac{\xi}{\tau}\int_0^1 \int_0^1 z_\rho(x,\rho,t)\, z d\rho dx \\
&= -\frac{\xi}{2\tau}\int_0^1 \int_0^1 \frac{\partial}{\partial\rho} z^2(x,\rho,t)\, d\rho dx \\
&= -\frac{\xi}{2\tau}\int_0^1 \left(z^2(x,1,t) - z^2(x,0,t)\right) dx \\
&= \frac{\xi}{2\tau}\int_0^1 \left(z^2(x,0,t) - z^2(x,1,t)\right) dx \\
&= \frac{\xi}{2\tau}\int_0^1 \left(\psi_t^2 - z^2(x,1,t)\right) dx.
\end{aligned} \tag{2.17}$$

Summing (2.16) and (2.17), we find

$$\frac{1}{2}\frac{d}{dt}\left[\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + K(\varphi_x+\psi)^2 + \xi\int_0^1 z^2(x,\rho,t)\, d\rho dx\right) + \hat{f}(\psi)\right]$$
$$= -\left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^1 \psi_t^2 dx - \mu_2\int_0^1 z(x,1,t)\,\psi_t dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\, dx.$$

Then,

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + K(\varphi_x+\psi)^2 + \xi\int_0^1 z^2(x,\rho,t)\, d\rho dx\right) + \hat{f}(\psi).$$

Using Young's inequality, we get

$$-\mu_2\int_0^1 z(x,1,t)\,\psi_t dx \le \frac{\mu_2}{2}\int_0^1 \psi_t^2 dx + \frac{\mu_2}{2}\int_0^1 z^2(x,1,t)\, dx.$$

Then,

$$E'(t) \le -C\int_0^1 \psi_t^2(x,t)\, dx - C\int_0^1 z^2(x,1,t)\, dx \le 0.$$

Choosing $\gamma_0 = min\left\{\frac{\xi}{2}, \frac{1}{2}\right\}$ and using (2.3), we get

$$\begin{aligned}
E(t) \ge\ & \gamma_0\left(\int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx + \int_0^1 (\varphi_x+\psi)^2 dx + \int_0^1 \psi_x^2 dx\right. \\
&\left. + \int_0^1\int_0^1 z^2(x,\rho)\, d\rho dx\right) - C_1.
\end{aligned}$$

□

2.3 The well-posedness

Lemma 2.2. *The operator A defined in (2.8) is the infinitesimal generator of a C_0- semigroup in $\mathcal{H}$.*

Proof. Let A and AU be defined previously, according to (2.11) and using integration by parts, we find that

$$\begin{aligned}\langle AU,U\rangle_{\mathcal{H}} = {} & K\int_0^1 u\left(\varphi_{xx}+\psi_x\right)dx + b\int_0^1 v\psi_{xx}dx - K\int_0^1 v\left(\varphi_x+\psi\right)dx \\ & -\mu_1\int_0^1 v^2dx - \mu_2\int_0^1 z\left(x,1\right)vdx + K\int_0^1\left(u_x+v\right)\left(\varphi_x+\psi\right)dx \\ & +b\int_0^1 v_x\psi_x dx - \frac{\xi}{\tau}\int_0^1\int_0^1 z_\rho z d\rho dx \\ = {} & -\mu_1\int_0^1 v^2dx - \mu_2\int_0^1 z\left(.,1\right)vdx - \frac{\xi}{\tau}\int_0^1\int_0^1 z_\rho z d\rho dx.\end{aligned}$$

We have,

$$\int_0^1\int_0^1 z_\rho z d\rho dx = \frac{1}{2}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2\left(x,\rho\right)d\rho dx = \frac{1}{2}\int_0^1\left(z^2\left(x,1\right)-z^2\left(x,0\right)\right)dx.$$

Then by using (2.5), the fact that $v=\psi_t$ and Young's inequality, we obtain

$$\begin{aligned}\langle AU,U\rangle_{\mathcal{H}} = {} & -\mu_1\int_0^1 v^2dx - \mu_2\int_0^1 z\left(x,1\right)vdx - \frac{\xi}{2\tau}\int_0^1\left(z^2\left(x,1\right)-z^2\left(x,0\right)\right)dx \\ = {} & -\mu_1\int_0^1 v^2dx - \mu_2\int_0^1 z\left(x,1\right)vdx - \frac{\xi}{2\tau}\int_0^1 z^2\left(x,1\right)dx + \frac{\xi}{2\tau}\int_0^1 z^2\left(x,0\right)dx \\ = {} & -\mu_1\int_0^1 v^2dx - \mu_2\int_0^1 z\left(x,1\right)vdx - \frac{\xi}{2\tau}\int_0^1 z^2\left(x,1\right)dx + \frac{\xi}{2\tau}\int_0^1 v^2dx \\ \leq {} & -\mu_1\int_0^1 v^2dx + \frac{\mu_2}{2}\int_0^1 z^2\left(x,1\right)dx + \frac{\mu_2}{2}\int_0^1 v^2dx \\ & -\frac{\xi}{2\tau}\int_0^1 z^2\left(x,1\right)dx + \frac{\xi}{2\tau}\int_0^1 v^2dx \\ \leq {} & \left(-\mu_1+\frac{\mu_2}{2}+\frac{\xi}{2\tau}\right)\int_0^1 v^2dx + \left(\frac{\mu_2}{2}-\frac{\xi}{2\tau}\right)\int_0^1 z^2\left(x,1\right)dx.\end{aligned}$$

It follows from (2.12) that

$$-\mu_1+\frac{\mu_2}{2}+\frac{\xi}{2\tau}<0 \quad\text{and}\quad \frac{\mu_2}{2}-\frac{\xi}{2\tau}<0,$$

which implies that $\langle AU,U\rangle_{\mathcal{H}}\leq 0$, then, the operator A is a disspative operator.

Next, we prove that the operator $\lambda I - A$ is surjective for $\lambda > 0$.

Let $(f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \psi, v, z)^T \in D(A)$ solution of the following system of equations

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_{xx} + \psi_x) = f_2, \\ \lambda\psi - v = f_3, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) + \dfrac{\mu_1}{\rho_2}v + \dfrac{\mu_2}{\rho_2}z(.,1) = f_4, \\ \lambda z + \dfrac{1}{\tau}z_\rho = f_5. \end{cases} \tag{2.18}$$

The solution of the last equation in (2.18) is

$$z(x,\rho) = v(x)\,e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho f_5(x,s)\,e^{-\lambda\tau s}ds. \tag{2.19}$$

The first and the third equations in (2.18) give

$$\begin{cases} u = \lambda\varphi - f_1, \\ v = \lambda\psi - f_3. \end{cases} \tag{2.20}$$

Then from (2.20), the solution (2.19) becomes

$$z(x,\rho) = \lambda\psi(x)\,e^{-\lambda\tau\rho} - f_3 e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho f_5(x,s)\,e^{-\lambda\tau s}ds. \tag{2.21}$$

By using (2.18) and (2.20), the functions φ and ψ satisfying the following system

$$\begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_{xx} + \psi_x) = f_2 + \lambda f_1, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) + \dfrac{\mu_1}{\rho_2}v + \dfrac{\mu_2}{\rho_2}z(.,1) = f_4 + \lambda f_3, \end{cases} \tag{2.22}$$

which is equivalent to the following system, by integrating over $(0,1)$ and using integration by parts

$$\begin{cases} \displaystyle\int_0^1 \left(\rho_1\lambda^2\varphi W + K(\varphi_x + \psi)W_x\right)dx = \int_0^1 \rho_1(f_2 + \lambda f_1)W dx, \\ \displaystyle\int_0^1 \left(\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x + \psi)\chi + \mu_1 v\chi + \mu_2 z(.,1)\chi\right)dx = \rho_2\int_0^1 (f_4 + \lambda f_3)\chi dx, \end{cases} \tag{2.23}$$

for all $(W,\chi) \in H_0^1(0,1) \times H_0^1(0,1)$.

From (2.21), we have

$$z(x,1) = \lambda\psi(x)\,e^{-\lambda\tau} + z_0(x),$$

where $x \in (0,1)$ and

$$z_0(x) = -f_3 e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^\rho f_5(x,s)\, e^{-\lambda\tau s} ds.$$

Consequently, the problem (2.23) is equivalent to the problem

$$\xi\left((\varphi,\psi),(W,\chi)\right) = l\,(W,\chi), \tag{2.24}$$

where the bilinear form $\xi : \left(H_0^1(0,1) \times H_0^1(0,1)\right)^2 \to \mathbb{R}$ and the linear form $l : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ are defined by

$$\begin{aligned}\xi\left((\varphi,\psi),(W,\chi)\right) = & \int_0^1 \rho_1 \lambda^2 \varphi W dx + K \int_0^1 (\varphi_x + \psi)(W_x + \chi)\, dx \\ & + \int_0^1 \left(\rho_2 \lambda^2 \psi \chi + b \psi_x \chi_x\right) dx + \int_0^1 \left(\mu_1 + \mu_2 e^{-\lambda\tau}\right) \lambda \psi \chi dx \end{aligned} \tag{2.25}$$

and

$$l\,(W,\chi) = \int_0^1 \left(\mu_1 f_3 \chi - \mu_2 z_0(x)\, \chi\right) dx + \int_0^1 \rho_1 (f_2 + \lambda f_1)\, W dx + \rho_2 \int_0^1 (f_4 + \lambda f_3)\, \chi dx.$$

It is easy to verify that ξ is continuous and coercive, and l is continuous. So applying the Lax-Milgram Theorem, we deduce that for all $(W,\chi) \in H_0^1(0,1) \times H_0^1(0,1)$, problem (2.24) admits a unique solution $(\varphi,\psi) \in H_0^1(0,1) \times H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (2.23) that $(\varphi,\psi) \in H^2(0,1) \times H^2(0,1)$.
Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Then we can infer that the operator A is m-dissipative in $\mathcal{H}$. Since $D(A)$is dense in $\mathcal{H}$, thus we can conclude that the operator A is the infinitesimal generator of a C_0 semigroup in $\mathcal{H}$ by the Lumer-Philips theorem. □

Lemma 2.3. *The operator F defined in (2.8) is locally Lipschitz in $\mathcal{H}$.*

Proof. Let $U = \left(\varphi^1, u^1, \psi^1, v^1, z^1\right)$ and $V = \left(\varphi^2, u^2, \psi^2, v^2, z^2\right)$, then by using (2.3), Holder's inequality and the embedding $H^1(0,1) \hookrightarrow L^{2(\theta+1)}$, for a fixed positive constant c_0, we obtain

$$\begin{aligned}\|F(U) - F(V)\|_{\mathcal{H}}^2 &= \frac{1}{\rho_2} \int_0^1 \left| f\left(\psi^1\right) - f\left(\psi^2\right)\right|^2 dx \\ &\le \frac{1}{\rho_2} K_0^2 \int_0^1 \left(\left|\psi^1\right|^\theta + \left|\psi^2\right|^\theta\right)^2 \left|\psi^1 - \psi^2\right|^2 dx\end{aligned}$$

$$
\begin{aligned}
&\leq c_0 \left(\int_0^1 \left(\left|\psi^1\right|^\theta + \left|\psi^2\right|^\theta \right)^{\frac{2(\theta+1)}{\theta}} dx \right)^{\frac{\theta}{(\theta+1)}} \left(\int_0^1 \left|\psi^1 - \psi^2\right|^{2(\theta+1)} dx \right)^{\frac{1}{(\theta+1)}} \\
&\leq c_0 \left\| \left|\psi^1\right|^\theta + \left|\psi^2\right|^\theta \right\|^2_{L^{\frac{2(\theta+1)}{\theta}}} \cdot \left\| \left|\psi^1 - \psi^2\right| \right\|^2_{L^{2(\theta+1)}} \\
&\leq c_0 \left(\left\|\psi^1\right\|^\theta_{L^{\frac{2(\theta+1)}{\theta}}} + \left\|\psi^2\right\|^\theta_{L^{\frac{2(\theta+1)}{\theta}}} \right)^2 \cdot \left\| \left|\psi^1 - \psi^2\right| \right\|^2_{L^{2(\theta+1)}} \\
&\leq c_0 \left(\left\|\psi^1\right\|^{2\theta}_{L^{\frac{2(\theta+1)}{\theta}}} + \left\|\psi^2\right\|^{2\theta}_{L^{\frac{2(\theta+1)}{\theta}}} \right) \cdot \left\|\psi^1 - \psi^2\right\|^2_{L^{2(\theta+1)}} \\
&\leq c_0 \left(\left\|\psi^1\right\|^{2\theta}_{H^1} + \left\|\psi^2\right\|^{2\theta}_{H^1} \right) \cdot \left\|\psi^1 - \psi^2\right\|^2_{H^1} \\
&\leq c_0 M \left\|\psi^1 - \psi^2\right\|^2_{H^1} \\
&\leq c_0 M \left\|U - V\right\|^2_{\mathcal{H}} .
\end{aligned}
$$

□

Theorem 2.1. *Assume that (2.3), (2.4) and $\mu_2 \leq \mu_1$ hold, then we have the following results:*

1. *If $U_0 \in \mathcal{H}$, then the problem (2.8) has a unique mild solution $U \in C([0,\infty), \mathcal{H})$ with $U(0) = U_0$.*
2. *If U_1 and U_2 are two mild solutions of problem (2.8), then there exists a positive constant $C_0 = C(U_1(0), U_2(0))$ such that*

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}} \leq e^{C_0 T} \|U_1(0) - U_2(0)\|_{\mathcal{H}} \quad 0 \leq t \leq T. \tag{2.26}$$

3. *If $U_0 \in D(A)$, then the above mild solution is a strong solution.*

Proof. (1) It follows from lemmas 2.2-2.3 that the cauchy problem has a unique local mild solution

$$U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} F(U(s))\, ds, \tag{2.27}$$

defined in a maximal interval $(0, t_{max})$. If $t_{max} < \infty$, then

$$\lim_{t \to t_{max}} \|U(t)\|_{\mathcal{H}} = +\infty. \tag{2.28}$$

Let $U(t)$ be a mild solution with $U_0(t) \in D(A)$. By using theorem 6.1.5 in Pazy [20], we conclude that it is a strong solution. It follows from (2.14) that $\|U(t)\|^2_{\mathcal{H}} \leq \frac{1}{\gamma_0}(E(0) + C_1)$, therefore $t_{max} = \infty$. Then the solution is global.

2. By using (2.27),the local Lipschitz behavior of F and Gronwall's inequality, we get (2.26). Then we can obtain the continuous dependence on the initial data for mild solutions.

3. By using theorem 6.1.5 in Pazy [20], we know that any mild solutions with initial data in $D(A)$ are strong.

□

2.4 Exponential stability

Lemma 2.4. *Let $(\varphi, \varphi_t, \psi, \psi_t, z)$ be the solution of problem (2.6)-(2.7). The functional I_1 defined by*

$$I_1 = -\int_0^1 (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t)\, dx - \frac{\mu_1}{2}\int_0^1 \psi^2 dx \tag{2.29}$$

satisfies for any $\epsilon > 0$

$$\begin{aligned} I_1'(t) \le\ & -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + K\int_0^1 (\varphi_x+\psi)^2\, dx + (b+c+\epsilon)\int_0^1 \psi_x^2 dx \\ & + \frac{\mu_2^2}{4\epsilon}\int_0^1 z^2(x,1,t)\, dx. \end{aligned} \tag{2.30}$$

Proof. By taking a derivative of I_1 and using equations in (2.6), we get

$$\begin{aligned} I_1'(t) =\ & -\rho_1\int_0^1 \varphi_{tt}\varphi dx - \rho_1\int_0^1 \varphi_t^2 dx - \rho_2\int_0^1 \psi_t^2 dx - \rho_2\int_0^1 \psi_{tt}\psi dx - \mu_1\int_0^1 \psi_t\psi dx \\ =\ & -K\int_0^1 \varphi(\varphi_x+\psi)_x\, dx - \rho_1\int_0^1 \varphi_t^2 dx - \rho_2\int_0^1 \psi_t^2 dx - b\int_0^1 \psi_{xx}\psi dx \\ & + K\int_0^1 (\varphi_x+\psi)\psi dx + \mu_2\int_0^1 \psi z(x,1,t)\, dx + \int_0^1 \psi f(\psi)\, dx. \end{aligned}$$

Using integration by parts, boundary conditions, Young's inequality, (2.3) and Poincaré's inequality, for a fixed positive constant C, we obtain

$$-K\int_0^1 \varphi(\varphi_x+\psi)_x\, dx = K\int_0^1 \varphi_x(\varphi_x+\psi)\, dx,$$

$$-b\int_0^1 \psi_{xx}\psi dx = b\int_0^1 \psi_x^2 dx,$$

$$\mu_2\int_0^1 \psi z(x,1,t)\, dx \le \epsilon\int_0^1 \psi_x^2 dx + \frac{\mu_2^2}{4\epsilon}\int_0^1 z^2(x,1,t)\, dx,$$

and

$$\int_0^1 \psi f(\psi)\,dx \le \int_0^1 |\psi|\,|\psi|^\theta\,|\psi|\,dx \le \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\|_{L^2} \le C\int_0^1 \psi_x^2 dx.$$

Then,

$$\begin{aligned} I_1'(t) \le & -\int_0^1 \left(\rho_1\varphi_t^2+\rho_2\psi_t^2\right)dx + K\int_0^1 (\varphi_x+\psi)^2\,dx + (b+C+\epsilon)\int_0^1 \psi_x^2 dx \\ & +\frac{\mu_2^2}{4\epsilon}\int_0^1 z^2(x,1,t)\,dx. \end{aligned}$$

□

Lemma 2.5. *Let $(\varphi,\varphi_t,\psi,\psi_t,z)$ be the solution of problem (2.6)-(2.7). The functional I_2 defined by*

$$I_2 = \int_0^1 (\rho_2\psi\psi_t + \rho_1\varphi_t\omega)\,dx + \frac{\mu_1}{2}\int_0^1 \psi^2 dx, \tag{2.31}$$

where ω is the solution of

$$-\omega_{xx} = \psi_x, \quad g\,|_{x=0,1} = 0. \tag{2.32}$$

Then I_2 satisfies for any $\lambda, \tilde{\lambda} > 0$

$$\begin{aligned} I_2'(t) \le\ & (\mu_2\lambda - b)\int_0^1 \psi_x^2 dx + \left(\rho_2+\frac{\rho_1}{4\tilde{\lambda}}\right)\int_0^1 \psi_t^2 dx + \rho_1\tilde{\lambda}\int_0^1 \varphi_t^2 dx \\ & +\frac{\mu_2^2}{4\lambda}\int_0^1 z^2(x,1,t)\,dx - \int_0^1 \hat{f}(\psi)\,dx. \end{aligned} \tag{2.33}$$

Proof. By taking a derivative of I_2 and using equations in (2.6), we get

$$\begin{aligned} I_2'(t) =\ & \rho_2\int_0^1 \psi_t^2 dx + \rho_2\int_0^1 \psi_{tt}\psi dx + \rho_1\int_0^1 \varphi_{tt}\omega dx + \rho_1\int_0^1 \varphi_t\omega_t dx + \mu_1\int_0^1 \psi_t\psi dx \\ =\ & \rho_2\int_0^1 \psi_t^2 dx + b\int_0^1 \psi_{xx}\psi dx - K\int_0^1 (\varphi_x+\psi)\,\psi dx - \mu_2\int_0^1 \psi z(x,1,t)\,dx \\ & -\int_0^1 \psi f(\psi)\,dx + K\int_0^1 \omega\,(\varphi_x+\psi)_x\,dx + \rho_1\int_0^1 \varphi_t\omega_t dx. \end{aligned}$$

From (2.32) and Poincaré's inequality, we get

$$\begin{cases} \int_0^1 \omega_x^2 dx \le \int_0^1 \psi^2 dx \le \int_0^1 \psi_x^2 dx, \\ \int_0^1 \omega_t^2 dx \le \int_0^1 \omega_{xt}^2 dx \le \int_0^1 \psi_t^2 dx. \end{cases} \tag{2.34}$$

By using integration by parts once and twice, boundary conditions and (2.32), we get

$$b\int_0^1 \psi_{xx}\psi dx = -b\int_0^1 \psi_x^2 dx,$$

$$-K\int_0^1 (\varphi_x+\psi)\,\psi dx = K\int_0^1 \varphi\psi_x dx - K\int_0^1 \psi^2 dx$$

and

$$\begin{aligned}K\int_0^1 \omega\,(\varphi_x+\psi)_x\,dx &= K\int_0^1 \omega\varphi_{xx}dx + K\int_0^1 \omega\psi_x^2 dx\\ &= K\int_0^1 \omega_{xx}\varphi dx - K\int_0^1 \omega\omega_{xx}dx\\ &= -K\int_0^1 \psi_x\varphi dx + K\int_0^1 \omega_x^2 dx.\end{aligned}$$

Substituting these results into $I_2'(t)$, we find

$$\begin{aligned}I_2'(t) &= \rho_2\int_0^1 \psi_t^2 dx - b\int_0^1 \psi_x^2 dx - K\int_0^1 \psi^2 dx + K\int_0^1 \omega_x^2 dx + \rho_1\int_0^1 \varphi_t\omega_t dx\\ &\quad -\mu_2\int_0^1 \psi z\,(x,1,t)\,dx - \int_0^1 \psi f\,(\psi)\,dx.\end{aligned}$$

Using Young's and Poincaré's inequality, (2.34) and (2.4), for some positive constants $\lambda, \tilde{\lambda} > 0$, we obtain

$$\begin{aligned}-\mu_2\int_0^1 \psi z\,(x,1,t)\,dx &\le \mu_2\lambda\int_0^1 \psi^2 dx + \frac{\mu_2}{4\lambda}\int_0^1 z^2\,(x,1,t)\,dx\\ &\le \mu_2\lambda\int_0^1 \psi_x^2 dx + \frac{\mu_2}{4\lambda}\int_0^1 z^2\,(x,1,t)\,dx\end{aligned}$$

and

$$\begin{aligned}\rho_1\int_0^1 \varphi_t\omega_t dx &\le \rho_1\tilde{\lambda}\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\tilde{\lambda}}\int_0^1 \omega_t^2 dx\\ &\le \rho_1\tilde{\lambda}\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\tilde{\lambda}}\int_0^1 \psi_t^2 dx.\end{aligned}$$

Then, we arrive at

$$\begin{aligned}I_2'(t) \le\ & (\mu_2\lambda - b)\int_0^1 \psi_x^2 dx + \left(\rho_2 + \frac{\rho_1}{4\tilde{\lambda}}\right)\int_0^1 \psi_t^2 dx + \rho_1\tilde{\lambda}\int_0^1 \varphi_t^2 dx\\ & + \frac{\mu_2^2}{4\lambda}\int_0^1 z^2\,(x,1,t)\,dx - \int_0^1 \hat{f}\,(\psi)\,dx.\end{aligned}$$

□

Lemma 2.6. *Let $(\varphi, \varphi_t, \psi, \psi_t, z)$ be the solution of problem (2.6)-(2.7). The functional J defined by*

$$J = \rho_2\int_0^1 \psi_t\,(\varphi_x+\psi)\,dx + \rho_2\int_0^1 \psi_x\varphi_t dx \tag{2.35}$$

satisfies

$$\begin{aligned}J'(t) \le\ & [b\varphi_x\psi_x]_{x=0}^{x=1} - \frac{K}{4}\int_0^1 (\varphi_x+\psi)^2\,dx + \left(\rho_2 + \frac{\mu_1^2}{K}\right)\int_0^1 \psi_t^2 dx\\ & + \frac{\mu_2^2}{K} + \int_0^1 z^2\,(x,1,t)\,dx + c\int_0^1 \psi_x^2 dx - \int_0^1 \hat{f}\,(\psi)\,dx.\end{aligned} \tag{2.36}$$

Proof. By taking a derivative of J and using equations in (2.6), we get

$$\begin{aligned} J'(t) &= \rho_2 \int_0^1 \psi_{tt}(\varphi_x+\psi)\,dx + \rho_2 \int_0^1 \psi_t(\varphi_x+\psi)_t\,dx + \rho_2 \int_0^1 \psi_{xt}\varphi_t dx + \rho_2 \int_0^1 \psi_x \varphi_{tt} dx \\ &= b\int_0^1 \psi_{xx}(\varphi_x+\psi)\,dx - K\int_0^1 (\varphi_x+\psi)^2\,dx - \mu_1 \int_0^1 \psi_t(\varphi_x+\psi)\,dx \\ &-\mu_2 \int_0^1 z(x,1,t)(\varphi_x+\psi)\,dx - \int_0^1 (\varphi_x+\psi) f(\psi)\,dx + \rho_2 \int_0^1 \psi_t(\varphi_x+\psi)_t\,dx \\ &+\rho_2 \int_0^1 \psi_t(\varphi_x+\psi)_t\,dx + \rho_2 \int_0^1 \psi_{xt}\varphi_t dx + \frac{\rho_2 K}{\rho_1}\int_0^1 \psi_x(\varphi_x+\psi)_x\,dx. \end{aligned}$$

Using integration by parts and boundary conditions, we get

$$\begin{aligned} b\int_0^1 \psi_{xx}(\varphi_x+\psi)\,dx &= [b\psi_x(\varphi_x+\psi)]_{x=0}^{x=1} - b\int_0^1 \psi_x(\varphi_x+\psi)_x\,dx \\ &= [b\psi_x\varphi_x]_{x=0}^{x=1} - b\int_0^1 \psi_x(\varphi_x+\psi)_x\,dx \end{aligned}$$

and

$$\begin{aligned} \rho_2 \int_0^1 \psi_t(\varphi_x+\psi)_t\,dx &= \rho_2\int_0^1 \psi_t\varphi_{xt}dx + \rho_2\int_0^1 \psi_t^2 dx \\ &= -\rho_2\int_0^1 \psi_{xt}\varphi_t dx + \rho_2\int_0^1 \psi_t^2 dx. \end{aligned}$$

Then, using $\dfrac{\rho_1}{\rho_2} = \dfrac{K}{b}$, J' becomes

$$\begin{aligned} J'(t) &= [b\psi_x\varphi_x]_{x=0}^{x=1} - K\int_0^1 (\varphi_x+\psi)^2\,dx - \mu_1\int_0^1 \psi_t(\varphi_x+\psi)\,dx \\ &-\mu_2\int_0^1 z(x,1,t)(\varphi_x+\psi)\,dx - \int_0^1 \varphi_x f(\psi)\,dx - \int_0^1 \psi f(\psi)\,dx \\ &+\rho_2\int_0^1 \psi_t(\varphi_x+\psi)_t^2\,dx. \end{aligned}$$

We have, by using Young's inequality

$$-\mu_1\int_0^1 \psi_t(\varphi_x+\psi)\,dx \le \frac{\mu_1^2}{K}\int_0^1 \psi_t^2 dx + \frac{K}{4}\int_0^1 (\varphi_x+\psi)^2\,dx$$

and

$$-\mu_2\int_0^1 z(x,1,t)(\varphi_x+\psi)\,dx \le \frac{\mu_2^2}{K}\int_0^1 z^2(x,1,t)\,dx + \frac{K}{4}\int_0^1 (\varphi_x+\psi)^2\,dx.$$

We have also, by using Holder's, Young's and Poincaré's inequalities,(2.3) and the facts

$\|\psi\|_{L^{2(\theta+1)}} \leq c\,\|\psi\|_{L^2}$ and $\varphi_x^2 \leq 2\left(\varphi_x+\psi\right)^2+2\psi^2$, we get

$$\begin{aligned}
\int_0^1 |\varphi_x f(\psi)|\,dx &\leq \int_0^1 |\varphi_x|\,|\psi|\,|\psi|^{\theta}\,dx \\
&\leq \|\psi\|_{2(\theta+1)}^{\theta}\,\|\psi\|_{2(\theta+1)}\,\|\varphi_x\|_2 \\
&\leq c\,\|\varphi_x\|_2\,\|\psi\|_2 \\
&\leq c\,\|\varphi_x\|_2\,\|\psi_x\|_2 \\
&\leq \frac{K}{8}\int_0^1 \varphi_x^2 dx+\frac{2c^2}{K}\int_0^1 \psi_x^2 dx \\
&\leq \frac{K}{4}\int_0^1 \left(\varphi_x+\psi\right)^2 dx+\frac{K}{4}\int_0^1 \psi_x^2 dx+\frac{2c^2}{K}\int_0^1 \psi_x^2 dx \\
&\leq \frac{K}{4}\int_0^1 \left(\varphi_x+\psi\right)^2 dx+c\int_0^1 \psi_x^2 dx.
\end{aligned}$$

Thus, substituting the above results into $J'(t)$ and using (2.4), we arrive at

$$\begin{aligned}
J'(t) \leq\ & \left[b\varphi_x\psi_x\right]_{x=0}^{x=1}-\frac{K}{4}\int_0^1 \left(\varphi_x+\psi\right)^2 dx+\left(\rho_2+\frac{\mu_1^2}{K}\right)\int_0^1 \psi_t^2 dx \\
&+\frac{\mu_2^2}{K}+\int_0^1 z^2(x,1,t)\,dx+c\int_0^1 \psi_x^2 dx-\int_0^1 \hat{f}(\psi)\,dx.
\end{aligned}$$

□

Lemma 2.7. *Let $(\varphi,\varphi_t,\psi,\psi_t,z)$ be the solution of problem (2.6)-(2.7), then the following estimate holds for any $\epsilon>0$*

$$\begin{aligned}
\left[b\varphi_x\psi_x\right]_{x=0}^{x=1} \leq\ & -\frac{\epsilon\rho_1}{K}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x dx-\frac{\rho_2 b}{4\epsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x dx+2\frac{\epsilon\rho_1}{K}\int_0^1 \varphi_t^2 dx \\
&+\left(6\epsilon+\frac{\epsilon K^2}{4}\right)\int_0^1 \left(\varphi_x+\psi\right)^2 dx+\left(7\epsilon+\frac{c}{4\epsilon}+\frac{b^2}{2\epsilon}+\frac{b^2}{2}+\frac{b^2}{2\epsilon^3}\right)\int_0^1 \psi_x^2 dx \\
&+\left(\frac{b\rho_2}{2\epsilon}+\frac{\mu_1^2}{4\epsilon^2}\right)\int_0^1 \psi_t^2 dx+\frac{\mu_2^2}{4\epsilon^2}\int_0^1 z^2(x,1,t)\,dx,
\end{aligned} \tag{2.37}$$

such that the functional q is defined by

$$q(x)=-4x+2, \quad x\in(0,1).$$

Proof. First, it follows from Young's inequality, for any $\epsilon>0$ that

$$\left[b\varphi_x\psi_x\right]_{x=0}^{x=1} \leq \epsilon\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\frac{b^2}{4\epsilon}\left[\psi_x^2(1)+\psi_x^2(0)\right]. \tag{2.38}$$

We have, by using (2.6)

$$\begin{aligned} \frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x dx = & \int_0^1 b\rho_2\psi_{tt}\psi_x dx + \int_0^1 b\rho_2 q\psi_t\psi_{xt}dx \\ = & b^2\int_0^1 q\psi_{xx}\psi_x dx - bK\int_0^1 q\left(\varphi_x+\psi\right)\psi_x dx - b\mu_1\int_0^1 q\psi_t\psi_x dx \\ & -b\mu_2\int_0^1 qz\left(x,1,t\right)\psi_x dx - b\int_0^1 q\psi_x f\left(\psi\right)dx + b\int_0^1 \rho_2 q\psi_t\psi_{xt}dx, \end{aligned}$$

by using integration by parts, we get

$$\begin{aligned} b^2\int_0^1 q\psi_{xx}\psi_x dx & = b^2\int_0^1 q\frac{d}{dx}\left(\frac{\psi_x^2}{2}\right)dx \\ & = \frac{1}{2}b^2\left[q\psi_x^2\right]_{x=0}^{x=1} + 2b^2\int_0^1 \psi_x^2 dx \qquad (2.39) \\ & = -b^2\left[\psi_x^2\left(1\right)+\psi_x^2\left(0\right)\right] + 2b^2\int_0^1 \psi_x^2 dx, \end{aligned}$$

then,

$$\begin{aligned} \frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x dx & = -b^2\left[\psi_x^2\left(1\right)+\psi_x^2\left(0\right)\right] + 2b^2\int_0^1 \psi_x^2 dx + 2b\rho_2\int_0^1 \psi_t^2 dx \\ & -bK\int_0^1 q\left(\varphi_x+\psi\right)\psi_x dx - b\mu_1\int_0^1 q\psi_t\psi_x dx - b\mu_2\int_0^1 qz\left(x,1,t\right)\psi_x dx \\ & -b\int_0^1 q\psi_x f\left(\psi\right)dx, \end{aligned}$$

using Young's inequality, we obtain

$$-bK\int_0^1 q\left(\varphi_x+\psi\right)\psi_x dx \le \epsilon^2K^2\int_0^1 \left(\varphi_x+\psi\right)^2 dx + \frac{b^2}{\epsilon^2}\int_0^1 \psi_x^2 dx,$$

$$-b\mu_1\int_0^1 q\psi_t\psi_x dx \le \frac{\mu_1^2}{\epsilon}\int_0^1 \psi_t^2 dx + \epsilon b^2\int_0^1 \psi_x^2 dx,$$

$$-b\mu_2\int_0^1 qz\left(x,1,t\right)\psi_x dx \le \frac{\mu_2^2}{\epsilon}\int_0^1 z^2\left(x,1,t\right)dx + \epsilon b^2\int_0^1 \psi_x^2 dx$$

and

$$-b\int_0^1 q\psi_x f\left(\psi\right)dx \le c\int_0^1 \psi_x^2 dx.$$

Then,

$$\begin{aligned} \frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x dx \le & -b^2\left[\psi_x^2\left(1\right)+\psi_x^2\left(0\right)\right]\left(c+2b^2+2\epsilon b^2+\frac{b^2}{\epsilon^2}\right)\int_0^1 \psi_x^2 dx \\ & +\left(2b\rho_2+\frac{\mu_1^2}{\epsilon}\right)\int_0^1 \psi_t^2 dx + \epsilon^2K^2\int_0^1 \left(\varphi_x+\psi\right)^2 dx \qquad (2.40) \\ & +\frac{\mu_2^2}{\epsilon}\int_0^1 z^2\left(x,1,t\right)dx. \end{aligned}$$

In the other side, we have, by using (2.6)

$$\begin{aligned}\frac{d}{dt}\int_0^1 \rho_1 q\varphi_t\varphi_x dx &= \rho_1\int_0^1 q\varphi_{tt}\varphi_x dx+\rho_1\int_0^1 q\varphi_t\varphi_{xt}dx\\ &= K\int_0^1 q\left(\varphi_x+\psi\right)_x\varphi_x dx+\rho_1\int_0^1 q\varphi_t\varphi_{xt}dx,\end{aligned}$$

we have, by using integration by parts and boundary conditions

$$\rho_1\int_0^1 q\varphi_t\varphi_{xt}dx = \rho_1\int_0^1 q\frac{d}{dx}\left(\frac{\varphi_t^2}{2}\right)dx = 2\rho_1\int_0^1 \varphi_t^2 dx$$

and

$$\begin{aligned}K\int_0^1 q\left(\varphi_x+\psi\right)_x\varphi_x dx &= K\int_0^1 q\varphi_{xx}\varphi_x dx+K\int_0^1 q\psi_x\varphi_x dx\\ &= K\int_0^1 q\frac{d}{dx}\left(\frac{\varphi_x^2}{2}\right)dx+K\int_0^1 q\psi_x\varphi_x dx\\ &= \frac{K}{2}\left[q\varphi_x^2\right]_{x=0}^{x=1}+2K\int_0^1\varphi_x^2 dx+K\int_0^1 q\psi_x\varphi_x dx\\ &= -K\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+2K\int_0^1\varphi_x^2 dx+K\int_0^1 q\psi_x\varphi_x dx,\end{aligned}$$

but, by using Young's inequality

$$K\int_0^1 q\psi_x\varphi_x dx\le K\int_0^1\psi_x^2 dx+K\int_0^1\varphi_x^2 dx.$$

Then,

$$\begin{aligned}\frac{d}{dt}\int_0^1\rho_1 q\varphi_t\varphi_x dx \le\ & -K\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+3K\int_0^1\varphi_x^2 dx\\ & +K\int_0^1\psi_x^2 dx+2\rho_1\int_0^1\varphi_t^2 dx\\ \le\ & -K\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+6K\int_0^1\left(\varphi_x+\psi\right)^2 dx\\ & +7K\int_0^1\varphi_x^2 dx+2\rho_1\int_0^1\varphi_t^2 dx.\end{aligned}\tag{2.41}$$

Multiplying (2.40) and (2.41) by $\frac{1}{4\epsilon}$ and $\frac{\epsilon}{K}$ respectively, we find

$$\begin{aligned}\frac{b\rho_2}{4\epsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x dx \le\ & -\frac{b^2}{4\epsilon}\left[\psi_x^2(1)+\psi_x^2(0)\right]\left(\frac{c}{4\epsilon}+\frac{b^2}{2\epsilon}+\frac{b^2}{2}+\frac{b^2}{4\epsilon^3}\right)\int_0^1\psi_x^2 dx\\ & +\left(\frac{b\rho_2}{2\epsilon}+\frac{\mu_1^2}{4\epsilon^2}\right)\int_0^1\psi_t^2 dx+\frac{\epsilon K^2}{4}\int_0^1\left(\varphi_x+\psi\right)^2 dx\\ & +\frac{\mu_2^2}{4\epsilon^2}\int_0^1 z^2(x,1,t)\,dx\end{aligned}\tag{2.42}$$

and

$$\frac{\epsilon\rho_1}{K}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x dx \leq \; -\epsilon\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+6\epsilon\int_0^1(\varphi_x+\psi)^2 dx \\ +7\epsilon\int_0^1\varphi_x^2 dx+2\frac{\rho_1\epsilon}{K}\int_0^1\varphi_t^2 dx. \tag{2.43}$$

Then, collecting (2.38), (2.42) and (2.43), we arrive at

$$\begin{aligned}[b\varphi_x\psi_x]_{x=0}^{x=1} \leq & -\frac{\epsilon\rho_1}{K}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x dx-\frac{\rho_2 b}{4\epsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x dx+2\frac{\epsilon\rho_1}{K}\int_0^1\varphi_t^2 dx \\ &+\left(6\epsilon+\frac{\epsilon K^2}{4}\right)\int_0^1(\varphi_x+\psi)^2 dx+\left(7\epsilon+\frac{c}{4\epsilon}+\frac{b^2}{2\epsilon}+\frac{b^2}{2}+\frac{b^2}{2\epsilon^3}\right)\int_0^1\psi_x^2 dx \\ &+\left(\frac{b\rho_2}{2\epsilon}+\frac{\mu_1^2}{4\epsilon^2}\right)\int_0^1\psi_t^2 dx+\frac{\mu_2^2}{4\epsilon^2}\int_0^1 z^2(x,1,t)\,dx,\end{aligned}$$

□

Lemma 2.8. *Let $(\varphi,\varphi_t,\psi,\psi_t,z)$ be the solution of problem (2.6)-(2.7). The functional I_3 defined by*

$$I_3=\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx \tag{2.44}$$

satisfies for a positive constant c

$$I_3'(t)\leq-\frac{c}{\tau}\int_0^1 z^2(x,1,t)\,dx+\frac{1}{\tau}\int_0^1\psi_t^2 dx-2I_3(t). \tag{2.45}$$

Proof. By taking a derivative of I_3 and using the facts that $z_t=-\frac{1}{\tau}z_\rho$ and $z(x,0,t)=\psi_t(x,t)$, we get

$$\begin{aligned}I_3'(t) &= 2\int_0^1\int_0^1 e^{-2\tau\rho}z(x,\rho,t)\,z_t(x,\rho,t)\,d\rho dx \\ &= -\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}z(x,\rho,t)\,z_\rho(x,\rho,t)\,d\rho dx \\ &= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}\frac{d}{d\rho}\left(z^2(x,\rho,t)\right)d\rho dx \\ &= -\frac{1}{\tau}\int_0^1\left(\left[e^{-2\tau\rho}z^2(x,\rho,t)\right]_{\rho=0}^{\rho=1}+2\tau\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho\right)dx \\ &= -\frac{1}{\tau}\int_0^1\left(e^{-2\tau}z^2(x,1,t)-z^2(x,0,t)\right)dx-2\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx \\ &= -\frac{c}{\tau}\int_0^1 z^2(x,1,t)\,dx+\frac{1}{\tau}\int_0^1\psi_t^2 dx-2I_3(t).\end{aligned}$$

□

Lemma 2.9. *Let $(\varphi, \varphi_t, \psi, \psi_t, z)$ be the solution of problem (2.6)-(2.7). The following Lyapunov functional $\mathcal{L}$ defined by*

$$\begin{aligned} \mathcal{L}(t) = \;& ME(t) + \frac{1}{8} I_1(t) + N I_2(t) + J(t) + \frac{\epsilon \rho_1}{K} \int_0^1 q \varphi_t \varphi_x dx \\ & + \frac{\rho_2 b}{4\epsilon} \int_0^1 q \psi_t \psi_x dx + I_3(t) \end{aligned} \tag{2.46}$$

satisfies, for M large enough, that there exist two positive constants γ_1 and γ_2 depending on M, N and ϵ, such that for any $t \geq 0$,

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t). \tag{2.47}$$

Proof. We consider the functional

$$\begin{aligned} H(t) = \;& \mathcal{L}(t) - ME(t) \\ = \;& \frac{1}{8} I_1(t) + N I_2(t) + J(t) + \frac{\epsilon \rho_1}{K} \int_0^1 q \varphi_t \varphi_x dx \\ & + \frac{\rho_2 b}{4\epsilon} \int_0^1 q \psi_t \psi_x dx + I_3(t). \end{aligned}$$

We have, from (2.29),(2.31), (2.44) and (2.35)

$$\begin{aligned} |H(t)| \;& \leq \frac{1}{8} \left| - \int_0^1 (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) dx - \frac{\mu_1}{2} \int_0^1 \psi^2 dx \right| \\ & + N_2 \left| \int_0^1 (\rho_2 \psi \psi_t + \rho_1 \varphi_t \omega) dx + \frac{\mu_1}{2} \int_0^1 \psi^2 dx \right| \\ & + \left| \rho_2 \int_0^1 \psi_t (\varphi_x + \psi) dx + \rho_2 \int_0^1 \psi_x \varphi_t dx \right| + \left| \frac{\epsilon \rho_1}{K} \int_0^1 q \varphi_t \varphi_x dx \right. \\ & \left. + \frac{\rho_2 b}{4\epsilon} \int_0^1 q \psi_t \psi_x dx \right| + \left| \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx \right|. \end{aligned}$$

It follows from Young's and Poincaré's inequalities, (2.34) and $\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2$ that

$$\begin{aligned} |H(t)| \;& \leq \alpha_1 \int_0^1 \varphi_t^2 dx + \alpha_2 \int_0^1 \psi_t^2 dx + \alpha_3 \int_0^1 (\varphi_x + \psi)^2 dx + \alpha_4 \int_0^1 \psi_x^2 dx \\ & + \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx, \end{aligned}$$

such that $\alpha_i|_{i=\overline{1,4}}$ are positive constants determined as follows:

$$\begin{cases} \alpha_1 &= \frac{\rho_1}{16} + \frac{N\rho_1}{2} + \frac{\rho_2}{2} + \frac{\epsilon\rho_1}{K}, \\ \alpha_2 &= \frac{N\rho_2}{2} + \frac{\rho_2}{2} + \frac{\rho_2 b}{4\epsilon} + \frac{\rho_2}{16}, \\ \alpha_3 &= \frac{\rho_1}{8} + \frac{\rho_2}{2} + \frac{2\epsilon\rho_1}{K}, \\ \alpha_4 &= \frac{\rho_1}{8} + \frac{\mu_1}{16} + \frac{\rho_2}{2} + \frac{N\rho_2}{2} + \frac{N\mu_1}{2} + \frac{N\rho_1}{2} + \frac{\rho_2 b}{4\epsilon} + \frac{2\epsilon\rho_1}{K} + \frac{\rho_2}{16}. \end{cases}$$

Then,

$$|H(t)| \leq \tilde{C}E(t),$$

such that

$$\tilde{C} = \frac{2max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{min\{\rho_1, \rho_2, b, \xi, K\}}.$$

It means that

$$|\mathcal{L}(t) - ME(t)| \leq \tilde{C}E(t).$$

choosing M large enough so that $\gamma_1 = M - \tilde{C}, \gamma_1 = M + \tilde{C} > 0$.

Then,

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t).$$

□

Theorem 2.2. *Assume that (2.3), (2.4) , $\mu_2 < \mu_1$ and $\frac{\rho_1}{\rho_2} = \frac{K}{b}$ hold. Then, with respect to mild solution, there exist $C > 0$ and $\eta > 0$ such that*

$$E(t) \leq Ce^{-\eta t}, \quad t \geq 0. \tag{2.48}$$

Proof. By taking a derivative of $\mathcal{L}$ and using (2.13), (2.30), (2.33), (2.36), (2.42), (2.43) and (2.45), we get

$$\begin{aligned} \mathcal{L}'(t) = \; & \frac{1}{8}I_1'(t) + NI_2'(t) + J'(t) + \frac{\epsilon\rho_1}{K}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x dx \\ & + \frac{\rho_2 b}{4\epsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x dx + I_3'(t) \end{aligned}$$

$$\leq \left(-MC - \frac{\rho_2}{8} + \rho_2 + N\left(\frac{\rho_1}{4\tilde{\lambda}} + \rho_2\right) + \frac{\mu_1^2}{K} + \frac{b\rho_2}{2\epsilon} + \frac{\mu_1^2}{4\epsilon^2} + \frac{1}{\tau}\right)\int_0^1 \psi_t^2$$

$$+\left(-MC + \frac{\mu_2^2}{32\epsilon} + \frac{N\mu_2}{4\lambda} + \frac{\mu_2^2}{K} + \frac{\mu_2^2}{4\epsilon^2} - \frac{c}{\tau}\right)\int_0^1 z^2(x,1,t)\,dx$$

$$+\left(-\frac{\rho_1}{8} + N\rho_1\tilde{\lambda} + 2\frac{\rho_1\epsilon}{K}\right)\int_0^1 \varphi_t^2 dx + \left(-\frac{K}{8} + 6\epsilon + \frac{\epsilon K^2}{4}\right)\int_0^1 (\varphi_x + \psi)^2\,dx$$

$$+\left(\frac{b+\epsilon}{8} + c - N(b - \mu_2\lambda) + 7\epsilon + \frac{c}{4\epsilon} + \frac{b^2}{2\epsilon} + \frac{b^2}{2} + \frac{b^2}{4\epsilon^3}\right)\int_0^1 \psi_x^2 dx$$

$$+(-N-1)\int_0^1 \hat{f}(\psi)\,dx.$$

At this point we have to choose our constants carefully.

First, choosing λ small enough such that

$$\lambda < \frac{b}{2\mu_2}.$$

Then, selecting N large enough so that

$$\frac{Nb}{4} \geq \frac{b+\epsilon}{8} + c + 7\epsilon + \frac{c}{4\epsilon} + \frac{b^2}{2\epsilon} + \frac{b^2}{2} + \frac{b^2}{4\epsilon^3}.$$

Then,

$$\epsilon \leq min\left\{\frac{\frac{K}{16}}{6 + \frac{K^2}{4}}, \frac{K}{32}\right\}.$$

Also,

$$\tilde{\lambda} < \frac{1}{32N}.$$

Finally, choosing M large enough. So that, there exists a positive constant η, such that

$$\mathcal{L}' \leq -\eta E(t),$$

which yields with (2.47)

$$\mathcal{L}' \leq -\frac{\eta}{\gamma_2}\mathcal{L}(t),$$

an integration over $(0,t)$, we get

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{\eta}{\gamma_2}t},$$

implies that

$$E(t) \leq \frac{\gamma_2}{\gamma_1}E(0)e^{-\frac{\eta}{\gamma_2}t}, \qquad \forall t \geq 0.$$

□

CHAPTER 3

Existence and new general decay results for a viscoelastic Timoshenko system

3.1 Presentation of the problem

In this work, we consider the following viscoelastic-type Timoshenko system:

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t g(t-s)\,\psi_{xx}(s)\,ds = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\, \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\, \psi_t(x,0) = \psi_1(x), \end{cases} \tag{P}$$

where φ is the transverse displacement, ψ is the rotational angle of the Timoshenko beam $(x,t) \in (0,L) \times (0,\infty)$, b, K, ρ_1, ρ_2 are positive physical constants, φ_0, φ_1, ψ_0, ψ_1 are given data and g is a relaxation function sutisfying some conditions to be specified later on. In this chapter, we state some preliminary results, the existence and uniqueness theorem of the solution and we state and prove some technical lemmas that are necessary to establish the general stability of our problem.

3.2 Preliminaries

Assumptions

(A.1) We assume that the relaxation function g satisfies the following:
$g : [0, +\infty) \to (0, +\infty)$ is a non-increasing differentiable function such that

$$b - \int_0^{+\infty} g(s)ds = l > 0.$$

(A.2) There exists a non-increasing differentiable function $\xi : [0, \infty) \to (0, +\infty)$ and a C^1-function $H : [0, +\infty) \to [0, +\infty)$ which is either linear or strictly increasing and strictly convex C^2-function on $(0, r]$, $r < g(0)$, with $H(0) = H'(0) = 0$ such that

$$g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0. \tag{3.1}$$

Remark 3.1. [14]

1. From Assumption **(A.1)**, we can show that

$$\lim_{t\to+\infty} g(t) = 0 \text{ and } g(t) \leq \frac{b-l}{t}, \quad \forall t > 0.$$

Also, Assumption **(A.2)** entails that, that exists $t_0 > 0$ such that

$$g(t_0) = r \text{ and } g(t) \leq r, \quad \forall t \geq t_0.$$

The non-increasing property of g gives

$$0 < g(t_0) \leq g(t) \leq g(0), \quad \forall t \in [0, t_0] .$$

The continuity of H yields, for two positive constants a and b,

$$a \leq H(g(t)) \leq b, \quad \forall t \in [0, t_0] .$$

Consequently, for any $t \in [0, t_0]$, we have

$$\begin{aligned} g'(t) &\leq -\xi(t)H(g(t)) \\ &\leq -a\xi(t) \\ &= -\frac{a}{g(0)}\xi(t)g(0) \\ &\leq -a\xi(t) \\ &= -\frac{a}{g(0)}\xi(t)g(t), \end{aligned}$$

then,

$$\xi(t)g(t) \leq -\frac{g(0)}{a}g'(t), \quad \forall t \in [0, t_0]. \tag{3.2}$$

(2) Any strictly increasing and strictly convex C^2-function H defined on $(0, r]$, with $H(0) = H'(0) = 0$, has an extention $\overline{H}$ which is a strictly increasing and strictly convex C^2-function on $(0, +\infty)$. For example, we can define $\overline{H}$, for any $t > r$, by

$$\overline{H}(t) = \frac{H''(r)}{2}t^2 + \left(H'(r) - H''(r)\,r\right)t + \left(H(r) + \frac{H''(r)}{2}r^2 - H'(r)\,r\right).$$

Lemma 3.1. *[16] For $g, w \in C^1(\mathbb{R}_+; \mathbb{R})$, we have*

$$\begin{aligned} 2\int_0^L w_t \int_0^t g(t-s)w(s)ds = & \frac{d}{dt}\left[\int_0^L \left(\int_0^t g(s)ds\right)|w(t)|^2 dx - (g \circ w)(t)\right] \\ & -g(t)\int_0^L |w(t)|^2 dx + (g' \circ w)(t). \end{aligned} \tag{3.3}$$

Theorem 3.1. *[14] Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, L) \times L^2(0, L)$ be given. Assume that g satisfies hypothesis **(A.1)**. Then, problem (P) has a unique global (weak) solution*

$$\varphi, \psi \in C\left(\mathbb{R}_+; H_0^1(0, L)\right) \cap C^1\left(\mathbb{R}_+; L^2(0, L)\right).$$

Moreover, if $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H^2(0, L) \cap H_0^1(0, L) \times H_0^1(0, L)$, then the problem has a unique strong solution

$$\varphi, \psi \in C\left(\mathbb{R}_+; H^2(0, L) \cap H_0^1(0, L)\right) \cap C^1\left(\mathbb{R}_+; H_0^1(0, L)\right) \cap C^2\left(\mathbb{R}_+; H^2(0, L)\right).$$

Lemma 3.2. *[4] Let (φ, ψ) be the solution of (P). Then, we introduce the energy functional*

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \left(b - \int_0^t g(s)ds\right)\psi_x^2 + K(\varphi_x + \psi)^2\right)dx + \frac{1}{2}(g \circ \psi_x)(t) \tag{3.4}$$

and

$$E'(t) = -\frac{1}{2}g(t)\int_0^L \psi_x^2 dx + \frac{1}{2}(g' \circ \psi_x)(t) \leq \frac{1}{2}(g' \circ \psi_x)(t) \leq 0, \quad \forall t \geq 0, \tag{3.5}$$

where for any $v \in L^2_{loc}(\mathbb{R}_+; L^2(0, L))$,

$$(g \circ v)(t) = \int_0^L \int_0^t g(t-s)(v(t) - v(s))^2 dsdx.$$

Proof. By multiplying equations in (P) by φ_t and ψ_t respectively and integrating over $(0, L)$

$$\begin{cases} \rho_1 \int_0^L \varphi_{tt}\varphi_t dx - K\int_0^L (\varphi_x+\psi)_x \varphi_t dx = 0, \\ \rho_2 \int_0^L \psi_{tt}\psi_t dx - b\int_0^L \psi_{xx}\psi_t dx + K\int_0^L (\varphi_x+\psi)\psi_t dx \\ \quad + \int_0^L \psi_t \int_0^t g(t-s)\psi_{xx}(s)\, ds dx = 0. \end{cases}$$

By using integration by parts and boundary conditions, we obtain

$$\begin{cases} \frac{\rho_1}{2}\frac{d}{dt}\int_0^L \varphi_t^2 dx + K\int_0^L (\varphi_x+\psi)\varphi_{xt} dx = 0, \\ \frac{\rho_2}{2}\frac{d}{dt}\int_0^L \psi_t^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^L \psi_x^2 dx + K\int_0^L (\varphi_x+\psi)\psi_t dx \\ \quad - \int_0^L \psi_{xt}\int_0^t g(t-s)\psi_x(s)\, ds dx = 0. \end{cases} \tag{3.6}$$

By summing $(3.6)_1$ and $(3.6)_2$ and using (3.3) we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L \varphi_t^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^L \psi_t^2 dx + \frac{k}{2}\frac{d}{dt}\int_0^L (\varphi_x+\psi)^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^L \psi_x^2 dx$$
$$-\frac{1}{2}\frac{d}{dt}\int_0^t g(s)\, ds\int_0^L \psi_x^2 dx + \frac{1}{2}\frac{d}{dt}(g\circ\psi_x)(t) + \frac{1}{2}g(t)\int_0^L \psi_x^2 dx - \frac{1}{2}(g'\circ\psi_x)(t) = 0,$$

implies

$$\frac{d}{dt}\left[\frac{\rho_1}{2}\int_0^L \varphi_t^2 dx + \frac{\rho_2}{2}\int_0^L \psi_t^2 dx + \frac{k}{2}\int_0^L (\varphi_x+\psi)^2 dx + \left(\frac{b}{2} - \frac{1}{2}\int_0^t g(s)\, ds\right)\int_0^L \psi_x^2 dx \right.$$
$$\left. + \frac{1}{2}(g\circ\psi_x)\right] = -\frac{1}{2}g(t)\int_0^L \psi_x^2 dx + \frac{1}{2}(g'\circ\psi_x)(t).$$

Then,

$$\begin{aligned} E(t) = {} & \frac{\rho_1}{2}\int_0^L \varphi_t^2 dx + \frac{\rho_2}{2}\int_0^L \psi_t^2 dx + \frac{k}{2}\int_0^L (\varphi_x+\psi)^2 dx \\ & + \left(\frac{b}{2} - \frac{1}{2}\int_0^t g(s)\, ds\right)\int_0^L \psi_x^2 dx + \frac{1}{2}(g\circ\psi_x) \end{aligned}$$

and

$$E'(t) = -\frac{1}{2}g(t)\int_0^L \psi_x^2 dx + \frac{1}{2}(g'\circ\psi_x)(t) \le 0. \tag{3.7}$$

□

Lemma 3.3. *[17] Assume that condition **(A.1)** holds. Then for any $v \in L^2_{loc}(\mathbb{R}_+; L^2(0,L))$, we have*

$$\int_0^L \left(\int_0^t g(t-s)(v(t)-v(s))ds\right)^2 dx \le C_\alpha (h\circ v)(t), \quad \forall t \ge 0, \tag{3.8}$$

such that for any $0 < \alpha < 1$,

$$C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)} ds \quad and \quad h(t) = \alpha g(t) - g'(t).$$

3.3 Some important lemmas

Lemma 3.4. *Assume that **(A.1)** and **(A.2)** hold. Then, the functional F is given by*

$$F(t) = -\rho_2 \int_0^L \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))\, dsdx$$

satisfies, along the solution of (P) and for any $0 < \delta < 1$, *the estimate*

$$\begin{aligned} F'(t) \leq\ & -\rho_2 \left(\int_0^t g(s)\, ds - \delta \right) \int_0^L \psi_t^2 dx + \delta K \int_0^L (\varphi_x + \psi)^2 dx \\ & + \delta \int_0^L \psi_x^2 dx + \frac{c}{\delta}(C_\alpha + 1)(h \circ \psi_x)(t). \end{aligned} \tag{3.9}$$

Proof. By differentiating F we obtain

$$\begin{aligned} F'(t) =\ & -\rho_2 \int_0^L \psi_{tt} \int_0^t g(t-s)(\psi(t) - \psi(s))\, dsdx \\ & -\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))\, dsdx - \rho_2 \int_0^t \psi_t^2 \int_0^t g(t-s)\, dsdx. \end{aligned}$$

By using equation $(P)_2$, we get

$$\begin{aligned} F'(t) =\ & \int_0^L \left(-b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t g(t-s)\psi_{xx}(s)\, ds \right) \int_0^t g(t-s)(\psi(t) - \psi(s))\, dsdx \\ & -\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))\, dsdx - \rho_2 \int_0^L \psi_t^2 \int_0^t g(t-s)\, dsdx. \\ =\ & - \int_0^L b\psi_{xx} \int_0^t g(t-s)(\psi(t) - \psi(s))\, dsdx \\ & + K \int_0^L (\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s))\, dsdx \\ & + \int_0^L \left(\int_0^t g(t-s)\psi_{xx}(s)\, ds \right) \int_0^t g(t-s)(\psi(t) - \psi(s))\, dsdx \\ & -\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))\, dsdx - \rho_2 \int_0^L \psi_t^2 \int_0^t g(t-s)\, dsdx. \end{aligned}$$

By using integration by parts and boundary conditions and adding and subtracting the term $\psi_{xx}(t)$, we get

$$F'(t) = b\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx$$
$$-\int_0^L \int_0^t g(t-s)(\psi_{xx}(t)-\psi_{xx}(s))\,ds \int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx$$
$$+\int_0^L \int_0^t g(t-s)\psi_{xx}(t)\,ds \int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx$$
$$+K\int_0^L (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx$$
$$-\rho_2\int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx - \rho_2\int_0^L \psi_t^2 \int_0^t g(t-s)\,dsdx.$$

By using integration by parts and boundary conditions, we get

$$F'(t) = b\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx$$
$$+\int_0^L \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx$$
$$-\int_0^t g(s)\,ds \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx$$
$$+K\int_0^L (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx$$
$$-\rho_2\int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx - \rho_2\int_0^L \psi_t^2 \int_0^t g(t-s)\,dsdx.$$

Then

$$F'(t) = \left(b-\int_0^t g(s)\,ds\right)\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx$$
$$+K\int_0^L (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx$$
$$+\int_0^L \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx - \rho_2\left(\int_0^t g(t-s)\,ds\right)\int_0^L \psi_t^2 dx$$
$$-\rho_2\int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx.$$

Now, we estimate the terms in the right side of the above equation.

First, using Young's inequality and (3.8), for any $0<\delta<1$,

$$\left(b-\int_0^t g(s)\,ds\right)\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx$$
$$\leq \delta\int_0^L \psi_x^2 + \frac{1}{4\delta}\left(b-\int_0^t g(s)\,ds\right)^2 \int_0^L \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx$$
$$\leq \delta\int_0^L \psi_x^2 + \frac{c}{\delta}C_\alpha(h\circ\psi_x)(t).$$

Second, we have for any $\delta > 0$,

$$
\begin{aligned}
& K\int_0^L (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx \\
& \le \delta K\int_0^L (\varphi_x+\psi)^2\,dx + \frac{K}{4\delta}\int_0^L \left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)^2 dx \\
& \le \delta K\int_0^L (\varphi_x+\psi)^2\,dx + \frac{c}{\delta}C_\alpha (h\circ\psi)(t) \\
& \le \delta K\int_0^L (\varphi_x+\psi)^2\,dx + \frac{c}{\delta}C_\alpha (h\circ\psi_x)(t).
\end{aligned}
$$

Third, for $0 < \delta < 1$

$$
\int_0^L \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx \le C_\alpha (h\circ\psi_x)(t) \le \frac{c}{\delta}C_\alpha (h\circ\psi_x)(t).
$$

Fourth, exploiting Young's inequality and (3.8),we get, for any $0 < \delta < 1$,

$$
\begin{aligned}
& -\rho_2\int_0^L \psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx \\
& = \rho_2\int_0^L \psi_t\int_0^t h(t-s)(\psi(t)-\psi(s))\,dsdx - \rho_2\int_0^L \psi_t\int_0^t \alpha g(t-s)(\psi(t)-\psi(s))\,dsdx \\
& \le \frac{\delta}{2}\rho_2\int_0^L \psi_t^2 dx + \frac{\rho_2}{2\delta}\int_0^L \left(\int_0^t \sqrt{h(t-s)}\sqrt{h(t-s)}(\psi(t)-\psi(s))\,ds\right)^2 dx \\
& \quad + \frac{\delta}{2}\rho_2\int_0^L \psi_t^2 dx + \frac{\rho_2}{2\delta}\alpha^2\int_0^L \left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)^2 dx \\
& \le \delta\rho_2\int_0^L \psi_t^2 dx + \frac{\rho_2}{2\delta}\left(\int_0^t h(s)\,ds\right)(h\circ\psi)(t) + \frac{c}{\delta}C_\alpha (h\circ\psi)(t) \\
& \le \delta\rho_2\int_0^L \psi_t^2 dx + \frac{c}{\delta}(C_\alpha+1)(h\circ\psi)(t).
\end{aligned}
$$

Collecting all the above estimates, we obtain,

$$
\begin{aligned}
F'(t) \le\ & -\rho_2\left(\int_0^t g(s)\,ds-\delta\right)\int_0^L \psi_t^2 dx + \delta K\int_0^L (\varphi_x+\psi)^2\,dx \\
& + \delta\int_0^L \psi_x^2 dx + \frac{c}{\delta}(C_\alpha+1)(h\circ\psi_x)(t).
\end{aligned}
$$

□

Lemma 3.5. *Assume that **(A.1)** and **(A.2)** hold. Then, the functional I_1 is given by*

$$
I_1(t) = -\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)\,dx
$$

satisfies, along the solution of (P), the estimate

$$
I_1'(t) \le -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L (\varphi_x+\psi)^2\,dx + c\int_0^L \psi_x^2 dx + cC_\alpha (h\circ\psi_x)(t). \tag{3.10}
$$

Proof. Differentiating I_1 we obtain

$$I_1'(t) = -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx - \int_0^L \rho_1\varphi\varphi_{tt}dx - \int_0^L \rho_2\psi\psi_{tt}dx.$$

Using equations in (P)

$$\begin{aligned} I_1'(t) = & -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx - K\int_0^L \varphi\left(\varphi_x + \psi\right)_x dx - b\int_0^L \psi_{xx}dx \\ & + K\int_0^L \psi\left(\varphi_x + \psi\right) dx + \int_0^L \psi\int_0^t g\left(t-s\right)\psi_{xx}\left(s\right) dsdx. \end{aligned}$$

Using integration by parts and boundary conditions and adding and subtracting the term $\psi_{xx}(t)$, we get

$$\begin{aligned} I_1'(t) = & -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L \varphi_x\left(\varphi_x + \psi\right) dx + b\int_0^L \psi_x^2 dx \\ & + K\int_0^L \psi\left(\varphi_x + \psi\right) dx + \int_0^L \psi_x\int_0^t g\left(t-s\right)\psi_x\left(s\right) dsdx \\ = & -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L \left(\varphi_x + \psi\right)^2 dx + b\int_0^L \psi_x^2 dx \\ & + \int_0^L \psi_x\int_0^t g\left(t-s\right)\left(\psi_x(t) - \psi_x(s)\right) dsdx + \int_0^L \psi_x\int_0^t g\left(t-s\right)\psi_x\left(t\right) dsdx \\ = & -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L \left(\varphi_x + \psi\right)^2 dx + b\int_0^L \psi_x^2 dx \\ & - \int_0^L \psi_x^2\int_0^t g\left(t-s\right) dsdx + \int_0^L \psi_x\int_0^t g\left(t-s\right)\left(\psi_x(t) - \psi_x(s)\right) dsdx \\ = & -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L \left(\varphi_x + \psi\right)^2 dx + \left(b - \int_0^t g(s)\, ds\right)\int_0^L \psi_x^2 dx \\ & + \int_0^L \psi_x\int_0^t g\left(t-s\right)\left(\psi_x(t) - \psi_x(s)\right) dsdx. \end{aligned}$$

Using Young's inequality

$$\begin{aligned} I_1'(t) & \leq -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L \left(\varphi_x + \psi\right)^2 dx + \left(b - \int_0^t g(s)\, ds\right)\int_0^L \psi_x^2 dx \\ & \quad + \delta\int_0^L \psi_x^2 dx + \frac{1}{4\delta}\int_0^L \left(\int_0^t g\left(t-s\right)\left(\psi_x(t) - \psi_x(s)\right) ds\right)^2 dx \\ & \leq -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^L \left(\varphi_x + \psi\right)^2 dx + c\int_0^L \psi_x^2 dx + cC_\alpha\left(h\circ\psi_x\right)(t). \end{aligned}$$

□

Lemma 3.6. *Assume that **(A.1)** and **(A.2)** hold. Then, the functional $I_2(t)$ is given by*

$$I_2(t) = \rho_2\int_0^L \psi_t\left(\varphi_x + \psi\right) dx + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_x dx - \frac{\rho_1}{K}\int_0^L \varphi_t\int_0^t g\left(t-s\right)\psi_x\left(s\right) dsdx$$

satisfies, along the solution of (P) and for any $0 < \delta < 1$, *the estimate*

$$\begin{aligned}
I_2'(t) \leq & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\
& + c\epsilon\rho_1 \int_0^L \varphi_t^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\epsilon}\int_0^L \psi_x^2 dx \\
& + \frac{c}{\epsilon}(C_\alpha+1)(h\circ\psi_x)(t) + \left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx.
\end{aligned} \tag{3.11}$$

Proof. Differentiating I_2 we obtain

$$\begin{aligned}
I_2'(t) = & \rho_2\int_0^L \psi_{tt}(\varphi_x+\psi)\,dx + \rho_2\int_0^L \psi_t(\varphi_{xt}+\psi_t)\,dx + \frac{b\rho_1}{K}\int_0^L \varphi_{tt}\psi_x dx \\
& + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_{xt}dx - \frac{\rho_1}{K}\int_0^L \varphi_{tt}\int_0^t g(t-s)\psi_x(s)\,dsdx \\
& - \frac{\rho_1}{K}\int_0^L \varphi_t\int_0^t g'(t-s)\psi_x(s)\,dsdx - \frac{\rho_1}{K}g(0)\int_0^L \varphi_t\psi_x(s)\,dx.
\end{aligned}$$

Using equations in (P), adding and subtracting the term $\psi_x(t)$, we get

$$\begin{aligned}
I_2'(t) = & \int_0^L b\psi_{xx}(\varphi_x+\psi)\,dx - K\int_0^L (\varphi_x+\psi)^2 dx - \int_0^L (\varphi_x+\psi)\int_0^t g(t-s)\psi_{xx}(s)\,dsdx \\
& + \rho_2\int_0^L \psi_t(\varphi_{xt}\psi_t)\,dx + b\int_0^L (\varphi_x+\psi)_x\psi_x dx \\
& + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_{xt}dx - \int_0^L (\varphi_x+\psi)_x\int_0^t g(t-s)\psi_x(s)\,dsdx \\
& - \frac{\rho_1}{K}\int_0^L \varphi_t\int_0^t g'(t-s)\psi_x(s)\,dsdx - \frac{\rho_1}{K}g(0)\int_0^L \varphi_t\psi_x(s)\,dx \\
= & \int_0^L b\psi_{xx}\varphi_x dx + \int_0^L b\psi_{xx}\psi dx - K\int_0^L (\varphi_x+\psi)^2 dx \\
& - \int_0^L \varphi_x\int_0^t g(t-s)\psi_{xx}(s)\,dsdx - \int_0^L \psi\int_0^t g(t-s)\psi_{xx}(s)\,dsdx \\
& + \rho_2\int_0^L \varphi_{xt}\psi_t dx + \rho_2\int_0^L \psi_t^2 dx + b\int_0^L \psi_x^2 dx + b\int_0^L \psi_x\varphi_{xx}dx \\
& + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_{xt}dx - \int_0^L \psi_x\int_0^t g(t-s)\psi_x(s)\,dsdx \\
& - \int_0^L \varphi_{xx}\int_0^t g(t-s)\psi_x(s)\,dsdx - \frac{\rho_1}{K}g(0)\int_0^L \varphi_t\psi_x(s)\,dx \\
& + \frac{\rho_1}{K}\int_0^L \varphi_t\psi_x dx\int_0^t g'(t-s)\,ds + \frac{\rho_1}{K}\int_0^L \varphi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx.
\end{aligned}$$

Using integration by parts and boundary conditions

$$\begin{aligned}
I_2'(t) = \; & [b\psi_x\varphi_x]_{x=0}^{x=L} - b\int_0^L \psi_x\varphi_{xx}dx - b\int_0^L \psi_x^2 dx - K\int_0^L (\varphi_x+\psi)^2 dx \\
& - \left[\varphi_x \int_0^t g(t-s)\psi_x(s)\,ds\right]_{x=0}^{x=L} + \int_0^L \varphi_{xx}\int_0^t g(t-s)\psi_x(s)\,dsdx \\
& + \int_0^L \psi_x \int_0^t g(t-s)\psi_x(s)\,dsdx - \rho_2\int_0^L \varphi_t\psi_{xt}dx + \rho_2\int_0^L \psi_t^2 dx + b\int_0^L \psi_x^2 dx \\
& + b\int_0^L \psi_x\varphi_{xx}dx + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_{xt}dx - \int_0^L \psi_x\int_0^t g(t-s)\psi_x(s)\,dsdx \\
& - \int_0^L \varphi_{xx}\int_0^t g(t-s)\psi_x(s)\,dsdx - \frac{\rho_1}{K}g(0)\int_0^L \varphi_t\psi_x(s)\,dx \\
& + \frac{\rho_1}{K}g(0)\int_0^L \varphi_t\psi_x dx - \frac{\rho_1}{K}g(t)\int_0^L \varphi_t\psi_x dx \\
& + \frac{\rho_1}{K}\int_0^L \varphi_t \int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx.
\end{aligned}$$

Then,

$$\begin{aligned}
I_2'(t) = \; & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\
& + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \rho_2\int_0^L \psi_t^2 dx - \frac{\rho_1}{K}g(t)\int_0^L \varphi_t\psi_x dx \\
& + \frac{\rho_1}{K}\int_0^L \varphi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx.
\end{aligned}$$

Using Young's inequality and the decrease of g, we obtain

$$\begin{aligned}
I_2'(t) \leq \; & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\
& + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \rho_2\int_0^L \psi_t^2 dx + \frac{\rho_1}{K}g(0)\epsilon\int_0^L \varphi_t^2 dx \\
& + \frac{\rho_1}{4K\epsilon}g(0)\epsilon\int_0^L \psi_x^2 dx + \frac{\rho_1}{K}\epsilon\int_0^L \varphi_t^2 dx \\
& + \frac{\rho_1}{4K\epsilon}\int_0^L \left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx \\
\leq \; & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\
& + c\epsilon\rho_1\int_0^L \varphi_t^2 dx + \rho_2\int_0^L \psi_t^2 dx + \frac{c}{\epsilon}\int_0^L \psi_x^2 dx \\
& + \frac{c}{\epsilon}(C_\alpha+1)(h\circ\psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx,
\end{aligned}$$

because, by using Cauchy-Schwarz's inequality, we obtain that

$$\begin{aligned}
&\int_0^L \left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx\\
&=\int_0^L \left(\int_0^t (\alpha g(t-s)-h(t-s))(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx\\
&=\int_0^L \left(\int_0^t \alpha g(t-s)(\psi_x(t)-\psi_x(s))\,ds-\int_0^t h(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx\\
&\le 2\int_0^L \left(\int_0^t \alpha g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx+2\int_0^L \left(\int_0^t h(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx\\
&\le 2C_\alpha (h\circ\psi_x)(t)+\int_0^L \left(\int_0^t \left(\sqrt{h(t-s)}\right)^2 ds\int_0^t \left(\sqrt{h(t-s)}\right)^2 (\psi_x(t)-\psi_x(s))^2\,ds\right) dx\\
&\le 2C_\alpha (h\circ\psi_x)(t)+\int_0^t h(s)\,ds\,(h\circ\psi_x)(t)\\
&\le c\,(C_\alpha+1)(h\circ\psi_x)(t).
\end{aligned}$$

□

Lemma 3.7. *Assume that **(A.1)** and **(A.2)** hold. Let* $m(x)=2-\frac{4}{L}x$, *for* $x\in[0,L]$. *Then, for any* $0<\epsilon<1$, *the functional* I_3 *defined by*

$$I_3(t)=\frac{\rho_2}{4\epsilon}\int_0^L m(x)\,\psi_t\left(b\psi_x-\int_0^t g(t-s)\,\psi_x(s)\,ds\right)dx+\epsilon\frac{\rho_1}{K}\int_0^L m(x)\,\varphi_t\varphi_x dx$$

satisfies, along the solution of (P), the estimate

$$\begin{aligned}
I_3'(t)\le\ & -\frac{1}{4\epsilon}\left(b\psi_x(L,t)-\int_0^t g(t-s)\,\psi_x(L,s)\,ds\right)^2\\
&-\frac{1}{4\epsilon}\left(b\psi_x(0,t)-\int_0^t g(t-s)\,\psi_x(0,s)\,ds\right)^2-\epsilon\left(\varphi_x^2(L,t)+\varphi_x^2(0,t)\right)\\
&+\left(\frac{1}{4}+c\epsilon\right)K\int_0^L(\varphi_x+\psi)^2dx+c\epsilon\rho_1\int_0^L\varphi_t^2dx\\
&+\frac{c}{\epsilon}\rho_2\int_0^L\psi_t^2dx+\frac{c}{\epsilon^2}\int_0^L\psi_x^2dx+\frac{c}{\epsilon^2}(C_\alpha+1)(h\circ\psi_x)(t).
\end{aligned}\tag{3.12}$$

Proof. Differentiating I_3 we obtain

$$\begin{aligned}
I_3'(t)=\ &\frac{\rho_2}{4\epsilon}\int_0^L m(x)\,\psi_{tt}\left(b\psi_x-\int_0^t g(t-s)\,\psi_x(s)\,ds\right)dx\\
&+\frac{\rho_2}{4\epsilon}\int_0^L m(x)\,\psi_t\left(b\psi_{xt}-\int_0^t g'(t-s)\,\psi_x(s)\,ds-g(0)\,\psi_x(t)\right)dx\\
&+\frac{\epsilon\rho_1}{K}\int_0^L m(x)\,\varphi_{tt}\varphi_x dx+\frac{\epsilon\rho_1}{K}\int_0^L m(x)\,\varphi_t\varphi_{xt}dx.
\end{aligned}$$

Using equations in (P), we get

$$\begin{aligned}
I_3'(t) = & \frac{1}{4\epsilon}\int_0^L m(x)\left(b\psi_{xx} - K(\varphi_x+\psi) - \int_0^t g(t-s)\psi_{xx}(s)\,ds\right)\left(b\psi_x\right. \\
& \left.-\int_0^t g(t-s)\psi_x(s)\,ds\right)dx + \frac{\rho_2 b}{4\epsilon}\int_0^L m(x)\psi_t\psi_{xt}dx - \frac{\rho_2}{4\epsilon}g(0)\int_0^L m(x)\psi_t\psi_x dx \\
& -\frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)\psi_x(s)\,dsdx + \epsilon\int_0^L m(x)(\varphi_x+\psi)_x\varphi_x dx \\
& +\frac{\epsilon\rho_1}{K}\int_0^L m(x)\varphi_t\varphi_{xt}dx \\
= & \frac{1}{4\epsilon}\int_0^L m(x)\left(b\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)\,ds\right)\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)dx \\
& -\frac{K}{4\epsilon}\int_0^L m(x)(\varphi_x+\psi)\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)dx \\
& +\frac{\rho_2 b}{4\epsilon}\int_0^L m(x)\psi_t\psi_{xt}dx - \frac{\rho_2}{4\epsilon}g(0)\int_0^L m(x)\psi_t\psi_x dx + \epsilon\int_0^L m(x)\varphi_{xx}\varphi_x dx \\
& +\epsilon\int_0^L m(x)\psi_x\varphi_x dx + \frac{\epsilon\rho_1}{K}\int_0^L m(x)\varphi_t\varphi_{xt}dx - \frac{\rho_2}{4\epsilon}g(0)\int_0^L m(x)\psi_t\psi_x dx \\
& -\frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)\psi_x(s)\,dsdx.
\end{aligned}$$

We have, by adding and subtracting $\psi_x(t)$

$$\begin{aligned}
& -\frac{\rho_2}{4\epsilon}g(0)\int_0^L m(x)\psi_t\psi_x dx - \frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)\psi_x(s)\,dsdx \\
& = -\frac{\rho_2}{4\epsilon}g(0)\int_0^L m(x)\psi_t\psi_x dx - \frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\psi_x\int_0^t g'(t-s)\,dsdx \\
& +\frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx \\
& = -\frac{\rho_2}{4\epsilon}g(0)\int_0^L m(x)\psi_t\psi_x dx - \frac{\rho_2}{4\epsilon}(g(t)-g(0))\int_0^L m(x)\psi_t\psi_x dx \\
& +\frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx \\
& = \frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx - \frac{\rho_2}{4\epsilon}g(t)\int_0^L m(x)\psi_t\psi_x dx
\end{aligned}$$

and we have, by using integration by parts and boundary conditions:

$$\begin{aligned}
\frac{\rho_2 b}{4\epsilon}\int_0^L m(x)\psi_t\psi_{xt}dx & = \frac{\rho_2 b}{4\epsilon}\int_0^L m(x)\frac{d}{2dx}\psi_t^2 dx \\
& = \frac{\rho_2 b}{4\epsilon}\left[\frac{1}{2}m(x)\psi_t^2\right]_{x=0}^{x=L} - \frac{\rho_2 b}{4\epsilon}\int_0^L \frac{1}{2}m'(x)\psi_t^2 dx \\
& = -\frac{\rho_2 b}{4\epsilon}\int_0^L \frac{1}{2}m'(x)\psi_t^2 dx.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\frac{1}{4\epsilon}\int_0^L m(x)\left(b\psi_{xx}-\int_0^t g(t-s)\psi_{xx}(s)\,ds\right)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)dx\\
&=\frac{1}{4\epsilon}\int_0^L m(x)\frac{d}{2dx}\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2dx\\
&=\frac{1}{4\epsilon}\left[\frac{1}{2}m(x)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2\right]_{x=0}^{x=L}\\
&\quad-\frac{1}{4\epsilon}\int_0^L\frac{1}{2}m'(x)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2dx\\
&=\frac{1}{4\epsilon}\left[-\left(b\psi_x(L,t)-\int_0^t g(t-s)\psi_x(L,s)\,ds\right)^2-\left(b\psi_x(0,t)-\int_0^t g(t-s)\psi_x(0,s)\,ds\right)^2\right.\\
&\quad\left.-\frac{1}{2}\int_0^L m'(x)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2dx\right],
\end{aligned}$$

$$\begin{aligned}
\epsilon\int_0^L m(x)\varphi_{xx}\varphi_x dx &= \epsilon\int_0^L m(x)\frac{d}{2dx}\varphi_x^2dx\\
&= \left[\frac{1}{2}\epsilon m(x)\varphi_x^2\right]_{x=0}^{x=L}-\frac{1}{2}\epsilon\int_0^L m'(x)\varphi_x^2dx\\
&= \epsilon\left(-\varphi_x^2(L,t)+\varphi_x^2(0,t)\right)-\frac{1}{2}\epsilon\int_0^L m'(x)\varphi_x^2dx
\end{aligned}$$

and

$$\begin{aligned}
\frac{\epsilon\rho_1}{K}\int_0^L m(x)\varphi_t\varphi_{xt}dx &= \frac{\epsilon\rho_1}{K}\int_0^L m(x)\frac{d}{2dx}\varphi_t^2dx\\
&= \left[\frac{\epsilon\rho_1}{2K}m(x)\varphi_t^2\right]_{x=0}^{x=L}-\frac{\epsilon\rho_1}{2K}\int_0^L m'(x)\varphi_t^2dx\\
&= -\frac{\epsilon\rho_1}{2K}\int_0^L m'(x)\varphi_t^2dx.
\end{aligned}$$

Then,

$$\begin{aligned}
I_3'(t) = {} & \frac{1}{4\epsilon}\left[-\left(b\psi_x(L,t)-\int_0^t g(t-s)\psi_x(L,s)\,ds\right)^2-\left(b\psi_x(0,t)\right.\right.\\
&\left.-\int_0^t g(t-s)\psi_x(0,s)\,ds\right)^2-\frac{1}{2}\int_0^L m'(x)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2dx\\
&-K\int_0^L m(x)(\varphi_x+\psi)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)dx\\
&-\frac{\rho_2 b}{2}\int_0^L m'(x)\psi_t^2dx-\rho_2 g(t)\int_0^L m(x)\psi_t\psi_x dx
\end{aligned}$$

$$+\rho_2 \int_0^L m(x)\psi_t \int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\Big]$$
$$+\epsilon\left[-\varphi_x^2(L,t)+\varphi_x^2(0,t)-\frac{1}{2}\int_0^L m'(x)\varphi_x^2 dx\right.$$
$$\left.+\int_0^L m(x)\psi_x\varphi_x dx-\frac{\rho_1}{2K}\int_0^L m'(x)\varphi_t^2 dx\right].$$

Now, we estimate the terms of the above equation.

From the boundedness of m and g, we get

$$-\frac{1}{2}\int_0^L m'(x)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2 dx$$
$$\leq c\int_0^L \left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2 dx$$
$$\leq c\int_0^L \psi_x^2 dx + c\int_0^L \left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))\,ds\right)^2 dx$$
$$+c\int_0^L \left(\int_0^t g(t-s)\psi_x(t)\,ds\right)^2 dx$$
$$\leq c\int_0^L \psi_x^2 dx + cC_\alpha (h\circ\psi_x)(t).$$

Then,

$$\frac{1}{4\epsilon}\left(-\frac{1}{2}\int_0^L m'(x)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2 dx\right)$$
$$\leq \frac{c}{\epsilon}\int_0^L \psi_x^2 dx+\frac{c}{\epsilon}C_\alpha (h\circ\psi_x)(t),$$
$$\frac{\rho_1}{2K}\epsilon\int_0^L m'(x)\varphi_t^2 dx \leq c\epsilon\rho_1\int_0^L \varphi_t^2 dx$$

and

$$-\frac{\rho_2 b}{8\epsilon}\int_0^L m'(x)\psi_t^2 dx \leq \frac{c\rho_2}{\epsilon}\int_0^L \psi_t^2 dx.$$

It follows from Young's inequality that

$$\frac{K}{4\epsilon}\int_0^L m(x)(\varphi_x+\psi)\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)dx$$
$$\leq \frac{1}{4}K\int_0^L (\varphi_x+\psi)^2 dx+\frac{c}{\epsilon^2}\int_0^L\left(b\psi_x-\int_0^t g(t-s)\psi_x(s)\,ds\right)^2 dx$$
$$\leq \frac{1}{4}K\int_0^L (\varphi_x+\psi)^2 dx+\frac{c}{\epsilon^2}\int_0^L \psi_x^2 dx+\frac{c}{\epsilon^2}\int_0^L\left(\int_0^t g(t-s)\psi_x(s)\,ds\right)^2 dx$$
$$\leq \frac{1}{4}K\int_0^L (\varphi_x+\psi)^2 dx+\frac{c}{\epsilon^2}\int_0^L \psi_x^2 dx+\frac{c}{\epsilon^2}\int_0^L\left(\int_0^t g(t-s)\psi_x(t)\,ds\right)^2 dx$$

$$+\frac{c}{\epsilon^2}\int_0^L\left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))\,ds\right)^2dx$$
$$\le \frac{1}{4}K\int_0^L(\varphi_x+\psi)^2dx+\frac{c}{\epsilon^2}\int_0^L\psi_x^2dx+\frac{c}{\epsilon^2}C_\alpha(h\circ\psi_x)(t),$$
$$\frac{-\rho_2}{4\epsilon}g(t)\int_0^L m(x)\psi_t\psi_x dx\le\frac{c\rho_2}{\epsilon}\int_0^L\psi_t^2dx+\frac{c}{\epsilon}\int_0^L\psi_x^2dx$$

and

$$\frac{\rho_2}{4\epsilon}\int_0^L m(x)\psi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,dsdx$$
$$\le c\rho_2\int_0^L\psi_t^2dx+\frac{c}{\epsilon^2}\int_0^L\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2dx$$
$$\le c\rho_2\int_0^L\psi_t^2dx+\frac{c}{\epsilon^2}(C_\alpha+1)(h\circ\psi_x)(t).$$

It follows from Young's and Poincaré's inequalities that

$$\epsilon\int_0^L m(x)\varphi_x\psi_x dx$$
$$\le c\epsilon K\int_0^L\varphi_x^2dx+c\epsilon\int_0^L\psi_x^2dx$$
$$\le c\epsilon K\int_0^L(\varphi_x+\psi)^2dx+c\epsilon K\int_0^L\psi^2dx+c\epsilon\int_0^L\psi_x^2dx$$
$$\le c\epsilon K\int_0^L(\varphi_x+\psi)^2dx+c\epsilon\int_0^L\psi_x^2dx$$

and

$$\begin{aligned}-\frac{\epsilon}{2}\int_0^L m'(x)\varphi_x^2dx &\le c\epsilon\int_0^L\varphi_x^2dx\\ &\le c\epsilon K\int_0^L(\varphi_x+\psi)^2dx+c\epsilon\int_0^L\psi^2dx\\ &\le c\epsilon K\int_0^L(\varphi_x+\psi)^2dx+c\epsilon\int_0^L\psi_x^2dx.\end{aligned}$$

Then,

$$\begin{aligned}I_3'(t)\le\ & -\frac{1}{4\epsilon}\left(b\psi_x(L,t)-\int_0^t g(t-s)\psi_x(L,s)\,ds\right)^2\\ &-\frac{1}{4\epsilon}\left(b\psi_x(0,t)-\int_0^t g(t-s)\psi_x(0,s)\,ds\right)^2-\epsilon\left(\varphi_x^2(L,t)+\varphi_x^2(0,t)\right)\\ &+\left(\frac{1}{4}+c\epsilon\right)K\int_0^L(\varphi_x+\psi)^2dx+c\epsilon\rho_1\int_0^L\varphi_t^2dx\\ &\frac{c}{\epsilon}\rho_2\int_0^L\psi_t^2dx+\frac{c}{\epsilon^2}\int_0^L\psi_x^2dx+\frac{c}{\epsilon^2}(C_\alpha+1)(h\circ\psi_x)(t).\end{aligned}$$

□

Lemma 3.8. *Assume that* ***(A.1)*** *and* ***(A.2)*** *hold, after fixing ϵ small enough, the functional I defined by*

$$I(t) = 3c\epsilon I_1 + I_2 + I_3$$

satisfies, along the solution of (P) and for some constant $c_1 > 0$, the estimate

$$\begin{aligned} I'(t) \leq & -\frac{K}{2}\int_0^L (\varphi_x + \psi)^2 dx - c_1\rho_1 \int_0^L \varphi_t^2 dx + c\rho_2 \int_0^L \psi_t^2 dx \\ & + c\int_0^L \psi_x^2 dx + c(C_\alpha + 1)(h \circ \psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx. \end{aligned} \tag{3.13}$$

Proof. Differentiating I_3 we obtain

$$\begin{aligned} I'(t) = & \; 3c\epsilon I_1' + I_2' + I_3' \\ \leq & -3c\epsilon \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + 3c\epsilon K \int_0^L (\varphi_x + \psi)^2 dx + c\int_0^L \psi_x^2 dx + cC_\alpha (h \circ \psi_x)(t) \\ & + \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x + \psi)^2 dx \\ & + c\epsilon\rho_1 \int_0^L \varphi_t^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\epsilon}\int_0^L \psi_x^2 dx + \frac{c}{\epsilon}(C_\alpha + 1)(h \circ \psi_x)(t) \\ & + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt} dx - \frac{1}{4\epsilon}\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)\,ds\right)^2 \\ & - \frac{1}{4\epsilon}\left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)\,ds\right)^2 - \epsilon\left(\varphi_x^2(L,t) + \varphi_x^2(0,t)\right) \\ & + \left(\frac{1}{4} + c\epsilon\right) K \int_0^L (\varphi_x + \psi)^2 dx + c\epsilon\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\epsilon}\rho_2 \int_0^L \psi_t^2 dx \\ & + \frac{c}{\epsilon^2}\int_0^L \psi_x^2 dx + \frac{c}{\epsilon^2}(C_\alpha + 1)(h \circ \psi_x)(t). \end{aligned}$$

By using the following inequality

$$\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x \leq \frac{1}{4\epsilon}\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)^2 + \epsilon\varphi_x^2,$$

we get

$$\begin{aligned}
&\left[\left(b\psi_x - \int_0^t g(t-s)\,\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} \\
&= \left(b\psi_x(L,t) - \int_0^t g(t-s)\,\psi_x(L,s)\,ds\right)\varphi_x(L,t) \\
&\quad - \left(b\psi_x(0,t) - \int_0^t g(t-s)\,\psi_x(0,s)\,ds\right)\varphi_x(0,t) \\
&\leq \frac{1}{4\epsilon}\left(b\psi_x(L,t) - \int_0^t g(t-s)\,\psi_x(L,s)\,ds\right)^2 + \epsilon\varphi_x^2(L,t) \\
&\quad \frac{1}{4\epsilon}\left(b\psi_x(0,t) - \int_0^t g(t-s)\,\psi_x(0,s)\,ds\right)^2 + \epsilon\varphi_x^2(0,t).
\end{aligned}$$

Then, choosing ϵ so that $4c\epsilon - \frac{3}{4} \leq -\frac{1}{2}$, and setting $c_1 = c\epsilon$, we obtain the result that

$$\begin{aligned}
I'(t) \leq\ & -\frac{K}{2}\int_0^L (\varphi_x+\psi)^2\,dx - c_1\rho_1\int_0^L \varphi_t^2 dx + c\rho_2\int_0^L \psi_t^2 dx \\
& + c\int_0^L \psi_x^2 dx + c\,(C_\alpha+1)\,(h\circ\psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx.
\end{aligned}$$

□

Lemma 3.9. *Assume that **(A.1)** and **(A.2)** hold. Then, the functional J is given by*

$$J(t) = \int_0^L (\rho_1 w\varphi_t + \rho_2\psi\psi_t)\,dx$$

satisfies, along the solution of (P), the estimate

$$J'(t) \leq \epsilon_0\rho_1\int_0^L \varphi_t^2 dx + \frac{c}{\epsilon_0}\rho_2\int_0^L \psi_t^2 dx - \frac{l}{2}\int_0^L \psi_x^2 dx + \frac{C_\alpha}{2l}(h\circ\psi_x)(t), \quad \forall \epsilon_0 > 0, \tag{3.14}$$

where

$$w(x,t) = \frac{1}{L}\left(\int_0^L \psi(y,t)\,dy\right)x - \int_0^x \psi(y,t)\,dy,$$

which solves

$$-w_{xx} = \psi_x, \quad w(0) = w(L) = 0 \tag{3.15}$$

and satisfies for some $c_2 > 0$,

$$\int_0^L w_x^2 dx \leq \int_0^L \psi^2 dx \quad \text{and} \quad \int_0^L w_t^2 dx \leq c_2\int_0^L \psi_t^2 dx. \tag{3.16}$$

Proof. Differentiating J and using equations in (P), we obtain

$$\begin{aligned}
J'(t) &= \int_0^L \rho_1 w_t \varphi_t dx + \int_0^L \rho_1 w \varphi_{tt} dx + \int_0^L \rho_2 \psi_t^2 dx + \int_0^L \rho_2 \psi \psi_{tt} dx \\
&= \int_0^L \rho_1 w_t \varphi_t dx + \int_0^L \rho_2 \psi_t^2 dx + K \int_0^L w \left(\varphi_x + \psi\right)_x dx \\
&\int_0^L \psi \left(b\psi_{xx} - K\left(\varphi_x + \psi\right) - \int_0^t g\left(t-s\right) \psi_{xx}\left(s\right) ds \right) dx \\
&= \int_0^L \rho_1 w_t \varphi_t dx + \int_0^L \rho_2 \psi_t^2 dx + K \int_0^L w \left(\varphi_x + \psi\right)_x dx + \int_0^L b \psi \psi_{xx} dx \\
&\quad - K \int_0^L \psi \left(\varphi_x + \psi\right) dx - \int_0^L \psi \int_0^t g\left(t-s\right) \psi_{xx}\left(s\right) ds dx.
\end{aligned}$$

We have, by using integration by parts and boundary conditions

$$b \int_0^L \psi \psi_{xx} dx = -b \int_0^L \psi_x^2 dx$$

and

$$\begin{aligned}
& - \int_0^L \psi \int_0^t g\left(t-s\right) \psi_{xx}\left(s\right) ds dx \\
&= \int_0^L \psi_x \int_0^t g\left(t-s\right) \psi_x\left(s\right) ds dx \\
&= \int_0^L \psi_x \int_0^t g\left(t-s\right) \left(\psi_x\left(s\right) - \psi_x\left(t\right)\right) ds dx + \int_0^L \psi_x \int_0^t g\left(t-s\right) \psi_x\left(t\right) ds dx \\
&= \int_0^L \psi_x \int_0^t g\left(t-s\right) \left(\psi_x\left(s\right) - \psi_x\left(t\right)\right) ds dx + \int_0^L \psi_x^2 \int_0^t g\left(t-s\right) ds dx \\
&= \int_0^L \psi_x \int_0^t g\left(t-s\right) \left(\psi_x\left(s\right) - \psi_x\left(t\right)\right) ds dx + \int_0^L \psi_x^2 \int_0^t g\left(s\right) ds dx,
\end{aligned}$$

by using integration by parts and (3.15)

$$\begin{aligned}
& K \int_0^L w \left(\varphi_x + \psi\right)_x dx - K \int_0^L \psi \left(\varphi_x + \psi\right) dx \\
&= K \int_0^L w \varphi_{xx} dx + K \int_0^L w \psi_x dx - K \int_0^L \psi^2 dx - K \int_0^L \psi \varphi_x dx \\
&= K \int_0^L w_{xx} \varphi dx + K \int_0^L w \psi_x dx + K \int_0^L \psi_x \varphi dx - K \int_0^L \psi^2 dx \\
&= K \int_0^L \left(w_{xx} + \psi_x\right) \varphi dx - K \int_0^L w w_{xx} dx - K \int_0^L \psi^2 dx \\
&= K \int_0^L w_x^2 dx - K \int_0^L \psi^2 dx \\
&= K \int_0^L \left(w_x^2 - \psi^2\right) dx.
\end{aligned}$$

Then,

$$J'(t) = \rho_1 \int_0^L w_t \varphi_t dx + \rho_2 \int_0^L \psi_t^2 dx - \left(b - \int_0^t g(s)\,ds\right) \int_0^L \psi_x^2 dx$$
$$+ \int_0^L \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right) dsdx + K \int_0^L \left(w_x^2 - \psi^2\right) dx.$$

Now, for the estimation of J',we have for all $\epsilon_0 > 0$,

using Young's inequality

$$\begin{aligned}\rho_1 \int_0^L w_t \varphi_t dx \leq\ & \epsilon_0 \rho_1 \int_0^L \varphi_t^2 dx + \frac{1}{4\epsilon} \rho_1 \int_0^L w_t^2 dx \\ \leq\ & \epsilon_0 \rho_1 \int_0^L \varphi_t^2 dx + \frac{1}{\epsilon_0} \rho_2 \int_0^L w_t^2 dx,\end{aligned}$$

$$-\left(b - \int_0^t g(s)\,ds\right) \int_0^L \psi_x^2 dx \leq -l \int_0^L \psi_x^2 dx,$$

using (3.16)

$$K \int_0^L \left(w_x^2 - \psi^2\right) dx \leq K \int_0^L \psi^2 dx - K \int_0^L \psi^2 dx \leq 0$$

and, using Young's inequality

$$\begin{aligned}&\int_0^L \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right) dsdx \\ &\leq \frac{l}{2} \int_0^L \psi_x^2 dx + \frac{1}{2l} \int_0^L \left(\int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right) ds\right)^2 dx \\ &\leq \frac{l}{2} \int_0^L \psi_x^2 dx + \frac{C_\alpha}{2l}\left(h \circ \psi_x\right).\end{aligned}$$

Then,

$$J'(t) \leq \epsilon_0 \rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\epsilon_0} \rho_2 \int_0^L \psi_t^2 dx - \frac{l}{2} \int_0^L \psi_x^2 dx + \frac{C_\alpha}{2l}\left(h \circ \psi_x\right).$$

□

Lemma 3.10. *[17] Assume that* ***(A.1)*** *and* ***(A.2)*** *hold. Then, the functional J_1 is defined by*

$$J_1(t) = \int_0^L \int_0^t f(t-s)\, \psi_x^2(s)\, dsdx,$$

where $f(t) = \int_t^\infty g(s)\,ds$, satisfies, along the solution of (P), the estimate

$$J_1'(t) \leq -\frac{1}{2}\left(g \circ \psi_x\right)(t) + 3(b-l) \int_0^L \psi_x^2 dx. \tag{3.17}$$

Proof. Differentiating J_1, we obtain

$$\begin{aligned} J_1'(t) = & \int_0^L \frac{d}{dt}\int_0^t f(t-s)\,\psi_x^2(s)\,dsdx \\ = & \int_0^L \int_0^t f'(t-s)\,\psi_x^2(s)\,dsdx + f(0)\int_0^L \psi_x^2 dx \\ = & -\int_0^L \int_0^t g(t-s)\,\psi_x^2(s)\,dsdx + f(0)\int_0^L \psi_x^2 dx \\ = & -\int_0^L \int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))^2\,dsdx - \left(\int_0^t g(s)\,ds\right)\int_0^L \psi_x^2 dx \\ & -2\int_0^L \psi_x \int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))\,dsdx + f(0)\int_0^L \psi_x^2 dx. \end{aligned}$$

We have

$$\begin{aligned} & f(0)\int_0^L \psi_x^2 dx - \left(\int_0^t g(s)\,ds\right)\int_0^L \psi_x^2 dx \\ & = \left(f(0) - \int_0^t g(s)\,ds\right)\int_0^L \psi_x^2 dx \\ & = \left(f(0) + \int_0^t f'(s)\,ds\right)\int_0^L \psi_x^2 dx \\ & = \left(f(0) + [f(s)]_{s=0}^{s=t}\right)\int_0^L \psi_x^2 dx \\ & = f(t)\int_0^L \psi_x^2 dx. \end{aligned}$$

Then,

$$J_1'(t) = -(g\circ\psi_x)(t) - 2\int_0^L \psi_x \int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))\,dsdx + f(t)\int_0^L \psi_x^2 dx.$$

To estimate J_1', using Young's and Cauchy-Schwarz's inequalities, we get

$$\begin{aligned} & -2\int_0^L \psi_x \int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))\,dsdx \\ & \leq 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2(b-l)}\int_0^L \left(\int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))\,ds\right)^2 dx \\ & \leq 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2(b-l)}\int_0^L \left(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}\,(\psi_x(s)-\psi_x(t))\,ds\right)^2 dx \\ & \leq 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2(b-l)}\int_0^L \left(\left(\int_0^t g(t-s)\,ds\right)^{\frac{1}{2}}\left(\int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))^2\,ds\right)^{\frac{1}{2}}\right)^2 dx \\ & \leq 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2(b-l)}\int_0^t g(s)\,ds\int_0^L \int_0^t g(t-s)\,(\psi_x(s)-\psi_x(t))^2\,dsdx. \end{aligned}$$

Then,

$$\begin{aligned} J_1'(t) \leq & -(g\circ\psi_x)(t) + f(t)\int_0^L \psi_x^2 dx + 2(b-l)\int_0^L \psi_x^2 dx \\ & +\frac{1}{2}\int_0^L\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))^2 dsdx \\ \leq & -(g\circ\psi_x)(t) + (b-l)\int_0^L \psi_x^2 dx + 2(b-l)\int_0^L \psi_x^2 dx \\ & +\frac{1}{2}(g\circ\psi_x)(t) \\ \leq & -\frac{1}{2}(g\circ\psi_x)(t) + 3(b-l)\int_0^L \psi_x^2 dx. \end{aligned}$$

□

Lemma 3.11. *The functional $\mathcal{L}$ is defined by*

$$\mathcal{L}(t) = NE(t) + N_1F(t) + I(t) + N_2J(t)$$

satisfies, for a suitable choice of $N, N_1, N_2 \geq 1$,

$$\mathcal{L}(t) \sim E(t) \tag{3.18}$$

and the estimate

$$\begin{aligned} \mathcal{L}'(t) \leq & -\frac{K}{4}\int_0^L (\varphi_x+\psi)^2 dx - \frac{\rho_1}{4}\int_0^L \varphi_t^2 dx - \frac{\rho_2}{4}\int_0^L \psi_t^2 dx - 4(b-l)\int_0^L \psi_x^2 dx \\ & \frac{1}{4}(g\circ\psi_x)(t) + \left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx, \quad \forall t\geq t_0. \end{aligned} \tag{3.19}$$

Proof. Differentiating $\mathcal{L}$, and setting $g_0 = \int_0^{t_0} g(s)\,ds$ and $\delta = \frac{1}{4N_1}$, and recalling that $g' = \alpha g - h$ we obtain, for all $t \geq t_0$,

$$\begin{aligned} \mathcal{L}'(t) = & NE'(t) + N_1F'(t) + I'(t) + N_2J'(t) \\ \leq & -\frac{N}{2}g(t)\int_0^L \psi_x^2 dx + \frac{N}{2}(g'\circ\psi_x)(t) - N_1\rho_2\left(\int_0^t g(s)\,ds - \delta\right)\int_0^L \psi_t^2 dx \\ & +N_1\delta K\int_0^L (\varphi_x+\psi)^2 dx + N_1\delta\int_0^L \psi_x^2 dx + N_1\frac{c}{\delta}(C_\alpha+1)(h\circ\psi_x)(t) \\ & -\frac{K}{2}\int_0^L (\varphi_x+\psi)^2 dx - c_1\rho_1\int_0^L \varphi_t^2 dx + c\rho_2\int_0^L \psi_t^2 dx + c\int_0^L \psi_x^2 dx \\ & +c(C_\alpha+1)(h\circ\psi_x)(t) + \left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + N_2\epsilon_0\rho_1\int_0^L \varphi_t^2 dx \\ & +N_2\frac{c}{\epsilon_0}\rho_2\int_0^L \psi_t^2 dx - N_2\frac{l}{2}\int_0^L \psi_x^2 dx + N_2\frac{C_\alpha}{2l}(h\circ\psi_x)(t) \end{aligned}$$

$$\begin{aligned}\leq\ & -\frac{K}{4}\int_0^L (\varphi_x+\psi)^2\,dx-(c_1-N_2\epsilon_0)\,\rho_1\int_0^L \varphi_t^2dx \\ & -\left(\frac{lN_2}{2}-\frac{1}{4}-c\right)\int_0^L \psi_x^2dx+\frac{\alpha}{2}N\left(g\circ\psi_x\right)(t)+\left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx \\ & -\left(\frac{N}{2}-c\left(4N_1^2+1\right)-C_\alpha\left(\frac{N_2}{2l}+c+4cN_1^2\right)\right)\left(h\circ\psi_x\right)(t). \\ & -\left(g_0N_1-\frac{1}{4}-c\left(1+\frac{N_2}{\epsilon_0}\right)\right)\rho_2\int_0^L \psi_t^2dx.\end{aligned} \tag{3.20}$$

Choosing N_2 large enough so that $\dfrac{lN_2}{2}-\dfrac{1}{4}-c>4\,(b-l)$, then pick ϵ_0 so small that $c_1-N_2\epsilon_0>\dfrac{1}{4}$. Next, we select N_1 so large that $g_0N_1-\dfrac{1}{4}-c\left(1+\dfrac{N_2}{\epsilon_0}\right)>\dfrac{1}{4}$.
As $\dfrac{\alpha g^2(s)}{\alpha g(s)-g'(s)}<g(s)$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{\alpha\to 0^+}\alpha C_\alpha=\lim_{\alpha\to 0^+}\int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s)-g'(s)}=0.$$

Consequently, there exists $0<\alpha_0<1$ such that if $\alpha<\alpha_0$, then

$$\alpha C_\alpha<\frac{1}{8\left(\dfrac{N_2}{2l}+c(1+4N_1^2)\right)}.$$

Now, we take N large enough so that $N>max\left\{4c\left(4N_1^2+1\right),\dfrac{1}{2\alpha_0}\right\}$ and set $\alpha=\dfrac{1}{2N}$, so, we get

$$\frac{N}{4}-c(4N_1^2+1)>0\text{and }\alpha=\frac{1}{2N}<\alpha_0.$$

This gives,

$$\frac{N}{2}-c(4N_1^2+1)-C_\alpha\left(\frac{N_2}{2l}+c(1+4N_1^2)\right)>\frac{N}{2}-c(1+4N_1^2)-\frac{1}{8\alpha}=\frac{N}{4}-c(1+4N_1^2)>0.$$

Then, we arrive at the estimate

$$\begin{aligned}\mathcal{L}'(t)\leq\ & -\frac{K}{4}\int_0^L (\varphi_x+\psi)^2\,dx-\frac{\rho_1}{4}\int_0^L \varphi_t^2dx-\frac{\rho_2}{4}\int_0^L \psi_t^2dx-4\,(b-l)\int_0^L \psi_x^2dx \\ & \frac{1}{4}\left(g\circ\psi_x\right)(t)+\left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx,\quad \forall t\geq t_0.\end{aligned}$$

□

3.4 A decay result for equal speeds of wave propagation

Theorem 3.2. *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, L) \times L^2(0, L)$. *Assume that* ***(A.1)*** *and* ***(A.2)*** *hold and*

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{3.21}$$

Then, there exists two positive constants k_1 *and* k_2 *such that the energy functional associated to problem (P) satisfies the estimate*

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_{t_0}^{t} \xi(s)\, ds\right), \quad t > t_0, \tag{3.22}$$

where H_1 *is given by*

$$H_1(t) = \int_t^r \frac{1}{sH'(s)} ds \quad and \quad t_0 = g^{-1}(r).$$

Proof. Starting by using estimates (3.2) and (3.7) to deduce, for any $t \geq t_0$,

$$\begin{aligned}
&\int_0^L \int_0^{t_0} g(s)(\psi_x(t) - \psi_x(t-s))^2 dsdx \\
&\leq \frac{1}{\xi(t_0)} \int_0^L \int_0^{t_0} \xi(s) g(s)(\psi_x(t) - \psi_x(t-s))^2 dsdx \\
&\leq -\frac{g(0)}{a\xi(t_0)} \int_0^L \int_0^{t_0} g'(s)(\psi_x(t) - \psi_x(t-s))^2 dsdx \\
&\leq -\frac{g(0)}{a\xi(t_0)} \int_0^L \int_0^{t} g'(s)(\psi_x(t) - \psi_x(t-s))^2 dsdx \\
&\leq -cE'(t).
\end{aligned}$$

Combining this last estimate with identity (3.21), the estimate (3.19) becomes, for some $m > 0$ and for any $t \geq t_0$,

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - \frac{\rho_1}{4}\int_0^L \varphi_t^2 dx - \frac{\rho_2}{4}\int_0^L \psi_t^2 dx - 4(b-l)\int_0^L \psi_x^2 dx \\
& + \frac{1}{4}(g \circ \psi_x)(t) \\
\leq\ & -mE(t) + c(g \circ \psi_x)(t) \\
\leq\ & -mE(t) + c\int_0^L \int_0^t g(s)(\psi_x(t) - \psi_x(t-s))^2 dsdx
\end{aligned}$$

$$\begin{aligned}
&\leq -mE(t)+c\int_0^L\int_0^{t_0} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx \\
&\quad +c\int_0^L\int_{t_0}^{t} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx \\
&\leq -mE(t)-cE'(t)+c\int_0^L\int_{t_0}^{t} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx.
\end{aligned}$$

By setting $\mathcal{F}=\mathcal{L}+cE\sim E$, we obtain

$$\mathcal{F}'(t)\leq -mE(t)+c\int_0^L\int_{t_0}^{t} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx,\quad \forall t\geq t_0. \tag{3.23}$$

Case 01. H is linear: Multiplying (3.23) by $\xi(t)$, then recalling **(A.2)** and (3.5), we get

$$\begin{aligned}
\xi(t)\mathcal{F}'(t) &\leq -m\xi(t)E(t)+c\xi(t)\textstyle\int_0^L\int_{t_0}^{t} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx \\
&\leq -m\xi(t)E(t)+c\int_0^L\int_{t_0}^{t}\xi(s)g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx \\
&\leq -m\xi(t)E(t)-c\int_0^L\int_{t_0}^{t} g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx \\
&\leq -m\xi(t)E(t)-cE'(t),\quad \forall t\geq t_0.
\end{aligned}$$

Using the non-increasing property of ξ, we have $\xi\mathcal{F}+cE\sim E$ and

$$(\xi\mathcal{F}+cE)'(t)\leq -m\xi(t)E(t),\quad \forall t\geq t_0.$$

Putting $\xi\mathcal{F}+cE=M$, then, $M\sim E$. So, for a positive constant k_1 we get

$$\frac{M'(t)}{M(t)}\leq -k_1\xi(t),\quad \forall t\geq t_0,$$

by a simple integration over (t_0,t), we get

$$E(t)\leq M(t_0)\,exp\left(-k_1\int_{t_0}^{t}\xi(s)\,ds\right).$$

Then,

$$E(t)\leq k_2 exp\left(-k_1\int_0^{t}\xi(s)\,ds\right),\quad \forall t>0,$$

such that $k_2=M(t_0)\,exp\left(-k_1\int_0^{t_0}\xi(s)\,ds\right)$.

Case 02. H is non-linear: Using lemmas 3.10 and 3.11, we conclude that

$$\mathcal{L}_*=\mathcal{L}(t)+J_1(t)$$

is non-negative and satisfies, for some $\beta > 0$ and for any $t \geq t_0$,

$$\begin{aligned} \mathcal{L}_*{}'(t) &\leq -\frac{K}{4}\int_0^L (\varphi_x+\psi)^2\,dx - \frac{\rho_1}{4}\int_0^L \varphi_t^2 dx - \frac{\rho_2}{4}\int_0^L \psi_t^2 dx - (b-l)\int_0^L \psi_x^2 dx \\ &\quad -\frac{1}{4}(g\circ\psi_x)(t) \\ &\leq -\beta E(t). \end{aligned} \tag{3.24}$$

Noticing that these estimates yield

$$\int_0^\infty E(s)\,ds < +\infty \tag{3.25}$$

and

$$E(t) \leq \frac{c}{t-t_0}, \quad \forall t > t_0. \tag{3.26}$$

Now, we define a functional η by

$$\eta(t) = \gamma\int_{t_0}^t \|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds,$$

where (3.25) allows us to choose $0 < \gamma < 1$ such that

$$\eta(t) < 1, \quad \forall t \geq t_0. \tag{3.27}$$

Also, we define another functional θ by

$$\theta(t) = -\int_{t_0}^t g'(s)\,\|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds.$$

Observing that

$$\theta(t) \leq -cE'(t), \quad \forall t \geq t_0. \tag{3.28}$$

It follows from the strict convexity of H and the fact that $H(0) = 0$ that

$$H(s\tau) \leq sH(\tau), \quad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0, r].$$

Using this fact, **(A.2)**, (3.27) and Jensen's inequality, we obtain

$$\begin{aligned} \theta(t) &= -\frac{1}{\gamma\eta(t)}\int_{t_0}^t \gamma\eta(t)\,g'(s)\,\|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds \\ &\geq \frac{1}{\gamma\eta(t)}\int_{t_0}^t \gamma\eta(t)\,\xi(s)\,H(g(s))\,\|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds \\ &\geq \frac{\xi(t)}{\gamma\eta(t)}\int_{t_0}^t \gamma H(\eta(t)\,g(s))\,\|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds \\ &\geq \frac{\xi(t)}{\gamma}H\left(\frac{1}{\eta(t)}\int_{t_0}^t \gamma\eta(t)\,g(s)\,\|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds\right) \\ &:= \frac{\xi(t)}{\gamma}H\left(\gamma\int_{t_0}^t g(s)\,\|\psi_x(t) - \psi_x(t-s)\|_2^2\,ds\right) \end{aligned}$$

$$:= \frac{\xi(t)}{\gamma}\overline{H}\left(\gamma\int_{t_0}^{t} g(s)\,\|\psi_x(t)-\psi_x(t-s)\|_2^2\,ds\right), \quad \forall t \geq t_0,$$

where $\overline{H}$ is a C^2- *extention* of H that is strictly increasing and strictly convex on $(0,\infty)$. This yields

$$\int_{t_0}^{t} g(s)\,\|\psi_x(t)-\psi_x(t-s)\|_2^2\,ds \leq \frac{1}{\gamma}\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)}\right), \quad \forall t \geq t_0,$$

and (3.23) becomes

$$\mathcal{F}'(t) \leq -mE(t) + c\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)}\right), \quad \forall t \geq t_0. \tag{3.29}$$

Now, for $0<\epsilon_1<r$, we define the functional $\mathcal{F}_1$ by

$$\mathcal{F}_1(t) = \overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + E(t), \quad \forall t \geq 0.$$

Then, using the facts that $E' \leq 0$, $\overline{H}' > 0$ and $\overline{H}'' > 0$, we find that $\mathcal{F}_1 \sim E$, and for $t \geq t_0$, we have

$$\begin{aligned}\mathcal{F}_1'(t) = &\ \epsilon_1\frac{E'(t)}{E(0)}\overline{H}''\left(\epsilon_1\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + \overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)\mathcal{F}'(t) + E'(t)\\ \leq &\ -mE(t)\,\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)}\right) + E'(t).\end{aligned} \tag{3.30}$$

Let $\overline{H}^*$ be the convex conjugate of $\overline{H}$ in the sense of Young [6] , which is given by

$$\overline{H}^*(s) = s\left(\overline{H}'\right)^{-1}(s) - \overline{H}\left[\left(\overline{H}'\right)^{-1}(s)\right] \tag{3.31}$$

and satisfies the following generalized Young's inequality

$$AB \leq \overline{H}^*(A) + \overline{H}(B). \tag{3.32}$$

By taking $A = \overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)$, $\quad B = \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)}\right)$ and combining (3.5), (3.30), (3.31) and (3.32), we obtain

$$\begin{aligned}\mathcal{F}_1'(t) &\leq -mE(t)\,\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)}\right) + E'(t)\\ &\leq -mE(t)\,\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\overline{H}^*\left(\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)\right) + c\overline{H}\left(\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)}\right)\right)\\ &\leq -mE(t)\,\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\overline{H}^*\left(\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right)\right) + c\gamma\frac{\theta(t)}{\xi(t)}\\ &\leq -mE(t)\,\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\epsilon_1\frac{E(t)}{E(0)}\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\gamma\frac{\theta(t)}{\xi(t)}\\ &\leq -\left(mE(0)-c\epsilon_1\right)\frac{E(t)}{E(0)}\overline{H}'\left(\epsilon_1\frac{E(t)}{E(0)}\right) + c\gamma\frac{\theta(t)}{\xi(t)}, \quad \forall t \geq 0.\end{aligned}$$

Multiplying this estimate by $\xi(t)$ and using $\overline{H}'\left(\epsilon_1 \frac{E(t)}{E(0)}\right) = H'\left(\epsilon_1 \frac{E(t)}{E(0)}\right)$, $\epsilon_1 \frac{E(t)}{E(0)} < R$ and (3.28), we get

$$\begin{aligned} \xi(t)\mathcal{F}_1'(t) &\leq -(mE(0) - c\epsilon_1)\xi(t)\frac{E(t)}{E(0)}\overline{H}'\left(\epsilon_1 \frac{E(t)}{E(0)}\right) + c\gamma\theta(t) \\ &\leq -(mE(0) - c\epsilon_1)\xi(t)\frac{E(t)}{E(0)}H'\left(\epsilon_1 \frac{E(t)}{E(0)}\right) - cE'(t) \\ &\leq -k\xi(t)\frac{E(t)}{E(0)}H'\left(\epsilon_1 \frac{E(t)}{E(0)}\right) - cE'(t), \quad \forall t \geq t_0, \end{aligned}$$

such that ϵ_1 is taken even smaller and k is a positive constant. So, by setting $\mathcal{F}_2 = \xi\mathcal{F}_1 + cE$, we obtain for two constants $\alpha, \beta > 0$

$$\alpha\mathcal{F}_2(t) \leq E(t) \leq \beta\mathcal{F}_2(t), \quad \forall t \geq t_0, \tag{3.33}$$

and by setting $H_2(\tau) = \tau H'(\epsilon_1\tau)$, we arrive at

$$\mathcal{F}_2'(t) \leq -k\xi(t)\frac{E(t)}{E(0)}H'\left(\epsilon_1 \frac{E(t)}{E(0)}\right) = -k\xi(t)H_2\left(\frac{E(t)}{E(0)}\right), \quad \forall t \geq t_0. \tag{3.34}$$

Since $H_2'(\tau) = H'(\epsilon_1\tau) + \epsilon_1\tau H''(\epsilon_1\tau)$, then the fact that H is increasing convex on $(0, r]$ lead to $H_2, H_2' > 0$ on $(0, 1]$. By setting

$$R(\tau) = \frac{\alpha\mathcal{F}_2(\tau)}{E(0)}, \tag{3.35}$$

and exploiting (3.33) and (3.34) to conclude that $R \sim E$ and for some $k_1 > 0$,

$$R'(t) \leq -k_1\xi(t)H_2(R(t)), \quad \forall t \geq t_0.$$

By integration over (t_0, t) we get

$$-\int_{t_0}^{t} \frac{R'(s)}{H_2(R(s))}ds \geq k_1 \int_{t_0}^{t} \xi(s)\,ds$$

or, by taking $\tau = R(s)$, then, taking $s = \epsilon_1\tau$

$$\int_{\epsilon_1 R(t)}^{\epsilon_1 R(t_0)} \frac{1}{sH'(s)}ds \geq k_1 \int_{t_0}^{t} \xi(s)\,ds.$$

We have from (3.35)

$$\epsilon_1 R(t_0) = \epsilon_1\alpha\frac{\mathcal{F}_2(t_0)}{E(0)} \leq \epsilon_1\frac{E(t_0)}{E(0)} < r.$$

Then,

$$\int_{\epsilon_1 R(t)}^{r} \frac{1}{sH'(s)} ds \geq k_1 \int_{t_0}^{t} \xi(s)\, ds \implies H_1(\epsilon_1 R(t)) \leq k_1 \int_{t_0}^{t} \xi(s)\, ds,$$
$$\implies \epsilon_1 R(t) \leq H_1^{-1}\left(k_1 \int_{t_0}^{t} \xi(s)\, ds\right),$$
$$\implies R(t) \leq \frac{1}{\epsilon_1} H_1^{-1}\left(k_1 \int_{t_0}^{t} \xi(s)\, ds\right),$$

where $H_1(t) = \int_t^r \frac{1}{sH'(s)} ds$. So, using the fact that $R \sim E$ we arrive at

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_{t_0}^{t} \xi(s)\, ds\right), \quad \forall t > t_0.$$

□

3.5 A decay result for non-equal speeds of wave propagation

Lemma 3.12. *[12] Let (φ, ψ) be the strong solution of (P).Then, the second energy functional is introduced by*

$$\begin{aligned} E_*(t) = {} & \frac{1}{2}\int_0^L \left(\rho_1 \varphi_{tt}^2 + \rho_2 \psi_{tt}^2 + \left(b - \int_0^t g(s)\, ds\right)\psi_{xt}^2 + K(\varphi_{xt} + \psi_t)^2\right) dx \\ & + \frac{1}{2}(g \circ \psi_{xt})(t) \end{aligned} \tag{3.36}$$

satisfies, for all $t \geq 0$,

$$E_*'(t) = -\frac{1}{2} g(t) \int_0^L \psi_{xt}^2 dx + \frac{1}{2}(g' \circ \psi_{xt}) - g(t) \int_0^L \psi_{tt} \psi_{0xx} dx \tag{3.37}$$

and

$$E_*'(t) \leq c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right). \tag{3.38}$$

Proof. By differentiating with respect to t, multiplying by φ_{tt} and ψ_{tt} respectively and integrating over $(0, L)$

$$\begin{cases} \rho_1 \int_0^L \varphi_{ttt}\varphi_{tt} dx - K \int_0^L (\varphi_x + \psi)_{xt} \varphi_{tt} dx = 0, \\ \rho_2 \int_0^L \psi_{ttt}\psi_{tt} dx - b \int_0^L \psi_{xxt}\psi_{tt} dx + K \int_0^L (\varphi_x + \psi)_t \psi_{tt} dx \\ \quad + \int_0^L \psi_{tt} \int_0^t g'(t-s)\psi_{xx}(s)\, ds dx + g(0) \int_0^L \psi_{xx}(t)\psi_{tt}(t)\, dx = 0. \end{cases}$$

By using integration by parts and boundary conditions, we get

$$\rho_1 \int_0^L \varphi_{ttt}\varphi_{tt}dx = \frac{\rho_1}{2}\frac{d}{dt}\int_0^L \varphi_{tt}^2 dx,$$

$$\rho_2 \int_0^L \psi_{ttt}\psi_{tt}dx = \frac{\rho_2}{2}\frac{d}{dt}\int_0^L \psi_{tt}^2 dx,$$

$$b \int_0^L \psi_{xxt}\psi_{tt}dx = \frac{b}{2}\frac{d}{dt}\int_0^L \psi_{xt}^2 dx,$$

$$\begin{aligned}
&-K\int_0^L (\varphi_x+\psi)_{xt}\,\varphi_{tt}dx + K\int_0^L (\varphi_x+\psi)_t\,\psi_{tt}dx \\
&= K\int_0^L (\varphi_x+\psi)_t\,\varphi_{xtt}dx + K\int_0^L (\varphi_x+\psi)_t\,\psi_{tt}dx \\
&= K\int_0^L (\varphi_x+\psi)_t\,(\varphi_{xtt}+\psi_{tt})\,dx \\
&= \frac{K}{2}\frac{d}{dt}\int_0^L \left((\varphi_x+\psi)_t\right)^2 dx
\end{aligned}$$

and, by using integration by parts and (3.3), we get

$$\begin{aligned}
&\int_0^t g'(t-s)\,\psi_{xx}(s)\,ds \\
&= -\left[g(t-s)\,\psi_{xx}\right]_{s=0}^{s=t} + \int_0^t g(t-s)\,\psi_{xxt}(s)\,ds \\
&= -g(0)\,\psi_{xx} + g(t)\,\psi_{xx0} + \int_0^t g(t-s)\,\psi_{xxt}(s)\,ds.
\end{aligned}$$

Implies

$$\begin{aligned}
&\int_0^L \psi_{tt}\int_0^t g'(t-s)\,\psi_{xx}(s)\,dsdx \\
&= -g(0)\int_0^L \psi_{xx}\psi_{tt}dx + g(t)\int_0^L \psi_{xx0}\psi_{tt}dx + \int_0^L \psi_{tt}\int_0^t g(t-s)\,\psi_{xxt}(s)\,dsdx \\
&= -g(0)\int_0^L \psi_{xx}\psi_{tt}dx + g(t)\int_0^L \psi_{xx0}\psi_{tt}dx - \int_0^L \psi_{ttx}\int_0^t g(t-s)\,\psi_{xt}(s)\,dsdx \\
&= -g(0)\int_0^L \psi_{xx}\psi_{tt}dx + g(t)\int_0^L \psi_{xx0}\psi_{tt}dx - \frac{1}{2}\frac{d}{dt}\left[\int_0^L\left(\int_0^t g(s)ds\right)|\psi_{xt}(t)|^2 dx\right. \\
&\quad \left. -(g\circ\psi_{xt})(t)\right] + \frac{1}{2}g(t)\int_0^L \psi_{xt}(t)^2 dx - \frac{1}{2}\left(g'\circ\psi_{xt}\right)(t).
\end{aligned}$$

Then, by summing these results, we obtain

$$\begin{aligned}
&\frac{d}{2dt}\left[\rho_1\int_0^L \varphi_{tt}^2 dx + \rho_2\int_0^L \psi_{tt}^2 dx + k\int_0^L \left((\varphi_x+\psi)_t\right)^2 dx\right. \\
&\left. + \left(b-\int_0^t g(s)\,ds\right)\int_0^L \psi_{xt}^2 dx + (g\circ\psi_{xt})(t)\right] \\
&= \frac{1}{2}\left(g'\circ\psi_{xt}\right)(t) - \frac{1}{2}g(t)\int_0^L \psi_{xt}^2 dx - g(t)\int_0^L \psi_{xx0}\psi_{tt}dx.
\end{aligned}$$

Then,

$$E_*(t) = \frac{1}{2}\left[\rho_1 \int_0^L \varphi_{tt}^2 dx + \rho_2 \int_0^L \psi_{tt}^2 dx + k \int_0^L ((\varphi_x + \psi)_t)^2 dx + \left(b - \int_0^t g(s)\,ds\right)\int_0^L \psi_{xt}^2 dx + (g \circ \psi_{xt})(t)\right]$$

and

$$E_*'(t) = \frac{1}{2}\left(g' \circ \psi_{xt}\right)(t) - \frac{1}{2} g(t) \int_0^L \psi_{xt}^2 dx - g(t) \int_0^L \psi_{xx0}\psi_{tt} dx.$$

We have, using Young's inequality

$$\begin{aligned} E_*(t) &\leq -g(t) \int_0^L \psi_{xx0}\psi_{tt} dx \\ &\leq \frac{1}{2} g(t) \left(\rho_2 \int_0^L \psi_{tt}^2 dx + \frac{1}{\rho_2}\int_0^L \psi_{xx0}^2 dx\right) \\ &\leq g(t) E_*(t) + \frac{1}{2\rho_2} g(t) \int_0^L \psi_{xx0}^2 dx, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dt}\left(E_*(t)\, e^{-\int_0^t g(s)ds}\right) &= E_*'(t)\, e^{-\int_0^t g(s)ds} - g(t) E_*(t)\, e^{-\int_0^t g(s)ds} \\ &\leq \frac{1}{2\rho_2} g(t)\, e^{-\int_0^t g(s)ds} \int_0^L \psi_{xx0}^2 dx \\ &\leq \frac{1}{2\rho_2} g(t) \int_0^L \psi_{xx0}^2 dx. \end{aligned}$$

An integration over $(0, t)$, we get

$$\int_0^t \frac{d}{ds}\left(E_*(s)\, e^{-\int_0^s g(\tau)d\tau}\right) ds = \left[E_*(s)\, e^{-\int_0^s g(\tau)d\tau}\right]_{s=0}^{s=t},$$

implies

$$E_*(t)\, e^{-\int_0^t g(s)ds} - E_*(0) \leq \frac{1}{2\rho_2}\int_0^t g(s)ds \int_0^L \psi_{xx0}^2 dx$$

and

$$E_*(t)\, e^{-\int_0^t g(s)ds} \leq E_*(0)\,\frac{1}{2\rho_2}\int_0^t g(s)ds \int_0^L \psi_{xx0}^2 dx,$$

then, for some constant $c > 0$

$$E_*(t) \leq e^{\int_0^t g(s)ds}\left(E_*(0) + \frac{1}{2\rho_2}\int_0^t g(s)ds \int_0^L \psi_{xx0}^2 dx\right) \leq c\left(E_*(0) + \int_0^L \psi_{xx0}^2 dx\right).$$

$\square$

Lemma 3.13. *[12] Let (φ,ψ) be the strong solution of (P). Then, for any $\epsilon>0$ we have*

$$\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx \le \epsilon E(t)+\frac{c}{\epsilon}\left((g\circ\psi_{xt})(t)-E'(t)+g(t)\right), \quad \forall t\ge t_0. \tag{3.39}$$

Proof. We have, by multiplying and dividing the term $\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx$ by $\int_0^t g(s)\,ds$ and adding and subtracting $\psi_{xt}(s)$

$$\begin{aligned}\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx = & \frac{\frac{\rho_1 b}{K}-\rho_2}{\int_0^t g(s)\,ds}\int_0^L \varphi_t\int_0^t g(t-s)\left(\psi_{xt}(t)-\psi_{xt}(s)\right)dsdx\\ &+\frac{\frac{d\rho_1 b}{K}-\rho_2}{\int_0^t g(s)\,ds}\int_0^L \varphi_t\int_0^t g(t-s)\,\psi_{xt}(s)\,dsdx.\end{aligned}$$

First, by using Young's inequality, we find

$$\begin{aligned}&\frac{\frac{\rho_1 b}{K}-\rho_2}{\int_0^t g(s)\,ds}\int_0^L \varphi_t\int_0^t g(t-s)\left(\psi_{xt}(t)-\psi_{xt}(s)\right)dsdx\\ &\le \frac{\epsilon}{2}\int_0^L \varphi_t^2dx+\frac{c}{\epsilon}\int_0^L\left(\int_0^t g(t-s)\left(\psi_{xt}(t)-\psi_{xt}(s)\right)ds\right)^2dx\\ &\le \frac{\epsilon}{2}E(t)+\frac{c}{\epsilon}(g\circ\psi_{xt})(t), \quad \forall t\ge t_0.\end{aligned} \tag{3.40}$$

On the other hand, by integrating by parts, the fact that E and g are non-increasing and $\psi(x,0)=\psi_0(x)$, we obtain

$$\begin{aligned}&\frac{\frac{\rho_1 b}{K}-\rho_2}{\int_0^t g(s)\,ds}\int_0^L \varphi_t\int_0^t g(t-s)\,\psi_{xt}(s)\,dsdx\\ &=\frac{\frac{\rho_1 b}{K}-\rho_2}{\int_0^t g(s)\,ds}\int_0^L \varphi_t\left(g(0)\,\psi_x-g(t)\,\psi_{0x}-\int_0^t g'(t-s)\,\psi_x(s)\,ds\right)dx\\ &=\frac{\frac{\rho_1 b}{K}-\rho_2}{\int_0^t g(s)\,ds}\int_0^L \varphi_t\left(g(t)\left(\psi_x-\psi_{0x}\right)-\int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)dx\\ &\le \frac{\epsilon}{2}\int_0^L \varphi_t^2dx+\frac{c}{\epsilon}\int_0^L\left(g(t)\left(\psi_x-\psi_{0x}\right)-\int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)^2dx\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\epsilon}{2}E(t)+\frac{c}{\epsilon}\left(\int_0^L g^2(t)\left(\psi_x^2-\psi_{0x}^2\right)dx+\int_0^L\left(\int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)^2dx\right)\\
&\leq \frac{\epsilon}{2}E(t)+\frac{c}{\epsilon}g(0)g(t)\left(\int_0^L \psi_x^2dx+\int_0^L \psi_0^2(x)\,dx\right)-\frac{c}{\epsilon}\left(g'\circ\psi_x\right)(t)\\
&\leq \frac{\epsilon}{2}E(t)+\frac{c}{\epsilon}g(t)\left(E(0)+c\right)-\frac{c}{\epsilon}\left(g'\circ\psi_x\right)(t)\\
&\leq \frac{\epsilon}{2}E(t)+\frac{c}{\epsilon}g(t)-\frac{c}{\epsilon}\left(g'\circ\psi_x\right)(t)\\
&\leq \frac{\epsilon}{2}E(t)+\frac{c}{\epsilon}g(t)-\frac{c}{\epsilon}E'(t), \qquad \forall t\geq t_0.
\end{aligned} \tag{3.41}$$

Collecting (3.40) and (3.5) , we arrive at

$$\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx\leq \epsilon E(t)+\frac{c}{\epsilon}\left(\left(g\circ\psi_{xt}\right)(t)-E'(t)+g(t)\right),\quad \forall t\geq t_0.$$

□

Lemma 3.14. *Assume that* ***(A.1)*** *and* ***(A.2)*** *hold with H being linear. Let* (φ,ψ) *be the strong solution of (P). Then, for some positive constant* c_1 *we have*

$$\xi(t)\left(g\circ\psi_{xt}\right)(t)\leq c\left(-E_*'(t)+c_1 g(t)\right),\quad \forall t\geq 0.$$

Proof. From (3.37) and (3.38), we have, for all $t\geq 0$ and some fixed positive constant c_1

$$\begin{aligned}
0\leq \quad -\left(g'\circ\psi_{xt}\right)(t) &= -2E_*'(t)-g(t)\int_0^L \psi_{xt}^2dx-2g(t)\int_0^L \psi_{xx0}\psi_{tt}dx\\
&\leq -2E_*'(t)-2g(t)\int_0^L \psi_{xx0}\psi_{tt}dx\\
&\leq -2E_*'(t)+g(t)\int_0^L\left(\psi_{xx0}^2+\psi_{tt}^2\right)dx\\
&\leq -2E_*'(t)+g(t)\left(\frac{2}{\rho_1}E_*(t)+\int_0^L \psi_{xx0}^2dx\right)\\
&\leq c\left(-E_*'(t)+c_1 g(t)\right).
\end{aligned} \tag{3.42}$$

Using the fact that ξ is non-increasing, the estimates (3.1) and (3.42), we get

$$\begin{aligned}
\xi(t)\left(g\circ\psi_{xt}\right)(t) &= \xi(t)\int_0^L\int_0^t g(t-s)\left(\psi_{xt}(t)-\psi_{xt}(s)\right)^2dsdx\\
&\leq \int_0^L\int_0^t \xi(t-s)g(t-s)\left(\psi_{xt}(t)-\psi_{xt}(s)\right)^2dsdx
\end{aligned}$$

$$
\begin{aligned}
&\leq -\int_0^L \int_0^t g'(t-s)\left(\psi_{xt}(t)-\psi_{xt}(s)\right)^2 dsdx \\
&\leq -\left(g' \circ \psi_{xt}\right)(t) \\
&\leq c\left(-E_*'(t)+c_1 g(t)\right), \quad \forall t \geq 0.
\end{aligned}
$$

□

Theorem 3.3. *[14] Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in \left(H^2(0,L) \cap H_0^1(0,L)\right) \times H_0^1(0,L)$. Assume that **(A.1)** and **(A.2)** hold and*

$$
\frac{K}{\rho_1} \neq \frac{b}{\rho_2}
$$

Then, there exist positive constants k_1, k_2 and $t_2 > t_0 = g^{-1}(r)$ such that the energy functional associated to problem (P) satisfies the estimate

$$
E(t) \leq k_2 (t-t_0) H_2^{-1}\left(\frac{k_1}{(t-t_0)\int_{t_2}^t \xi(s)\,ds}\right), \quad \forall t > t_2, \tag{3.43}
$$

where H_2 is given by

$$
H_2(\tau) = \tau H'(\tau).
$$

Proof. By applying lemma 3.13 to estimate (3.19), we have, for some $m > 0$

$$
\begin{aligned}
\mathcal{L}'(t) &\leq c\left(g \circ \psi_x\right)(t) + \left(\frac{\rho_1 b}{K} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} dx - mE(t) \\
&\leq -(m-\epsilon) E(t) + c\left(g \circ \psi_x\right)(t) + \frac{c}{\epsilon}\left(g \circ \psi_{xt} - E' + g\right)(t) \\
&\leq -m_1 E(t) + c\left(g \circ \psi_x\right)(t) + g \circ \psi_{xt}(t) - cE'(t) + cg(t), \qquad \forall t \geq t_0,
\end{aligned}
$$

such that ϵ is small enough and m_1 is a fixed positive constant.
By setting $\mathcal{F} = \mathcal{L} + cE \sim E$, we obtain

$$
\mathcal{F}'(t) \leq -m_1 E(t) + c\left(g \circ \psi_x(t) + g \circ \psi_{xt}(t)\right) + cg(t), \qquad \forall t \geq t_0. \tag{3.44}
$$

Case 01. H is linear: Multiplying (3.44) by $\xi(t)$, then recalling **(A.2)** and lemma 3.14, we obtain

$$
\begin{aligned}
\xi(t)\mathcal{F}'(t) &\leq -m_1 \xi(t) E(t) + c\xi(t)\left(g \circ \psi_x(t) + g \circ \psi_{xt}(t)\right) + c\xi(t) g(t) \\
&\leq -m_1 \xi(t) E(t) + c\xi(t)\, g \circ \psi_x(t) + c\xi(t)\, g \circ \psi_{xt}(t) + c\xi(t) g(t) \\
&\leq -m_1 \xi(t) E(t) - cE'(t) + c\left(-E_*'(t) + c_1 g(t)\right) + c\xi(0) g(t), \quad \forall t \geq t_0.
\end{aligned}
$$

It follows from the non-increasing property of ξ ($\xi' \leq 0$) that

$$(\xi\mathcal{F} + cE + cE_*)'(t) \leq \xi(t)\,\mathcal{F}'(t) + cE'(t) + cE_*'(t).$$

Implies that

$$\begin{aligned}(\xi\mathcal{F} + cE + cE_*)'(t) &\leq -m_1\xi(t)\,E(t) + cg(t) + c\xi(0)\,g(t)\\ &\leq -m_1\xi(t)\,E(t) + c_2 g(t), \quad \forall t \geq t_0,\end{aligned}$$

where c_2 is a fixed positive constant.
Then,

$$m_1\xi(t)\,E(t) \leq -(\xi\mathcal{F} + cE + cE_*)'(t) + c_2 g(t), \quad \forall t \geq t_0.$$

An integration over (t_0, t) and exploiting the non-increasing of E, we get

$$\begin{aligned}m_1 E(t)\int_{t_0}^{t}\xi(s)\,ds &\leq -\int_{t_0}^{t}(\xi\mathcal{F} + cE + cE_*)'(s)\,ds + c_2\int_{t_0}^{t}g(s)\,ds\\ &\leq \left[-(\xi\mathcal{F} + cE + cE_*)(s)\right]_{s=t_0}^{s=t} + c_2\int_{t_0}^{t}g(s)\,ds\\ &\leq -(\xi\mathcal{F} + cE + cE_*)(t) + (\xi\mathcal{F} + cE + cE_*)(t_0) + c_2\int_{t_0}^{t}g(s)\,ds.\end{aligned}$$

Using the estimate (3.38), we arrive at

$$\begin{aligned}m_1 E(t)\int_{t_0}^{t}\xi(s)\,ds &\leq (\xi\mathcal{F} + cE)(0) + cE_*(0) + \int_0^L \psi_{xx0}^2\,dx + c_2(b-l)\\ &\leq (\xi\mathcal{F} + cE + cE_*)(0) + \int_0^L \psi_{xx0}^2\,dx + c_2(b-l)\end{aligned}$$

Then,

$$E(t) \leq \frac{(\xi\mathcal{F} + cE + cE_*)(0) + \int_0^L \psi_{xx0}^2\,dx + c_2(b-l)}{m_1\int_{t_0}^{t}\xi(s)\,ds}.$$

Thus, for some fixed positive constant C

$$E(t) \leq \frac{C}{\int_{t_0}^{t}\xi(s)\,ds}.$$

Case 02. H is nonlinear: It follows from the estimates (3.7), (3.5) and (3.42) and the non-increasing property of ξ that , for a fixed positive constant c_2 and any $t \geq t_0$

$$\begin{aligned}&\int_0^L\int_0^{t_0} g(s)\,(\psi_x(t) - \psi_x(t-s))^2\,dsdx + \int_0^L\int_0^{t_0} g(s)\,(\psi_{xt}(t) - \psi_{xt}(t-s))^2\,dsdx\\ &\leq \frac{1}{\xi(t_0)}\int_0^L\int_0^{t_0}\xi(s)\,g(s)\,(\psi_x(t) - \psi_x(t-s))^2\,dsdx\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\xi(t_0)}\int_0^L\int_0^{t_0}\xi(s)\,g(s)\left(\psi_{xt}(t)-\psi_{xt}(t-s)\right)^2 dsdx\\
&\leq -\frac{g(0)}{a\xi(t_0)}\int_0^L\int_0^{t_0}g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx\\
&\quad-\frac{g(0)}{a\xi(t_0)}\int_0^L\int_0^{t_0}g'(s)\left(\psi_{xt}(t)-\psi_{xt}(t-s)\right)^2 dsdx\\
&\leq -\frac{g(0)}{a\xi(t_0)}\int_0^L\int_0^{t_0}g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx\\
&\quad-\frac{g(0)}{a\xi(t_0)}\int_0^L\int_0^{t_0}g'(s)\left(\psi_{xt}(t)-\psi_{xt}(t-s)\right)^2 dsdx\\
&\leq -\frac{g(0)}{a\xi(t_0)}\int_0^L\int_0^{t}g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 dsdx\\
&\quad-\frac{g(0)}{a\xi(t_0)}\int_0^L\int_0^{t}g'(s)\left(\psi_{xt}(t)-\psi_{xt}(t-s)\right)^2 dsdx\\
&\leq -c\left(E'(t)+E'_*(t)\right)+c_2 g(t)\,.
\end{aligned}
$$

Applying this estimate to (3.44), we find, for a fixed positive constant c_3 and $t \geq t_0$

$$
\begin{aligned}
\mathcal{F}'(t) \leq\;& -m_1E(t)+c\left(g\circ\psi_x(t)+g\circ\psi_{xt}(t)\right)+cg(t)\\
\leq\;& -m_1E(t)-c\left(E'(t)+E'_*(t)\right)+c_2g(t)+cg(t)\\
\leq\;& -m_1E(t)-c\left(E'(t)+E'_*(t)\right)+c_3g(t)\\
&+c\int_0^L\int_0^t g(s)\left(\left(\psi_x(t)-\psi_x(t-s)\right)^2+\left(\psi_{xt}(t)-\psi_{xt}(t-s)\right)^2\right)dsdx\\
\leq\;& -m_1E(t)-c\left(E'(t)+E'_*(t)\right)+c_3g(t)\\
&+c\int_0^t g(s)\left(\|\psi_x(t)-\psi_x(t-s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right)ds.
\end{aligned}
\tag{3.45}
$$

Now, we define a functional η by

$$
\eta(t)=\frac{\gamma}{t-t_0}\int_{t_0}^t\left(\|\psi_x(t)-\psi_x(t-s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right)ds.
$$

It follows from the estimates (3.4), (3.5), (3.36) and (3.38) that

$$
\begin{aligned}
&\frac{1}{t-t_0}\int_{t_0}^t\left(\|\psi_x(t)-\psi_x(t-s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right)ds\\
&\leq \frac{2}{t-t_0}\int_{t_0}^t\left(\|\psi_x(t)\|_2^2+\|\psi_x(t-s)\|_2^2+\|\psi_{xt}(t)\|_2^2+\|\psi_{xt}(t-s)\|_2^2\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{4}{l\,(t-t_0)} \int_{t_0}^{t} \left(E(t) + E(t-s) + E_*(t) + E_*(t-s)\right) ds \\
&\leq \frac{8}{l\,(t-t_0)} \int_{t_0}^{t} \left(E(0) + c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right)\right) ds \\
&\leq \frac{8}{l\,(t-t_0)} \left(E(0) + c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right)\right) \int_{t_0}^{t} ds \\
&\leq \frac{8}{l} \left(E(0) + c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right)\right) < +\infty, \quad \forall t > t_0,
\end{aligned}
$$

which allows us to choose $0 < \gamma < 1$ such that

$$\eta\,(t) < 1, \quad \forall t \geq t_0. \tag{3.46}$$

We define an other functional θ by

$$\theta\,(t) = -\int_{t_0}^{t} g'\,(s) \left(\|\psi_x\,(t) - \psi_x\,(t-s)\|_2^2 + \|\psi_{xt}\,(t) - \psi_{xt}\,(t-s)\|_2^2\right) ds.$$

Noticing that, for a fixed positive constant c_4

$$
\begin{aligned}
\theta\,(t) &\leq -g' \circ \psi_x\,(t) - g' \circ \psi_{xt}\,(t) \\
&\leq -cE'\,(t) - c\left(E_*'\,(t) + c_1 g(t)\right) \qquad (3.47) \\
&\leq -c\left(E'\,(t) + E_*'\,(t) + c_4 g(t)\right), \quad \forall t > t_0.
\end{aligned}
$$

It follows from the strict convexity of H and the fact that $H(0) = 0$ that

$$H\,(s\tau) \leq sH\,(\tau), \quad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0, r]\,.$$

Exploiting this fact with (3.1), the non-increasing property of ξ and Jensen's inequality, we get, for any $t > t_0$

$$
\begin{aligned}
\theta\,(t) &= -\int_{t_0}^{t} g'\,(s) \left(\|\psi_x\,(t) - \psi_x\,(t-s)\|_2^2 + \|\psi_{xt}\,(t) - \psi_{xt}\,(t-s)\|_2^2\right) ds \\
&= -\frac{1}{\eta\,(t)} \int_{t_0}^{t} \eta\,(t)\, g'\,(s) \left(\|\psi_x\,(t) - \psi_x\,(t-s)\|_2^2 + \|\psi_{xt}\,(t) - \psi_{xt}\,(t-s)\|_2^2\right) ds \\
&\geq \frac{1}{\eta\,(t)} \int_{t_0}^{t} \eta\,(t)\, \xi\,(s)\, H\,(g\,(s)) \left(\|\psi_x\,(t) - \psi_x\,(t-s)\|_2^2 + \|\psi_{xt}\,(t) - \psi_{xt}\,(t-s)\|_2^2\right) ds \\
&\geq \frac{\xi\,(t)}{\eta\,(t)} \int_{t_0}^{t} H\,(\eta\,(t)\, g\,(s)) \left(\|\psi_x\,(t) - \psi_x\,(t-s)\|_2^2 + \|\psi_{xt}\,(t) - \psi_{xt}\,(t-s)\|_2^2\right) ds \\
&\geq \frac{\gamma\,(t-t_0)\, \xi\,(t)}{\gamma\,(t-t_0)\, \eta\,(t)} \int_{t_0}^{t} H\,(\eta\,(t)\, g\,(s)) \left(\|\psi_x\,(t) - \psi_x\,(t-s)\|_2^2\right. \\
&\quad \left. + \|\psi_{xt}\,(t) - \psi_{xt}\,(t-s)\|_2^2\right) ds
\end{aligned}
$$

$$\geq \frac{(t-t_0)\,\xi(t)}{\gamma} H\left(\frac{\gamma}{(t-t_0)\,\eta(t)} \int_{t_0}^{t} \eta(t)\, g(s) \left(\|\psi_x(t)-\psi_x(t-s)\|_2^2 \right.\right.$$
$$\left.\left. + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right) ds\right)$$
$$\geq \frac{(t-t_0)\,\xi(t)}{\gamma} H\left(\frac{\gamma}{(t-t_0)} \int_{t_0}^{t} g(s) \left(\|\psi_x(t)-\psi_x(t-s)\|_2^2 \right.\right.$$
$$\left.\left. + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right) ds\right)$$
$$:= \frac{(t-t_0)\,\xi(t)}{\gamma} \overline{H}\left(\frac{\gamma}{(t-t_0)} \int_{t_0}^{t} g(s) \left(\|\psi_x(t)-\psi_x(t-s)\|_2^2 \right.\right.$$
$$\left.\left. + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right) ds\right),$$

where $\overline{H}$ is a $C^2 - extention$ of H that is strictly increasing and strictly convex on $(0,\infty)$. This yields, for all $t > t_0$

$$\int_{t_0}^{t} g(s) \left(\|\psi_x(t)-\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\right) ds \leq \frac{t-t_0}{\gamma} \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right).$$

Then, the estimate (3.45) becomes, for all $t > t_0$

$$\mathcal{F}'(t) \leq -m_1 E(t) - c\left(E'(t) + E_*'(t)\right) + c_3 g(t) + c\frac{t-t_0}{\gamma} \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right).$$

Setting $\mathcal{F}_1 = \mathcal{F} + cE + cE_*$, we obtain, for all $t > t_0$

$$\mathcal{F}_1'(t) \leq -m_1 E(t) + c_3 g(t) + c\frac{t-t_0}{\gamma} \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right). \tag{3.48}$$

Let $\mathcal{F}_2$ be a functional defined for $0 < \epsilon_1 < r$ by

$$\mathcal{F}_2(t) = \overline{H}'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\mathcal{F}_1, \quad \forall t > t_0.$$

Exploiting (3.48), the facts that $H' > 0$, $H'' > 0$ and $E' \leq 0$, we arrive at

$$\begin{aligned}
\mathcal{F}_2'(t) &= \left(\frac{\epsilon_1}{t-t_0}\frac{E'(t)}{E(0)} - \frac{\epsilon_1}{(t-t_0)^2}\frac{E(t)}{E(0)}\right)\overline{H}''\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) \\
&\quad + \mathcal{F}_1'(t)\,\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) \\
&\leq \left(-m_1 E(t) + c_3 g(t) + c\frac{t-t_0}{\gamma}\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right)\right)\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) \\
&\leq -m_1 E(t)\,\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) + c_3 g(t)\,\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) \\
&\quad + c\frac{t-t_0}{\gamma}\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right)\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right).
\end{aligned} \tag{3.49}$$

Let $\overline{H}^*$ be the convex conjugate of $\overline{H}$ as in (3.31), by taking

$$A = \overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) \quad \text{and} \quad B = \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right),$$

then, with using (3.31), (3.32) and (3.49), we find that

$$\begin{aligned}
\mathcal{F}_2'(t) \leq\ & -m_1 E(t)\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) + c_3\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)g(t) \\
& + c\frac{t-t_0}{\gamma}\overline{H}\left(\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right)\right) + \overline{H}^*\left(\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)\right) \\
\leq\ & -m_1 E(t)\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) + c_3\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)g(t) \\
& + c\frac{t-t_0}{\gamma}.\frac{\gamma\theta(t)}{\xi(t)(t-t_0)} + \overline{H}^*\left(\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)\right) \\
\leq\ & -m_1 E(t)\overline{H}'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + c_3\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)g(t) \\
& + c\frac{\theta(t)}{\xi(t)} + c\epsilon_1\frac{E(t)}{E(0)}\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) \\
\leq\ & \left(-m_1 E(0) + c\epsilon_1\right)\frac{E(t)}{E(0)}\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right) + c\frac{\theta(t)}{\xi(t)} \\
& + c_3\overline{H}'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)g(t), \quad \forall t > t_0,
\end{aligned}$$

then, for $m_2 > 0$ and ϵ_1 is fixed small enough

$$\mathcal{F}_2'(t) \leq -m_2\frac{E(t)}{E(0)}\overline{H}'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + c\frac{\theta(t)}{\xi(t)} + c_3\overline{H}'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)g(t), \quad \forall t > t_0.$$

Multiplying this estimate by $\xi(t)$ and using (3.47) and $\epsilon_1\frac{E(t)}{E(0)} < r$, we find that

$$\begin{aligned}
\xi(t)\mathcal{F}_2'(t) \leq\ & -m_2\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\xi(t) + c\theta(t) + c_3 H'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)g(t)\xi(t) \\
\leq\ & -m_2\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\xi(t) - c\left(E'(t) + E_*'(t)\right) + cc_4 g(t) \\
& + c_3 H'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)g(t)\xi(t), \quad \forall t > t_0.
\end{aligned} \tag{3.50}$$

Combining the strictly increasing property of H' and the non-increasing property of ξ and E with the fact that there exists $t_2 > t_0$ such that $\frac{1}{t-t_0} < 1$ and $\lim_{t\to\infty}\frac{1}{t-t_0} = 0, \quad \forall t > t_2$, we obtain

$$H'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\xi(t) \leq H'(\epsilon_1)\xi(0), \quad \forall t > t_2.$$

Then, for some positive constant c_5, (3.50) becomes

$$\xi(t)\mathcal{F}_2'(t)+c\left(E'(t)+E_*'(t)\right)\le -m_2\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)\xi(t)+c_5g(t),\quad \forall t>t_2.$$

By setting $\mathcal{F}_3=\xi\mathcal{F}_2+c(E+E_*)$ and using the non-increasing property of ξ, we arrive at

$$m_2\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)\xi(t)\le -\mathcal{F}_3'+c_5g(t),\quad \forall t>t_2. \tag{3.51}$$

Noticing that the following map

$$t\longmapsto E(t)H'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)$$

is non-increasing because of the fact that $H''>0$ and the non-increasing property of E. Thus, an integration of (3.51) over (t_2,t) gives

$$\begin{aligned} m_2\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}\frac{E(t)}{E(0)}\right)\int_{t_2}^t\xi(s)\,ds &\le \int_{t_2}^t m_2\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\xi(s)\,ds\\ &\le -\int_{t_2}^t\mathcal{F}_3'(s)\,ds+c_5\int_{t_2}^t g(s)\,ds\\ &\le -\mathcal{F}_3'(t)+\mathcal{F}_3'(t_2)+c_5\int_{t_2}^t g(s)\,ds\\ &\le \mathcal{F}_3'(t_2)+c_5(b-l),\quad \forall t>t_2, \end{aligned} \tag{3.52}$$

multiplying this estimate by $\dfrac{1}{t-t_0}>0$, we obtain, for $t>t_2$

$$m_2\frac{1}{t-t_0}\frac{E(t)}{E(0)}H'\left(\frac{\epsilon_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\int_{t_2}^t\xi(s)\,ds\le\frac{1}{t-t_0}\left(\mathcal{F}_3'(t_2)+c_5(b-l)\right),\qquad \forall t>t_2.$$

Setting $H_2(\tau)=\tau H'(\tau)$, which is strictly increasing because
$H_2'(\tau)=H'(\tau)+\tau H''(\tau)$, we arrive for two positive constants k_1 and k_2 at

$$\begin{aligned} E(t) &\le \frac{E(0)(t-t_0)}{\epsilon_1}H_2^{-1}\left(\frac{\epsilon_1}{m_2}\frac{\mathcal{F}_3'(t_2)+c_5(b-l)}{(t-t_0)\int_{t_2}^t\xi(s)\,ds}\right)\\ &\le k_2(t-t_0)H_2^{-1}\left(\frac{k_1}{(t-t_0)\int_{t_2}^t\xi(s)\,ds}\right),\quad \forall t>t_2. \end{aligned}$$

□

Conclusion

Timoshenko system is a field of research that aroused a lot of interest in recent years and became an attracting field of research ever since. After a few years of great interest, the study of its stability properties has become an active area of research in the field of partial differential equations. It is widely used in structural engineering and mechanics to analyze the behavior of beams, columns, and other structural elements under various loading conditions. This work is an analytic study of only two types of Timoshenko systems, the nonlinear type with delay and the viscoelastic type with type-memory, the existence and uniqueness of their solutions and the exponential and general stability of these systems, respectively. Finally, there is other Timoshenko systems to study, like Thermoviscoelastic Timoshenko system, Timoshenko system with past history, Timoshenko system with diffusion effect..., which can exhibit either general, exponential or polynomial decay depending on the system parameters and the presence of frictional dissipative terms.

Bibliography

[1] G. Allaire. *Numerical analysis and optimization: an introduction to mathematical modelling and numerical simulation*. OUP Oxford, 2007.

[2] D. d. S. Almeida Júnior, M. Santos, and J. Muñoz Rivera. Stability to 1-d thermoelastic timoshenko beam acting on shear force. *Zeitschrift für angewandte Mathematik und Physik*, 6.

[3] F. Ammar-Khodja, A. Benabdallah, J. M. Rivera, and R. Racke. Energy decay for timoshenko systems of memory type. *Journal of Differential Equations*, 194(1):82–115, 2003.

[4] F. Ammar-Khodja, A. Benabdallah, J. M. Rivera, and R. Racke. Energy decay for timoshenko systems of memory type. *Journal of Differential Equations*, 194(1):82–115, 2003.

[5] T. A. Apalara, S. A. Messaoudi, and A. A. Keddi. On the decay rates of timoshenko system with second sound. *Mathematical Methods in the Applied Sciences*, 3(10):2671–2684, 2016.

[6] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer, 1989.

[7] H. a. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. 2011.

[8] R. Datko, J. Lagnese, and M. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM journal on control and optimization*, 24(1):152–156, 1986.

[9] K.-J. Engel, R. Nagel, and S. Brendle. *One-parameter semigroups for linear evolution equations*, volume 194. Springer, 2000.

[10] B. Feng and X.-G. Yang. Long-time dynamics for a nonlinear timoshenko system with delay. *Applicable Analysis*, 96(4):606–625, 2017.

[11] A. Guesmia and S. A. Messaoudi. On the control of a viscoelastic damped timoshenko-type system. *Applied mathematics and computation*, 206(2):589–597, 2008.

[12] A. Guesmia and S. A. Messaoudi. On the stabilization of timoshenko systems with memory and different speeds of wave propagation. *Applied Mathematics and Computation*, 219(17):9424–9437, 2013.

[13] A. Guesmia, S. A. Messaoudi, and A. Soufyane. Stabilization of a linear timoshenko system with infinite history and applications to the timoshenko-heat systems. *Electronic Journal of Differential Equations*, (193), 2012.

[14] J. H. Hassan, S. A. Messaoudi, and M. Zahri. Existence and new general decay results for a viscoelastic timoshenko system. *Zeitschrift für Analysis und ihre Anwendungen*, 39(2):185–222, 2020.

[15] K.-P. Jin, J. Liang, and T.-J. Xiao. Coupled second order evolution equations with fading memory: Optimal energy decay rate. *Journal of Differential Equations*, 257(5):1501–1528, 2014.

[16] J. E. Mu {\~ n} oz Rivera and R. Racke. Magneto-thermo-elasticity—large-time behavior for linear systems. 2001.

[17] M. I. Mustafa. General decay result for nonlinear viscoelastic equations. *Journal of Mathematical Analysis and Applications*, 457(1):134–152, 2018.

[18] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM Journal on Control and Optimization*, 45(5):1561–1585, 2006.

[19] S. Nicaise and C. Pignotti. Stabilization of the wave equation with boundary or internal distributed delay. 2008.

[20] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[21] B. Said-Houari and Y. Laskri. A stability result of a timoshenko system with a delay term in the internal feedback. *Applied Mathematics and Computation*, 217(6):2857–2869, 2010.

[22] T. SP. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Phil Mag*, 41:744–746.

[23] K. Yoshida and M. S. Narasimhan. *Lectures on semi-group theory and its application to Cauchy's problem in partial differential equations*, volume 8. Tata Institute of Fundamental Research Bombay, India, 1957.