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**Nonlinear Delay Integral Inequalities of the Volterra-Fredholm
Type on Time Scales**

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ABSTRACT

The theory of integral inequalities constitutes a fundamental component of modern mathematical analysis. It plays a key role in examining the qualitative properties of solutions to complex differential and integral equations. As such, it is crucial for stability analysis, the investigation of existence and uniqueness, and the estimation of solutions in various dynamic systems.

This study aims to investigate the theory of time scales, then to provide some generalisations of the Volterra-Fredholm type integral inequalities on time scales with delay.

Including a time delay element into these inequalities makes them more realistic in modeling natural and physical systems, as it reflects the natural delay in response and cumulative effects over time.

Furthermore, this work will include illustrative examples and numerical simulation to demonstrate the practical utility and effectiveness of the derived inequalities.

key words: Time Scales, integral inequality, Volterra-Fredholm inequalities, retarded inequality.

RÉSUMÉ

La théorie des inégalités intégrales constitue un élément fondamental de l'analyse mathématique moderne. Elle joue un rôle clé dans l'étude des propriétés qualitatives des solutions des équations différentielles et intégrales complexes. À ce titre, elle est essentielle pour l'analyse de la stabilité, la recherche de l'existence et de l'unicité, ainsi que pour l'estimation des solutions dans divers systèmes dynamiques.

Cette étude vise en premier à investiguer la théorie des échelles de temps, puis à établir quelques généralisations des inégalités intégrales de type Volterra-Fredholm aux échelles du temps avec retard.

L'intégration d'un élément de retard dans ces inégalités les rends plus réalistes dans la modélisation des systèmes naturels et physiques, car elle reflète le délai naturel de réponse et les effets cumulatifs au fil du temps.

De plus, ce travail comprendra des exemples illustratifs et une simulation numérique pour démontrer l'utilité pratique et l'efficacité des inégalités établies.

Mots clés: Echelles de temps, inégalité intégrale, inégalité de Volterra Fredholm, inégalité à retard.

ملخص

تُعدُّ نظرية المتباينات التكاملية من الركائز الأساسية في التحليل الرياضي المعاصر. حيث تؤدي دورًا محوريًا في الدراسة النوعية لحلول المعادلات التفاضلية والتكاملية المعقدة. وتعد هذه النظرية أداة فعالة في معالجة قضايا تحليل الاستقرار، واثبات نظريات الوجود والوحدانية، بالإضافة إلى تقديم تقديرات دقيقة للحلول ضمن مختلف أصناف الأنظمة الديناميكية.

تتمحور هذه الدراسة حول استكشاف نظرية المقاييس الزمنية، تليها صياغة تعميمات جديدة للمتباينات التكاملية من نوع فولتيرا-فريدهولم على المقاييس الزمنية مع إدراج عنصر التأخير الزمني.

إن إدراج عنصر التأخير الزمني في هذه المتباينات يجعلها أكثر واقعية في نمذجة الأنظمة الطبيعية والفيزيائية، حيث يعكس التأخير الطبيعي في الاستجابة والتأخيرات التراكمية بمرور الزمن.

إضافة إلى ذلك، تتضمن هذه الدراسة أمثلة توضيحية وتطبيقات تهدف إلى إبراز الجدوى العملية والفعالية التطبيقية للمتباينات المطروحة في هذا العمل.

الكلمات المفتاحية: مقاييس زمنية، متباينات تكاملية، متباينات فولتيرا-فريدهولم، متباينات بتأخر.

NOTATIONS

To help the reader navigate the content more easily, this section gathers the key symbols and notations that will appear throughout this work.

- $\mathbb{T}$: Denotes a time scale.
- $\sigma(t)$: The forward jump operator at a point t .
- $\rho(t)$: The backward jump operator at a point t .
- $\mu(t)$: The graininess function.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}_+$: The set of positive real numbers.
- $\mathfrak{R}$: The set of regressive functions.
- $\mathfrak{R}_+$: The set of positive regressive functions.
- $\oplus$: The circle plus addition.
- C_{rd} : Continuous right-dense functions.
- I : $I = [t_0, T] \cap \mathbb{T}^k$.
- Δ : The delta (Hilger) derivative of a function.
- $C(\mathbb{R}_+, \mathbb{R}_+)$: The set of all real-valued functions defined on the positive real numbers that are continuous and take only positive real values.

- $D_1k(t, s)$: The partial derivative of a function $K(t, s)$ with respect to its first variable, which is t .
- $\mathbb{N}_0$: Denotes the set of natural numbers including zero.
- $\mathbb{N}$: Denotes the set of strictly positive natural numbers.

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INTRODUCTION

Historically, inequalities became the heart of mathematical analyses, therefore, inequalities played an important role in many branches of modern mathematics, including the theory of probability and statistics, real analysis, complex analysis, and numerical analysis. Furthermore, the importance of these integral inequalities largely extends to the study of integral equations and more generally, in the study of ordinary differential equations.

These integral inequalities were introduced by Gronwall in 1919, where he established their applications in various domains. Since that time, the theory of these inequalities evolved rapidly. The application of integral inequalities, as well as differential inequalities, have been remarkably developed in the study of the asymptotic behavior of solutions, including aspects such as existence, uniqueness, comparison, and stability. New inequalities have been demonstrated according to a generalisations for some well known integral inequalities, among these generalisations, we can mention the works of Bellman [2], Pachpatte [14], and Volterra-Fredholm [15].

Furthermore, the issue of unification and simplification in mathematics has occupied the attention of many researchers [9]. As a result the theory of time scales played the fundamental role to unify between discrete and continuous analyses. This new theory was proposed by Stephan Hilger in 1988 in his thesis of doctorat [8]. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice, once for differential equations, and once for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scales, which is an arbitrary nonempty closed subset

of reals, more details are shown in [10].

The purpose of this work is to give a brief summary on integral inequalities, and to establish some generalizations of integral inequalities of the Volterra-Fredholm type on time scales, then to present some examples and a numerical simulation.

- The first chapter, presents a terminology, which contains some basic notions related to time scales theory, which are necessary and essential for the understanding of our work.
- The second chapter represents some classical inequalities such those of: Grönwall, Bellman, Pachpatte, Volterra-Fredholm. Then, it introduces several known generalisations on time scales given by Bohner and Paterson, Pachpatte, and Gu-Meng. Finally, we present, with full proofs some generalised Volterra-Fredholm type integral inequalities on time scales with delay.
- The third chapter provides some examples of retarded Volterra-Fredholm type integral inequalities on time scales, then a numerical simulation with MATLAB.

CHAPTER

1

PRELIMINARIES ON TIME SCALES

Before proving inequalities or applying them to dynamic equations, it is necessary to establish the foundational concepts of time scales calculus. This chapter introduces the main elements of the theory, we presented definitions, examples, theorems, proposition and some illustrations in this chapter according to the books [0] and [16], while the lemmas were quoted from [10], we started by a general definition for the time scales theory, then we established jump operators, graininess function, rd-continuity, Delta differentiation, Delta integration, and exponential functions on time scales. These preliminaries will allow us to rigorously understand the results that follow in later chapters. For more details, the reader can check also [0], [12], and [17].

1.1 Basic notions

1.1.1 The theory of time scales

The theory of time scales has received the attention of many scientists. It was introduced in 1988 by Stephan Hilger in his PHD work [8] in order to unify between discrete and continuous analyses. Time scales was defined as a nonempty closed subset. It contributed fundamentally to the study of difference and differential equations, in addition, the time scales calculus has a various mathematical applications, it is applicable to domains where process dynamics are described by discrete-time or continuous-time

models or a combination theorem, for example, it can model insect populations that are continuous while in season (and may follow a difference scheme with variable step-size), die out in (say) winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population; we quoted this from [11]. time scales applications are not limited to: macroeconomic models, and the agronomic cycle of seasonal plants.

1.1.2 Terminology

Definition 1.1.1 *Time scale is an arbitrary nonempty closed subset of the real numbers, the time scale's symbol chosen in our research is $\mathbb{T}$.*

Example 1.1.1 *The sets $\mathbb{R}, \mathbb{Z}, \mathbb{Z}^+, [0, 1] \cup [2, 3]$, and $[0, 1] \cup \mathbb{N}$ are time scales.*

Example 1.1.2 *The sets $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{C},]0, 1[$ are not time scales.*

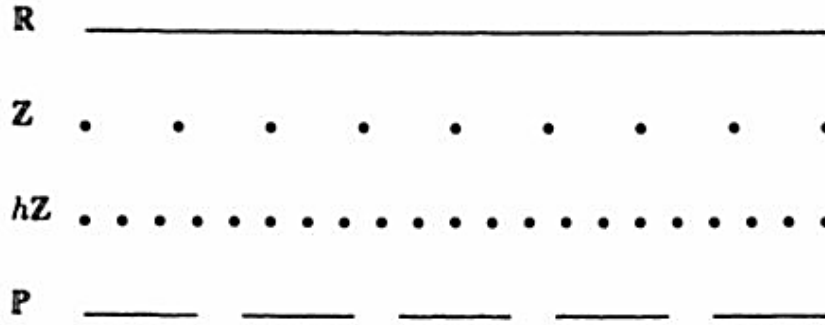


Fig1: Some time scales

1.1.3 Forward and backward jump operators

Definition 1.1.2 *Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, is defined by:*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}. \quad (1.1)$$

Definition 1.1.3 *Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$, is defined by:*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \quad (1.2)$$

Definition 1.1.4 Suppose that $t \in \mathbb{T}$, the points in a an arbitrary time scale are classified as follows:

1. If $\sigma(t) > t$, then we say that t is a right scattered point.
2. If $t < \sup(\mathbb{T})$ and $\sigma(t) = t$, then we say that t is a right-dense point.
3. If $\rho(t) < t$, then we say that t is left-scattered.
4. If $t > \inf(\mathbb{T})$ and $\rho(t) = t$, then we say that t is a left-dense point.
5. If t is left-scattered and right-scattered at the same time, then we say that t is isolated.
6. If t is left-dense and right-dense at the same time, then we say that t is dense.

For more understanding, we presented the following figure:

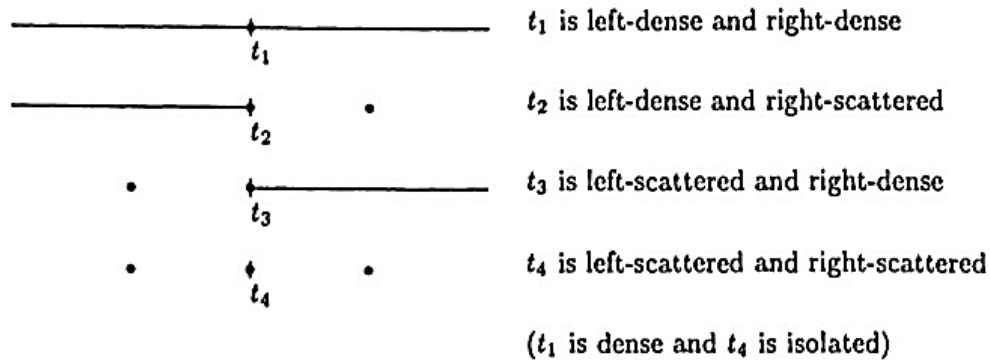


Fig2: Classification of points

Example 1.1.3 Let's consider the following time scale: $\mathbb{T} = \{\sqrt{2n+1} : n \in \mathbb{N}\}$, If $t = \sqrt{2n+1}$, for some $n \in \mathbb{N}$, then $n = \frac{t^2-1}{2}$ and

$$\sigma(t) = \inf \{l \in \mathbb{N} : \sqrt{2l+1} > \sqrt{2n+1}\} = \sqrt{2n+3} = \sqrt{t^2+2} \quad \text{for } n \in \mathbb{N},$$

$$\rho(t) = \sup \{l \in \mathbb{N} : \sqrt{2l+1} < \sqrt{2n+1}\} = \sqrt{2n-1} = \sqrt{t^2-2} \quad \text{for } n \in \mathbb{N}, \quad n \geq 2.$$

For $n = 1$ we have

$$\rho(\sqrt{3}) = \sup(\emptyset) = \inf(\mathbb{T}) = \sqrt{3},$$

since

$$\sqrt{t^2-2} < t < \sqrt{t^2+2} \quad \text{for } n \geq 2.$$

We conclude that every point $\sqrt{2n+1}$, $n \in \mathbb{N}$, $n \geq 2$, is right-scattered and left-scattered, i.e., every point $\sqrt{2n+1}$, $n \in \mathbb{N}$, $n \geq 2$ is isolated. Because

$$\sqrt{3} = \rho(\sqrt{3}) < \sigma(\sqrt{3}) = \sqrt{5},$$

We have that the point $\sqrt{3}$ is right-scattered.

Example 1.1.4 Let's consider the following time scale $\mathbb{T} = \left\{ \frac{n}{3} : n \in \mathbb{N}_0 \right\}$ and $t = \frac{n}{3}$, $n \in \mathbb{N}_0$, be arbitrarily chosen,

1. If $n \in \mathbb{N}$. Then

$$\sigma\left(\frac{n}{3}\right) = \inf \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 > \frac{n}{3}, l \in \mathbb{N}_0 \right\} = \frac{n+1}{3} > \frac{n}{3}$$

$$\rho\left(\frac{n}{3}\right) = \sup \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 < \frac{n}{3}, l \in \mathbb{N}_0 \right\} = \frac{n-1}{3} < \frac{n}{3}$$

Hence all points $t = \frac{n}{3}$, $n \in \mathbb{N}$, are right-scattered and left-scattered,

i.e., all points $t = \frac{n}{3}$, $n \in \mathbb{N}$, are isolated.

2. For $n = 0$. Then

$$\sigma(0) = \inf \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 > 0, l \in \mathbb{N}_0 \right\} = \frac{1}{3} > 0$$

$$\rho(0) = \sup \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 < 0, l \in \mathbb{N}_0 \right\} = \sup(\emptyset) = \inf(\mathbb{T}) = 0.$$

i.e., $t = 0$ is right-scattered.

Definition 1.1.5 Assume that $\mathbb{T}$ is a time scale, for $t \in \mathbb{T}$, the graininess function is defined as follows:

$$\mu : \mathbb{T} \longrightarrow \mathbb{T} \quad , \quad \mu(t) = \sigma(t) - t. \quad (1.3)$$

Definition 1.1.6 Assume that f is a function, such that $f : \mathbb{T} \rightarrow \mathbb{R}$, then for any $t \in \mathbb{T}$ we can define the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t))$$

i.e.,

$$f^\sigma = f \circ \sigma.$$

Example 1.1.5 Let $\mathbb{T} = \{t = 2^{n+2} : n \in \mathbb{N}\}$, $f(t) = t^2 + t - 1$. Then

$$\sigma(t) = \inf \left\{ 2^{l+2} : 2^{l+2} > 2^{n+2}, l \in \mathbb{N} \right\} = 2^{n+3} = 2t.$$

Therefore

$$f^\sigma(t) = f(\sigma(t)) = \sigma^2(t) + \sigma(t) - 1 = (2t)^2 + 2t - 1, \quad t \in \mathbb{T}. \quad (1.4)$$

Example 1.1.6 suppose that $\mathbb{T} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$, $f(t) = t^3 - t$, $t \in \mathbb{T}$.

1. If $t = \frac{1}{n}$, $n \geq 2$. Then $n = \frac{1}{t}$ and

$$\sigma(t) = \inf \left\{ \frac{1}{l}, 0 : \frac{1}{l}, 0 > \frac{1}{n}, l \in \mathbb{N} \right\} = \frac{1}{n-1} = \frac{t}{1-t}$$

Therefore

$$\begin{aligned} f(\sigma(t)) &= \sigma^3(t) - \sigma(t) = \left(\frac{t}{1-t} \right)^3 - \frac{t}{1-t} \\ &= \frac{t^3}{(1-t)^3} - \frac{t}{1-t} = \frac{t^3 - t(1-t)^2}{(1-t)^3} \\ &= \frac{t^3 - t(1-2t+t^2)}{(1-t)^3} = \frac{t^3 - t + 2t^2 - t^3}{(1-t)^3}. \end{aligned}$$

2. For $t = 1$. We find

$$\sigma(1) = \inf \left\{ \frac{1}{l}, 0 : \frac{1}{l}, 0 > 1, l \in \mathbb{N} \right\} = \inf(\emptyset) = \sup(\mathbb{T}) = 1,$$

$$f(\sigma(1)) = \sigma^3(1) - \sigma(1) = 1 - 1 = 0,$$

3. In the case $t = 0$. We have

$$\sigma(0) = \inf \left\{ \frac{1}{l}, 0 : \frac{1}{l}, 0 > 0 \right\} = 0,$$

$$f(\sigma(0)) = \sigma^3(0) - \sigma(0) = 0.$$

Now, we will illustrate the previous definitions by some sets of time scales and their characteristics (jump operators, graininess function, etc.), see Table [1.1](#).

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$k\mathbb{Z}$	$\mathbb{N}_0^2$	$q^{\mathbb{N}}$	$[0, 1] \cup [2, 3]$
$\sigma(t)$	t	$t + 1$	$t + k$	$(\sqrt{t} + 1)^2$	qt	$\begin{cases} t & \text{if } t \in [0, 1[\cup [2, 3] \\ 2 & \text{if } t = 1 \end{cases}$
$\rho(t)$	t	$t - 1$	$t - k$	$(\sqrt{t} - 1)^2$	$\frac{t}{q}$	$\begin{cases} t & \text{if } t \in [0, 1[\cup]2, 3] \\ 1 & \text{if } t = 2 \end{cases}$
$\mu(t)$	0	1	k	$2\sqrt{t} + 1$	$(q - 1)t$	$\begin{cases} 0 & \text{if } t \in [0, 1[\cup [2, 3] \\ 1 & \text{if } t = 1 \end{cases}$

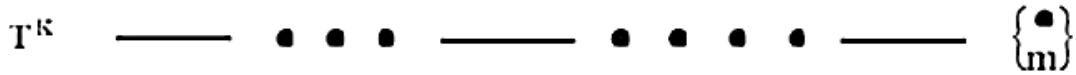
Table1.1 Some time scales and their characteristics

1.1.4 Subsets derived from a time scale

Let m be a maximum element in a time scale $\mathbb{T}$.

If the point m is left-scattered then, $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. Alternatively, note that from each time scale we can extract the following subset :

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus]\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases}$$

Fig3: Illustration of the set $\mathbb{T}^k$

Example 1.1.7 let $\mathbb{T} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. Then $\sup(\mathbb{T}) = 1$, and

$$\rho(1) = \sup \left\{ \frac{1}{l}, 0 : \frac{1}{l}, 0 < 1, \quad l \in \mathbb{N} \right\} = \frac{1}{2}.$$

Therefore

$$\mathbb{T}^k = \mathbb{T} \setminus \left(\frac{1}{2}, 1 \right] = \left\{ \frac{1}{n} : n \in \mathbb{N}, n \geq 2 \right\} \cup \{0\}.$$

Definition 1.1.7 For two points $a, b \in \mathbb{T}$, the time scale interval is defined by :

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

1.1.5 Differentiation on time scales

First, we define the so-called delta (Hilger) derivative.

Definition 1.1.8 Let the function $f : \mathbb{T} \rightarrow \mathbb{R}$, we say that f is Δ -differentiable (or simply differentiable) at $t \in \mathbb{T}^k$, provided there exists $f^\Delta(t)$ such that for any $\varepsilon > 0$, there exists a neighbourhood U of t , i.e, $(U = (t - \delta, t + \delta) \cap \mathbb{T}, \text{ for some } \delta > 0)$ with:

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t) [\sigma(t) - s] \right| \leq \varepsilon |[\sigma(t) - s]|. \quad (1.5)$$

We say $f^\Delta(t)$ the delta or hilger derivative of f at t , or shortly differentiable.

If $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$, then the function $f^\Delta : \mathbb{T} \rightarrow \mathbb{R}$ is said to be delta derivative or hilger derivative, shortly derivative of f on $\mathbb{T}^k$.

Example 1.1.8 Let's consider the function $f(t) = t^2$, $t \in \mathbb{T}$. We will prove that $f^\Delta(t) = \sigma(t) + t$, $t \in \mathbb{T}^k$.

Really, for $t \in \mathbb{T}^k$ and for any $\epsilon > 0$, and for any $s \in (t - \epsilon, t + \epsilon) \cap \mathbb{T}$, $s \neq \sigma(t)$. we have $|t - s| < \epsilon$ and

$$\begin{aligned} |f(\sigma(t)) - f(s) - (\sigma(t) + t)(\sigma(t) - s)| &= |\sigma^2(t) - s^2 - (\sigma(t) + t)(\sigma(t) - s)| \\ &= |(\sigma(t) - s)(\sigma(t) + s) - (\sigma(t) + t)(\sigma(t) - s)| \\ &= |\sigma(t) - s||t - s| \\ &\leq \epsilon |\sigma(t) - s|. \end{aligned} \quad (1.6)$$

Theorem 1.1.1 [1] Let $f : \mathbb{T} \rightarrow \mathbb{R}$, is a function and $t \in \mathbb{T}^k$,

1. If f is differentiable at t , then f is continuous at t .
2. If f is continuous on t and if t is right-scattered, then f is Δ -differentiable on t , and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

3. If t right-dense, then f is Δ -differentiable on t if the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s},$$

exists and f as a finite number. In this case we have

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}.$$

4. If f is differentiable on t , then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

Example 1.1.9 Let's consider the following set $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$, and the function $f(t) = t^2, t \in \mathbb{T}$, for all $t \in \mathbb{T}^k, t = n^2, n = \sqrt{t}, n \in \mathbb{N}_0$ we have

$$\sigma(t) = \inf \{l^2 : l^2 > n^2, n \in \mathbb{N}_0\} = (n+1)^2 = (\sqrt{t}+1)^2 > t.$$

Therefore, every point in $\mathbb{T}$ is right-scattered. We note that the function $f(t)$ is continuous on $\mathbb{T}$. Thus,

$$\begin{aligned} f^\Delta(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{\sigma^2(t) - t^2}{\sigma(t) - t} \\ &= \sigma(t) + t = (\sqrt{t}+1)^2 + t \\ &= t + 2\sqrt{t} + 1 + t = 1 + 2\sqrt{t} + 2t. \end{aligned}$$

Example 1.1.10 Let's consider the following time scale $\mathbb{T} = \left\{\frac{1}{2n+1}, n \in \mathbb{N}_0\right\} \cup (0), f(t) = \sigma(t), t \in \mathbb{T}$. We will obtain $f^\Delta(t), t \in \mathbb{T}$. For $t \in \mathbb{T}, t = \frac{1}{2n+1}, n = \frac{1-t}{2t}, n \geq 1$, we have

$$\begin{aligned} \sigma(t) &= \inf \left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > \frac{1}{2n+1}, l \in \mathbb{N}_0 \right\} = \frac{1}{2n-1} \\ &= \frac{1}{2\frac{1-t}{2t} - 1} = \frac{t}{1-2t} > t, \end{aligned}$$

i.e, any point $\frac{1}{2n+1}, n \neq 1$, is right scattered at these points.

$$\begin{aligned} f^\Delta(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = 2 \frac{\sigma(t)}{(1 - 2\sigma(t)\sigma(t) - t)} \\ &= 2 \frac{\left(\frac{t}{1-2t}\right)^2}{\left(1 - \frac{2t}{1-2t}\right) \left(\frac{t}{1-2t} - t\right)} = 2 \frac{\frac{t^2}{(1-2t)^2}}{\frac{1-4t}{1-2t} \frac{2t^2}{1-2t}} = 2 \frac{t^2}{2t^2(1-4t)} = \frac{1}{1-4t}. \end{aligned}$$

Let $n = 0$, i.e, $t = 1$. Then

$$\sigma(1) = \inf \left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 1, l \in \mathbb{N}_0 \right\} = \inf(\emptyset) = \sup(\mathbb{T}) = 1,$$

i.e, $t = 1$ is right-dense point. Also,

$$\lim_{s \rightarrow 1} \frac{f(1) - f(s)}{1 - s} = \lim_{s \rightarrow 1} \frac{\sigma(1) - \sigma(s)}{1 - s} = \lim_{s \rightarrow 1} \frac{1 - \frac{s}{1-2s}}{1 - s} = \lim_{s \rightarrow 1} \frac{1 - 3s}{(1 - s)(1 - 2s)} = +\infty.$$

Hence, $\sigma'(1)$ does not exist.

Let's consider that $t = 0$. Then

$$\sigma(0) = \inf \left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 0, l \in \mathbb{N}_0 \right\} = 0$$

Consequently $t = 0$ is right-dense. Also

$$\lim_{h \rightarrow 0} \frac{\sigma(h) - \sigma(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h}{1-2h} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{1-2h} = 0.$$

Therefore $\sigma'(t) = 1$

Remark 1.1.1 :

1. If $\mathbb{T} = \mathbb{R}$, then we say that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$, if

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exists and, $f^\Delta(t) = f'(t)$.

2. If $\mathbb{T} = \mathbb{Z}$, then the function $f : \mathbb{Z} \rightarrow \mathbb{R}$ is Δ -differentiable at $t \in \mathbb{Z}$, and we have

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t).$$

This following theorem can indicates some basic identities for computing the Δ -derivative.

Theorem 1.1.2 [16] If $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are Δ -differentiables on $t \in \mathbb{T}^k$, then

1. $f + g$ is Δ -differentiable at t , and

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

2. (αf) is Δ -differentiable at t , for all $\alpha \in \mathbb{R}$ and we have

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

3. The product fg is Δ -differentiable at t and

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

4. If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is Δ -differentiable at t and

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$

5. If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ -differentiable at t and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

Example 1.1.11 Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are two functions defined by $f(t) = t, g(t) = t^2$. We obtain

$$\begin{aligned} (fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) \\ &= t^\Delta t^2 + \sigma(t)(t^2)^\Delta \\ &= t^\Delta t^2 + \sigma(t)(t^\Delta t + \sigma(t)t^\Delta) \\ &= t^2 + t\sigma(t) + \sigma^2(t), \end{aligned}$$

and

$$\begin{aligned} (f+g)^\Delta(t) &= f^\Delta + g^\Delta \\ &= t^\Delta + (t^2)^\Delta \\ &= 1 + t^\Delta t + \sigma(t)t^\Delta \\ &= 1 + t + \sigma(t). \end{aligned}$$

Theorem 1.1.3 [176] Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, we consider that $g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at $\mathbb{T}^k$, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, then there exists $c \in [t, \sigma(t)]$ with

$$(f \circ g)^\Delta = f'(g(c))g^\Delta(t)$$

Example 1.1.12 Let's consider the following time scale $\mathbb{T} = \{2^n : n \in \mathbb{N}_0\}$, $f(t) = t + 2$, $g(t) = t^2 - 1$. Knowing that $g : \mathbb{T} \rightarrow \mathbb{R}$ is Delta-differentiable, $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. For $t \in \mathbb{T}$, $t = 2^n$, $n \in \mathbb{N}_0$, $n = \log_2 t$, we have

$$\sigma(t) = \inf \left\{ 2^l : 2^l > 2^n, l \in \mathbb{N}_0 \right\} = 2^{2n+1} = 2t > t.$$

Therefore all point of $\mathbb{T}$ are right-scattered. since up $\mathbb{T} = \infty$ we have that $\mathbb{T}^k = \mathbb{T}$. Also for $t \in \mathbb{T}$, we have

$$(f \circ g)(t) = f(g(t)) = g(t) + 2 = t^2 - 1 + 2 = t^2 + 1$$

$$(f \circ g)^\Delta(t) = \sigma(t) + t = 2t + t = 3t.$$

Hence,

$$(f \circ g)^\Delta(2) = 6. \quad (1.7)$$

Now, applying theorem 1.1.3, we conclude that there is $c \in [2, \sigma(2)] = [2, 4]$, such that

$$(f \circ g)^\Delta(2) = f'(g(c))g^\Delta(2) = g^\Delta(2) = \sigma(2) + 2 = 4 + 2 = 6. \quad (1.8)$$

According to 1.8, we conclude that for every $c \in [2, 4]$

$$(f \circ g)^\Delta(2) = f'(g(c))g^\Delta(2). \quad (1.9)$$

Remark 1.1.2 We denote the rd-continuous fuctions by the symbol C_{rd} or $C_{rd}(\mathbb{T})$.

1.1.6 Integration on time scales

Definition 1.1.9 If the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is continuous at every right-dense point and if its left sided limit exists and is finite at every left dense-point, then f is called an rd-continuous function. For all $t \in \mathbb{T}$, if t is right-dense and if its left sided limit exists and is finite at every left-dense point, then f is regressive.

Definition 1.1.10 Let $F : \mathbb{T} \rightarrow \mathbb{R}$ is a function, F is said to be the primitive of $f : \mathbb{T} \rightarrow \mathbb{R}$, if it verifies $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}^k$.

Example 1.1.13 Let $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $t \in \mathbb{T}$, let $f(t) = 3t^3 + 5t + 2$, $t \in \mathbb{T}$. the primitive function of f is defined by $g(t) = t^3 + t^2$.

$$\begin{aligned} g^\Delta(t) &= (\sigma(t))^2 + t\sigma(t) + t^2 + \sigma(t) + t \\ &= (t+1)^2 + t(t+1) + t^2 + t + 1 + t \\ &= t^2 + 2t + 1 + t^2 + t + t^2 + 2t + 1 \\ &= 3t^2 + 5t + 2 \\ &= f(t). \end{aligned}$$

Theorem 1.1.4 Every rd-continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$, has a primitive function $F : \mathbb{T} \rightarrow \mathbb{R}$, in addition we denote

$$\int_s^r f(t)\Delta t = F(r) - F(s) \quad \text{for all } r, s \in \mathbb{T}.$$

Theorem 1.1.5 If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then we have :

$$\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu(t) f(t).$$

Theorem 1.1.6 [16] Let $a, b, c \in \mathbb{T}, \lambda \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$ then :

1. $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t.$
2. $\lambda \int_a^b f(t) \Delta t.$
3. $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t.$
4. $\int_a^c f(t) \Delta t = \int_a^b f(t) \Delta t + \int_b^c f(t) \Delta t.$
5. $\int_a^a f(t) \Delta t = 0.$
6. if $|f(t)| \leq g(t)$ on $[a, b]_{\mathbb{T}}$, then $\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b g(t) \Delta t.$

Proposition 1.1.1 We have the following cases:

1. For $\mathbb{T} = \mathbb{R}$, we denote :

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt.$$

2. For $\mathbb{T} = \mathbb{Z}$, we denote :

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b \\ - \sum_{t=b}^{a-1} f(t) & \text{if } a > b \\ 0 & \text{if } a = b \end{cases}$$

3. If the interval $[a, b]$ contains two isolated points then :

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{t \in [a, b[} \mu(t) f(t) & \text{if } a < b \\ - \sum_{t \in [b, a[} \mu(t) f(t) & \text{if } a > b \\ 0 & \text{if } a = b \end{cases}$$

Proposition 1.1.2 We define the integration by parts formula :

$$\begin{aligned}\int_a^b f^\sigma(t)g^\Delta(t) &= [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t. \\ \int_a^b f(t)g^\Delta(t) &= [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t.\end{aligned}$$

Theorem 1.1.7 [10] Suppose that, for each $\varepsilon > 0$, there exists a neighborhood U of t , independent of $\tau \in [t_0, \sigma(t)]$, such that

$$|w(\sigma(t), \tau) - w(s, \tau) - w_t^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad s \in U. \quad (1.10)$$

where $w : \mathbb{T} \times \mathbb{T}^k \rightarrow \mathbb{R}_+$, is continuous at (t, t) , $t \in \mathbb{T}^k$ with $t > t_0$, and $w_t^\Delta(t, \cdot)$ are rd-continuous on $[t_0, \sigma(t)]$. Then

$$g(t) := \int_{t_0}^t w(t, \tau) \Delta \tau$$

implies

$$g^\Delta(t) = \int_{t_0}^t w^\Delta(t, \tau) \Delta \tau + w(\sigma(t), t), \quad t \in \mathbb{T}^k.$$

1.1.7 Exponential function on time scales

Definition 1.1.11 We say that the function $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^k, \quad (1.11)$$

holds.

Definition 1.1.12 We denote the set of all regressive and rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ by $\mathfrak{R}(\mathbb{T})$ or $\mathfrak{R}$

Definition 1.1.13 We define the group of all positive regressive functions by

$$\mathfrak{R}_+ = \left\{ p \in \mathfrak{R}_+ : 1 + \mu(t)p(t) > 0, \quad \text{for all } t \in \mathbb{T}^k \right\}. \quad (1.12)$$

Definition 1.1.14 If $p \in \mathfrak{R}$, then the first order linear dynamic equation, which is defined by

$$y^\Delta(t) = p(t)y(t), \quad (1.13)$$

is said to be regressive

Theorem 1.1.8 Suppose that $\mathbb{T}$ is regressive, and fix $t_0 \in \mathbb{T}$, then $e_p(., t_0)$ is a solution of the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1 \quad (1.14)$$

on $\mathbb{T}$.

Definition 1.1.15 If $p \in \mathbb{R}$ and $\mu(t) \neq 0$, then $t, s \in \mathbb{T}$. We define the exponential function by

$$e_p(t, s) = \exp \left(\int_s^t \frac{\log(1 + \mu(\tau)p(\tau))}{\mu(\tau)} \Delta\tau \right). \quad (1.15)$$

If $\mu(t) = 0$, then $e_p(t, s) = \exp \left(\int_{t_0}^t p(s) ds \right)$.

Remark 1.1.3 Explicitly, this solution is said to be the exponential function corresponding to the given time scale,

$$y(t) = \exp_p(t, t_0) \quad (1.16)$$

Proposition 1.1.3 Let $p, q \in \mathbb{R}$ and t, t_0 in $\mathbb{T}$, then

1. $e_0(t, t_0) \equiv 1$ and $e_p(t, t) \equiv 1$,
2. $e_p(\sigma(t), t_0) = (1 + \mu(t)p(t)) e_p(t, t_0)$,
3. $e_p(t, t_0)e_p(t_0, s) = e_p(t, s)$,
4. $e_p(t, t_0) = \frac{1}{e_p(t_0, t)}$,
5. If $p \in \mathbb{R}_+$, $q \in \mathbb{R}_+$ and $p \leq q$, then $e_p(t, t_0) \leq e_q(t, t_0)$,
6. If $p \geq 0$, then $e_p(., t_0)$ is an increasing function and $e_p(t, t_0) \geq 1$,
7. If $p \in \mathbb{R}_+$, then $e_p(t, t_0) > 0$,
8. The function $e_p(., s)$ is said to be Delta-differentiable on t and $(e_p(., s))^\Delta(t) = p(t)e_p(t, s)$,
9. $e_p(t, t_0)e_q(t, t_0) = e_{p \oplus q}(t, t_0)$, where $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$, for all $t \in \mathbb{T}^k$.

Some examples are presented in the two tables bellow:

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$h\mathbb{Z}$	$\frac{1}{n}\mathbb{Z}$	$2^{\mathbb{N}_0}$
$e^\alpha(t, t_0)$	$e^{\alpha(t-t_0)}$	$(1+\alpha)^{t-t_0}$	$(1+h\alpha)^{\frac{t-t_0}{h}}$	$\left(1+\frac{\alpha}{n}\right)^{n(t-t_0)}$	$\prod_{s \in [t_0, t]} (1+\alpha s)$ if $t > t_0$

Table1.2 Some examples on exponential function where $p(t) = \alpha$

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$h\mathbb{Z}$	$\mathbb{N}^2$	$\frac{1}{n}\mathbb{Z}$
$e_\alpha(t, t_0)$	e^t	2^t	$(1+h)^{\frac{t}{h}}$	$2^{\sqrt{t}} (\sqrt{t})!$	$\left[\left(1+\frac{1}{n}\right)^n\right]^t$

Table1.3 Some examples on exponential function where $t_0 = 0$ and $p(t) = 1$

1.2 Some usefull lemmas and definitions

We present now some important lemmas wich will be usefull in the third section of the second chapter. In all this work we consider the interval $I = [t_0, T] \cap \mathbb{T}^k$, where $t_0, T \in \mathbb{T}^k$, $T > t_0$.

Lemma 1.2.1 Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}$. If f is differentiable at t , then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t), \quad (1.17)$$

Lemma 1.2.2 Suppose that $v : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function and $\tilde{\mathbb{T}} = v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \rightarrow \mathbb{R}$ is an rd-continuous function and is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$.

$$\int_a^b f(t)v^\Delta(t)\Delta t = \int_{v(a)}^{v(b)} (f \circ v^{-1})(s)\tilde{\Delta}s. \quad (1.18)$$

Lemma 1.2.3 Suppose that $\alpha : I \rightarrow I$ is a continuous and strictly increasing function where $\alpha(t) \leq t$, and α^Δ is rd-continuous. Assume that $f : I \rightarrow \mathbb{R}$ is an rd-continuous function, then

$$g(t) = \int_{\alpha(t_0)}^{\alpha(t)} f(s)\Delta s, \quad t \in I, \quad (1.19)$$

implies

$$g^\Delta(t) = f(\alpha(t))\alpha^\Delta(t), \quad t \in I. \quad (1.20)$$

Proof. Accordding to [1.19](#), we obtain, for any $t \in I$,

$$g(t) = \int_{\alpha(t_0)}^{\alpha(t)} f(s)\Delta s = \int_{\alpha(t_0)}^{\alpha(t)} f(\alpha(\alpha^{-1}(s)))\Delta s, \quad (1.21)$$

By lemma [1.22](#), we obtain

$$g(t) = \int_{t_0}^t f(\alpha(s))\alpha^\Delta(s)\Delta s, \quad t \in I, \quad (1.22)$$

then we get

$$g^\Delta(t) = f(\alpha(t))\alpha^\Delta(t). \quad (1.23)$$

□

Lemma 1.2.4 We define y and f as an rd-continuous function and $p \in \mathbb{R}_+$. Then

$$y^\Delta(t) \leq p(t)y(t) + f(t) \quad \text{for all } t \in \mathbb{T}, \quad (1.24)$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}. \quad (1.25)$$

Definition 1.2.1 A function $x : I \rightarrow I$ is said to belong to the class γ if

1. x is continuous and strictly increasing, and
2. $x(t) \leq t$ and x^Δ is rd-continuous.

CHAPTER

2

INTEGRAL INEQUALITIES: CLASSICAL FORMS AND GENERALIZATIONS

This chapter is presenting a selection of classical and generalised integral inequalities, we started by general historical overview for some basic inequalities that contributed essentially in the developpement of analyses and differential equations. This part includes the wellknown inequalities of Grönwall, Bellman, Pachpatte, and the VolterraFredholm type. They're powerful tools, allow us to estimate solutions for all sorts of integral and differential systems.

In the second part of chapter, we briefly presented some generalisations for these classical integral inequalities, these generalisations provides us to unify discrete and continuous analyses. Each generalised inequality is referanced and introduced in a context, with specified mathematical conditions. The proofs are omitted, for more details, the reader can see [1], [2], [13], [6], [15].

Lastly, the final part of the chapter presents and focuses on retarded Volterra-Fredholm type integral inequalities on time scales. We will see new classes of nonlinear inequalities that involve a delay, presented carrefully and equipped with upper bounds, corollaries and detailed proofs.

2.1 Classical Integral Inequalities on $\mathbb{R}$

- In 1919, Thomas H. Grönwall introduced a fundamental inequality of linear integral inequalities.

Theorem 2.1.1 [G](Grönwall, 1919)

Let u be a continuous function defined on the interval $J = [\alpha, \alpha + h]$, and

$$0 \leq u(t) \leq \int_{\alpha}^t [bu(s) + a] ds, \quad t \in J,$$

Where a and b are nonnegative constants. Then

$$0 \leq u(t) \leq ahe^{bh}, \quad t \in J.$$

- In 1943, Richard Bellman established a similar inequality in the case where u is a function depending on the variable b , which is known as the Gronwall-Bellman type inequality. The result is recalled in the following theorem

Theorem 2.1.2 [B](Bellman, 1943)

Let u and f be continuous and nonnegative functions defined on $J = [\alpha, \beta]$, and let c be a nonnegative constant. Then the inequality

$$u(t) \leq c + \int_{\alpha}^t f(s)u(s) ds, \quad t \in J,$$

implies that

$$u(t) \leq c \exp \left(\int_{\alpha}^t f(s) ds \right), \quad t \in J.$$

- In 2002, B.G. Pachpatte developed a more general inequality involving nested integrals.

Theorem 2.1.3 [P](Pachpatte, 2002)

Let $u(t), a(t), f(t), \in C(\mathbb{R}_+, \mathbb{R}_+)$, $D_1 k(t, s), k(t, s) \in C(G_1, \mathbb{R}_+)$, and c be a nonnegative constant, such that

$$G_1 = \left\{ (t, s) \in \mathbb{R}_+^2 : 0 \leq s \leq t \leq \infty \right\},$$

if

$$u(t) \leq a(t) + \int_{t_0}^t f(s) \left[u(s) + \int_0^s k(s, \tau) u(\tau) d\tau \right] ds,$$

for $t \in \mathbb{R}_+$, then

$$u(t) \leq a(t) + e(t) \left[1 + \int_0^t f(s) \exp \left(\int_0^s [f(\tau) + A(\tau)] d\tau \right) ds, \right.$$

for $t \in \mathbb{R}_+$, where

$$e(t) = \int_0^t f(s) \left[a(s) + \int_0^s k(s, \sigma) a(\tau) d\tau \right] ds,$$

for $t \in \mathbb{R}_+$, and

$$A(t) = k(t, t) + \int_0^t D_1 k(t, \tau) d\tau.$$

- Volterra and Fredholm introduced an inequality structure combining variable and fixed integral limits. These are foundational in integro-differential equation theory.

Theorem 2.1.4 [15](Volterra-Fredholm)

We define the domain $D = \{(t, s) \in I^2 : \alpha \leq s \leq t \leq \beta\}$.

Let $u(t), a(t), c(t) \in C([\alpha, \beta], \mathbb{R}_+)$ be continuous nonnegative functions on $[\alpha, \beta]$. Suppose:

$\frac{\partial}{\partial t} h(t, s) \in C(D, \mathbb{R}_+)$ is continuous and nonnegative on D .

If $u(t)$ satisfies the Volterra-Fredholm inequality:

$$u(t) \leq a(t) + \int_\alpha^t h(t, s) u(s) ds + \int_\alpha^\beta c(s) u(s) ds,$$

and the condition:

$$\int_\alpha^\beta c(s) \exp \left(\int_\alpha^s B(\sigma) d\sigma \right) ds < 1,$$

then the following bound holds:

$$u(t) \leq a(t) + M_3 \exp \left(\int_\alpha^t B(\sigma) d\sigma \right) + \int_\alpha^t A(s) \exp \left(\int_s^t B(\sigma) d\sigma \right) ds,$$

for $t \in [\alpha, \beta]$ where:

$$A(t) = h(t, t) a(t) + \int_\alpha^t \frac{\partial}{\partial t} h(t, s) a(s) ds,$$

$$B(t) = h(t, t) + \int_\alpha^t \frac{\partial}{\partial t} h(t, s) ds,$$

$$M_3 = \frac{1}{1 - p_3} \int_\alpha^\beta c(s) \left[a(s) + \int_\alpha^s A(\tau) \exp \left(\int_\tau^s B(\tau) d\tau \right) d\tau \right] ds.$$

2.2 Generalizations on Time Scales

In order to develop classical inequalities further, several authors have formulated versions that are valid on arbitrary time scales. Among them, we mention:

- Bohner and Paterson extended the classical Grönwall inequality to arbitrary time scales using the exponential function on $\mathbb{T}$.

Theorem 2.2.1 [11](Bohner, Paterson 2001)

Let $u \in C_{rd}$, $b \in \mathbb{R}_+$, $b > 0$, $a \in \mathbb{R}$. Then

$$u(t) \leq a + \int_{t_0}^t b(s)u(s) \Delta s, \quad \text{for all}$$

$t \in \mathbb{T}$,

implies

$$u(t) \leq a \cdot e_b(t, t_0), \quad \text{for all } t \in \mathbb{T}.$$

where $e_b(t, t_0)$ is the exponential function on the time scale $\mathbb{T}$.

- A discrete version of Pachpatte's inequality was adapted to $\mathbb{N}_0$ to handle difference inequalities with nested sums.

Theorem 2.2.2 [13](Pachpatte on, 2002, $\mathbb{T} = \mathbb{N}_0$)

Let $u(n), f(n), a(n)$ be nonnegative function defined on $\mathbb{N}_0$, $k(n, s), \Delta_1 k(n, s), 0 \leq s \leq n \leq \infty$, and $n, s \in \mathbb{N}_0$ be nonnegative functions and c be a nonnegative constant. If

$$u(n) \leq a(n) + \sum_{s=0}^{n-1} f(s) \left[u(s) + \sum_{\tau=0}^{s-1} k(s, \tau)u(\tau) \right],$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leq a(n) + E(n) \left[1 + \sum_{s=0}^{n-1} f(s) \prod_{\tau=0}^{s-1} [1 + f(\tau) + B(\tau)] \right],$$

for $n \in \mathbb{N}_0$, where

$$E(n) = \sum_{s=0}^{n-1} f(s) \left[a(s) + \sum_{\tau=0}^{s-1} k(s, \tau)a(\tau) \right],$$

for $n \in \mathbb{N}_0$ and

$$B(n) = k(n+1, n) \sum_{\tau=0}^{n-1} \Delta_1 k(n, \tau), \quad \text{for } n \in \mathbb{N}.$$

- Gu and Meng proposed a volterra type inequality on time scales.

Theorem 2.2.3 [6](Gu-Meng, 2014)

Let $u, a, f_1, f_2, g : [t_0, T]_{\mathbb{T}} \rightarrow \mathbb{R}_+$ be rd-continuous. If

$$u(t) \leq a(t) + \int_{t_0}^t \left[f_1(s)u(s) + f_2(s) \int_{t_0}^s g(\tau)u(\tau) \Delta\tau \right] \Delta s,$$

then

$$u(t) \leq a(t) + \int_{t_0}^t a(s) \left[f_1(s) + f_2(s) \int_{t_0}^s g(\tau) \Delta\tau \right] e_{(p \oplus q)} \Delta s.$$

where

$$p(s) = f_1(s),$$

$$q(s) = f_2(s) \int_{t_0}^s g(\tau) \Delta\tau.$$

2.3 Retarded Volterra-Fredholm Type Integral Inequalities on Time Scales

This section presents recent results concerning retarded integral inequalities of Volterra-Fredholm type on arbitrary time scales.

These inequalities allow us to obtain upper bounds for unknown functions involving delay arguments. Such forms generalize classical inequalities and provide tools for studying dynamic equations with memory effects on time scales.

The theorems and results presented in this section are cited from [10], we detailed and clarified them to gain a deeper understanding of this type of Volterra-Fredholm inequalities.

Theorem 2.3.1 Let's say that $\alpha \in \gamma$ and $u, a, b, f_1, f_2, g : I \rightarrow \mathbb{R}_+$ are rd-continuous functions, a is nondecreasing, $\lambda \geq 0$ is a constant, $\alpha^\Delta(t) \geq 0$, $b^\Delta(t) \geq 0$, $\mu(t)A(t) < 1$. Suppose

that u satisfies

$$\begin{aligned} u(t) \leq & a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\ & + \lambda b(t) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s, \quad t \in I. \end{aligned} \quad (2.1)$$

If

$$\beta = e_{B \oplus C}(T, t_0) < 1 + \frac{1}{\lambda}, \quad (2.2)$$

then

$$u(t) \leq \frac{a(T)}{\lambda + 1 - \beta\lambda} e_{B \oplus C}(t, t_0), \quad t \in I, \quad (2.3)$$

knowing that $\frac{1}{0} = +\infty$, then:

$$A(t) = b^\Delta(t) \int_{\alpha(t_0)}^{\alpha(\sigma(t))} \left[f_1(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)\Delta\tau \right] \Delta s, \quad (2.4)$$

$$B(t) = \frac{A(t)}{1 - \mu(t)A(t)}, \quad (2.5)$$

$$C(t) = b(t) \left[f_1(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)\Delta\tau \right] \alpha^\Delta(t). \quad (2.6)$$

Proof. Denote

$$\begin{aligned} z(t) = & a(T) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\ & + \lambda b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s, \quad t \in I. \end{aligned} \quad (2.7)$$

Since $u, a, b, f_1, f_2, g \geq 0$, a is nondecreasing, $\lambda \geq 0$, then we conclude that z is nondecreasing on I .

$$u(t) \leq z(t), \quad t \in I, \quad (2.8)$$

let's calculate $z(t_0)$

$$\begin{aligned}
 z(t_0) &= a(t) + b(t_0) \int_{\alpha(t_0)}^{\alpha(t_0)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
 &\quad + \lambda b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
 &= a(T) + b(t_0).0 + \lambda b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
 &= a(T) + \lambda b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s. \tag{2.9}
 \end{aligned}$$

Let's calculate now $z^\Delta(t)$, according to lemma 2.3 and 2.4-2.8 we get

$$\begin{aligned}
 z^\Delta(t) &= b^\Delta(t) \int_{\alpha(t_0)}^{\alpha(\sigma(t))} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
 &\quad + b(t) \left[f_1(\alpha(t))u(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)u(\tau)\Delta\tau \right] \alpha^\Delta(t) \\
 &\leq b^\Delta(t) \int_{\alpha(t_0)}^{\alpha(\sigma(t))} \left[f_1(s)z(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)z(\tau)\Delta\tau \right] \Delta s \\
 &\quad + b(t) \left[f_1(\alpha(t))z(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)z(\tau)\Delta\tau \right] \alpha^\Delta(t) \\
 &\leq z(\sigma(t))b^\Delta(t) \int_{\alpha(t_0)}^{\alpha(\sigma(t))} \left[f_1(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)\Delta\tau \right] \Delta s \\
 &\quad + z(t)b(t) \left[f_1(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)\Delta\tau \right] \alpha^\Delta(t) \\
 &= A(t)z(\sigma(t)) + C(t)z(t), \quad t \in I. \tag{2.10}
 \end{aligned}$$

It is noted that 2.5 implies

$$A(t) = \frac{B(t)}{1 + \mu(t)B(t)}, \tag{2.11}$$

let's clarify this more, we have

$$B(t) = \frac{A(t)}{1 - \mu(t)A(t)},$$

implies

$$\begin{aligned} A(t) &= B(t) (1 - \mu(t)A(t)) \\ &= B(t) - B(t)\mu(t)A(t), \end{aligned}$$

this gives us

$$A(t) + B(t)\mu(t)A(t) = B(t),$$

therefore

$$A(t) (1 + B(t)\mu(t)) = B(t),$$

this leads us to conclude that,

$$A(t) = \frac{B(t)}{1 + \mu(t)B(t)}. \quad (2.12)$$

From 2.10 we have

$$z^\Delta(t) \leq A(t)z(\sigma(t)) + C(t)z(t), \quad (2.13)$$

substituting 2.12 in 2.13 we get:

$$z^\Delta(t) \leq \frac{B(t)}{1 + \mu(t)B(t)}z(\sigma(t)) + C(t)z(t), \quad (2.14)$$

we know that

$$z(\sigma(t)) = z(t) + \mu(t)z^\Delta(t), \quad (2.15)$$

let's substitute 2.15 into 2.14,

$$\begin{aligned} z^\Delta(t) &\leq \frac{B(t)}{1 + \mu(t)B(t)} \left[z(t) + \mu(t)z^\Delta(t) \right] + C(t)z(t) \\ &\leq \frac{B(t)z(t)}{1 + \mu(t)B(t)} + \frac{B(t)\mu(t)z^\Delta(t)}{1 + \mu(t)B(t)} + C(t)z(t), \end{aligned}$$

implies

$$z^\Delta(t) - \frac{B(t)\mu(t)z^\Delta(t)}{1 + \mu(t)B(t)} \leq \frac{B(t)z(t)}{1 + \mu(t)B(t)} + C(t)z(t),$$

this provides us

$$z^\Delta(t) \left(1 - \frac{B(t)\mu(t)}{1 + \mu(t)B(t)} \right) \leq z(t) \left(\frac{B(t)}{1 + \mu(t)B(t)} + C(t) \right),$$

by simplifying the previous inequality we get

$$z^\Delta(t) \left(\frac{1 + \mu(t)B(t) - \mu(t)B(t)}{1 + \mu(t)B(t)} \right) \leq z(t) \left(\frac{B(t)}{1 + \mu(t)B(t)} + C(t) \right),$$

then we obtain

$$\frac{1}{1 + \mu(t)B(t)} z^\Delta(t) \leq \left(\frac{B(t)}{1 + \mu(t)B(t)} + C(t) \right) z(t),$$

this provides

$$\begin{aligned} z^\Delta(t) &\leq [B(t) + (1 + \mu(t)B(t)C(t))] z(t) \\ &= (B \oplus C)(t) z(t). \end{aligned} \quad (2.16)$$

We consider that z is rd-continuous and $B \oplus C \in \mathfrak{R}_+$, from lemma 2.2.4 and 2.16 we get

$$z(t) \leq z(t_0)e_{B \oplus C}(t, t_0), \quad t \in I. \quad (2.17)$$

From 2.7 and 2.9 we obtain

$$\begin{aligned} (z(t_0) - a(T)) \frac{\lambda + 1}{\lambda} + a(T) &= \frac{\lambda + 1}{\lambda} \left[\lambda b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \right] \\ &\quad + a(T) \\ &= (\lambda + 1) \left[b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \right] \\ &\quad + a(T) \\ &= \lambda \left[b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \right] + a(T) \\ &\quad + \left[b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s + a(T) \right] \\ &= z(T). \end{aligned} \quad (2.18)$$

Then we conclude that

$$(z(t_0) - a(T)) \frac{\lambda + 1}{\lambda} + a(T) = z(T). \quad (2.19)$$

Note that 2.18 equates

$$\frac{\lambda + 1}{\lambda} z(t_0) - \frac{1}{\lambda} a(T) = z(T). \quad (2.20)$$

From (2.1), (2.17) and (2.20) we have

$$\frac{\lambda+1}{\lambda}z(t_0) - \frac{1}{\lambda}a(T) = z(T) \leq z(t_0)e_{B \oplus C}(T, t_0) = z(t_0)\beta. \quad (2.21)$$

By using (2.21) we find

$$\frac{\lambda+1}{\lambda}z(t_0) - \frac{1}{\lambda}a(T) \leq z(t_0)\beta,$$

this implies

$$\frac{\lambda+1}{\lambda}z(t_0) - z(t_0)\beta \leq \frac{1}{\lambda}a(T),$$

implies

$$\left(\frac{\lambda+1}{\lambda} - \beta\right)z(t_0) \leq \frac{1}{\lambda}a(T),$$

then we obtain

$$\begin{aligned} z(t_0) &\leq \frac{\frac{1}{\lambda}a(T)}{\frac{\lambda+1}{\lambda} - \beta} \\ &= \frac{\frac{1}{\lambda}a(T)}{\frac{\lambda+1-\beta\lambda}{\lambda}} \end{aligned} \quad (2.22)$$

this could provide us

$$z(t_0) \leq \frac{a(T)}{\lambda+1-\beta\lambda}, \quad (2.23)$$

replacing (2.23) into (2.17) we obtain

$$z(t) \leq \frac{a(T)}{\lambda+1-\beta\lambda}e_{B \oplus C}(t, t_0), \quad t \in I, \quad (2.24)$$

see that $u(t) \leq z(t)$, so we obtain

$$u(t) \leq \frac{a(T)}{\lambda+1-\beta\lambda}e_{B \oplus C}(t, t_0), \quad t \in I. \quad (2.25)$$

This completes the proof. $\square$

Let's assume that $\lambda = \frac{1}{b(T)}$ in theorem 2.3.1, then we obtain this following corollary:

Corollary 2.3.1 Suppose that $\alpha, u, a, b, f_1, f_2, g, A, B, C$ have the same properties as in theo-

rem 2.3.1 and $b(t) \neq 0$. hypothesize that u satisfies

$$\begin{aligned} u(t) &\leq a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\ &\quad + \frac{1}{b(T)} b(T) \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\ &= a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)\Delta\tau \right] \Delta s. \end{aligned}$$

If

$$\begin{aligned} \beta &= e_{(B \oplus C)}(T, t_0) < 1 + \frac{1}{\frac{1}{b(T)}} \\ &= e_{B \oplus C}(T, t_0) < 1 + b(T), \end{aligned}$$

Then

$$\begin{aligned} u(t) &\leq \frac{a(T)}{\frac{1}{b(T)} + 1 - \frac{\beta}{b(T)}} e_{B \oplus C}(t, t_0) \\ &= \frac{a(T)}{\frac{1+b(T)-\beta}{b(T)}} e_{B \oplus C}(t, t_0) \\ &= \frac{a(T)b(T)}{1+b(T)-\beta} e_{B \oplus C}(t, t_0), \quad t \in I. \end{aligned}$$

Remark 2.3.1 If we take $a(t) \equiv u_0$, $b(t) \equiv 1$, $\alpha(t) \equiv t$, and $\lambda = 1$, then the theorem 2.3.1 is reducing to [[6], Theorem.3]. If we take $a(t) \equiv u_0$, $b(t) \equiv 1$, $\alpha(t) = t$, $f_1(t) = f_2(t)$ and $\lambda = 1$, then the theorem 2.3.1 is reducing to [[11], Theorem.2.3]

Theorem 2.3.2 Let's assume that $\alpha, u, a, b, g : I \rightarrow \mathbb{R}_+$ are rd-continuous functions, $\lambda \geq 0$, $w, v : \mathbb{T} \times \mathbb{T}^k \rightarrow \mathbb{R}_+$ are continuous functions at (t, t) , $t \in \mathbb{T}^k$ with $t > t_0$, $w_t^\Delta(t, \tau)$, $v_t^\Delta(t, \tau)$ are rd-continuous functions, and 1.10 holds. Suppose that u satisfies

$$\begin{aligned} u(t) &\leq a(t) + b(t) \int_{t_0}^t \left[v(t, s)u(s) + w(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s \\ &\quad + \lambda b(T) \int_{t_0}^T \left[v(T, s)u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s, \quad t \in I. \end{aligned} \quad (2.26)$$

If

$$\beta = e_{\tilde{B} \oplus \tilde{C}}(T, t_0) < 1 + \frac{1}{\lambda}, \quad (2.27)$$

then

$$u(t) \leq \frac{a(T)}{\lambda + 1 - \beta\lambda} e_{\tilde{B} \oplus \tilde{C}}(t, t_0), \quad t \in I, \quad (2.28)$$

such that

$$\tilde{A}(t) = b^\Delta(t) \int_{t_0}^{\sigma(t)} \left[v(\sigma(t), s) + w(\sigma(t), s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau) \Delta\tau \right] \Delta s, \quad (2.29)$$

$$\tilde{B}(t) = \frac{\tilde{A}(t)}{1 - \mu(t)\tilde{A}(t)}, \quad (2.30)$$

$$\begin{aligned} \tilde{C}(t) = b(t) & \left[v(\sigma(t), t) + w(\sigma(t), t) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau) \Delta\tau \right. \\ & \left. + \int_{t_0}^t \left[v_t^\Delta(t, s) + w_t^\Delta(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau) \Delta\tau \right] \Delta s \right], \quad t \in I. \end{aligned}$$

Proof. Denote

$$\begin{aligned} z(t) = a(T) & + b(t) \int_{t_0}^t \left[v(t, s)u(s) + w(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau) \Delta\tau \right] \Delta s \\ & + \lambda b(T) \int_{t_0}^T \left[v(T, s)u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau) \Delta\tau \right] \Delta s, \quad t \in I. \end{aligned} \quad (2.31)$$

Since $u, a, g : I \rightarrow \mathbb{R}_+$, $\lambda \geq 0$, $w, v : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}_+$, then z is nondecreasing on I .

$$u(t) \leq z(t), \quad t \in I, \quad (2.32)$$

and

$$\begin{aligned}
z(t_0) &= a(T) + b(t_0) \int_{t_0}^{t_0} \left[v(t_0, s)u(s) + w(t_0, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
&\quad + \lambda b(T) \int_{t_0}^T \left[v(T, s)u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
&= a(T) + b(t_0).0 + \lambda b(T) \int_{t_0}^T \left[v(T, s)u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
&= a(T) + \lambda b(T) \int_{t_0}^T \left[v(T, s)u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s. \tag{2.33}
\end{aligned}$$

Let's suppose that

$$N(t, s) = v(T, s)u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau, \tag{2.34}$$

Knowing that

$$z^\Delta(t) = b^\Delta(t) \int_{t_0}^{\sigma(t)} \left[v(\sigma(t), s)u(s) + w(\sigma(t), s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s + b(t)N^\Delta(t, s), \tag{2.35}$$

applying lemma [1.2.3](#), we find, for a function

$$g(t) = \int_{t_0}^t N(t, s)\Delta s, \tag{2.36}$$

we have

$$g^\Delta(t) = \int_{t_0}^t N_t^\Delta(t, s)\Delta s + N(\sigma(t), t), \tag{2.37}$$

hence

$$N_t^\Delta(t, s) = \int_{t_0}^t \left[v_t^\Delta(t, s)u(s) + w_t^\Delta(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s, \tag{2.38}$$

and

$$N(\sigma(t), t) = v(\sigma(t), t)u(t) + w(\sigma(t), t) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau, \tag{2.39}$$

substituting [2.38](#) and [2.39](#) into [2.35](#):

$$\begin{aligned}
z^\Delta(t) &= b^\Delta(t) \int_{t_0}^{\sigma(t)} \left[v(\sigma(t), s)u(s) + w(\sigma(t), s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)u(\tau)\Delta\tau \right] \Delta s \\
&\quad + b(t) \left[v(\sigma(t), t)u(t) + w(\sigma(t), t) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)u(\tau)\Delta\tau \right. \\
&\quad \left. + \int_{t_0}^t \left[v_t^\Delta(t, s)u(s) + w_t^\Delta(t, s) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)u(\tau)\Delta\tau \right] \Delta s \right] \\
&\leq b^\Delta(t) \int_{\alpha(t_0)}^{\sigma(t)} \left[v(\sigma(t), s)z(s) + w(\sigma(t), s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)z(\tau)\Delta\tau \right] \Delta s \\
&\quad + b(t) \left[v(\sigma(t), t)z(t) + w(\sigma(t), t) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)z(\tau)\Delta\tau \right. \\
&\quad \left. + \int_{t_0}^t \left[v_t^\Delta(t, s)z(s) + w_t^\Delta(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)z(\tau)\Delta\tau \right] \Delta s \right] \\
&\leq z(\sigma(t))b^\Delta(t) \int_{t_0}^{\sigma(t)} \left[v(\sigma(t), s) + w(\sigma(t), s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau)\Delta\tau \right] \Delta s \\
&\quad + z(t)b(t) \left[v(\sigma(t), t) + w(\sigma(t), t) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)\Delta\tau \right. \\
&\quad \left. + \int_{t_0}^t \left[v_t^\Delta(t, s) + w_t^\Delta(t, s) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)\Delta\tau \right] \Delta s \right] \\
&= \tilde{A}(t)z(\sigma(t)) + \tilde{C}(t)z(t).
\end{aligned} \tag{2.40}$$

Note that from [2.30](#) we have

$$\tilde{B}(t) = \frac{\tilde{A}(t)}{1 - \mu(t)\tilde{A}(t)}$$

then

$$\begin{aligned}
\tilde{A}(t) &= \tilde{B}(t) (1 - \mu(t)\tilde{A}(t)) \\
&= \tilde{B}(t) - \tilde{B}(t)\mu(t)\tilde{A}(t),
\end{aligned}$$

this provides

$$\tilde{A}(t) + \tilde{B}(t)\mu(t)\tilde{A}(t) = \tilde{B}(t),$$

then we obtain

$$\tilde{A}(t) (1 + \tilde{B}(t)\mu(t)) = \tilde{B}(t),$$

we get finally

$$\tilde{A}(t) = \frac{\tilde{B}(t)}{1 + \tilde{B}(t)\mu(t)}. \tag{2.41}$$

and from [2.40](#) we have

$$\begin{aligned} z^\Delta(t) &\leq \frac{\tilde{B}(t)}{1 + \tilde{B}(t)\mu(t)} z(\sigma(t)) + \tilde{C}(t)z(t) \\ &= \frac{\tilde{B}(t)}{1 + \tilde{B}\mu(t)} \left[z(t) + \mu(t)z^\Delta(t) \right] + \tilde{C}(t)z(t) \\ &= \frac{\tilde{B}(t)z(t)}{1 + \tilde{B}(t)\mu(t)} + \frac{\tilde{B}(t)\mu(t)z^\Delta(t)}{1 + \tilde{B}(t)\mu(t)} + \tilde{C}(t)z(t). \end{aligned}$$

Implies

$$z^\Delta(t) - \frac{\tilde{B}(t)\mu(t)z^\Delta(t)}{1 + \tilde{B}(t)\mu(t)} \leq \frac{\tilde{B}(t)z(t)}{1 + \tilde{B}(t)\mu(t)} + \tilde{C}(t)z(t).$$

This provides us

$$z^\Delta(t) \left(1 - \frac{\tilde{B}(t)\mu(t)}{1 + \tilde{B}(t)\mu(t)} \right) \leq z(t) \left(\frac{\tilde{B}(t)}{1 + \tilde{B}(t)\mu(t)} + \tilde{C}(t) \right), \quad (2.42)$$

implies

$$z^\Delta(t) \left(\frac{1 + \tilde{B}(t)\mu(t) - \tilde{B}(t)\mu(t)}{1 + \tilde{B}(t)\mu(t)} \right) \leq z(t) \left(\frac{\tilde{B}(t) + \tilde{C}(t)(\tilde{B}(t)\mu(t) + 1)}{\tilde{B}(t)\mu(t) + 1} \right), \quad (2.43)$$

then we obtain

$$z^\Delta(t) \leq z(t) (\tilde{B}(t) + \tilde{C}(t)(\tilde{B}(t)\mu(t) + 1)). \quad (2.44)$$

Note that z is rd-continuous and $B \oplus C \in \mathfrak{R}_+$
we have

$$z(t) \leq z(t_0)e_{\tilde{B} \oplus \tilde{C}}(t, t_0). \quad (2.45)$$

From [2.31](#) and [2.33](#) we get

$$(z(t_0) - a(T)) \frac{\lambda + 1}{\lambda} + a(T) = z(T) \quad (2.46)$$

, i.e

$$\frac{\lambda + 1}{\lambda} z(t_0) - \frac{1}{\lambda} a(T) = z(T), \quad (2.47)$$

according to (2.27), (2.45) and (2.46) we have

$$\frac{\lambda + 1}{\lambda} z(t_0) - \frac{1}{\lambda} a(T) = z(T) \leq z(t_0) e_{B \oplus C}(T, t_0) = z(t_0) \beta. \quad (2.48)$$

In view of (2.27) and (2.48), we get

$$z(t) \leq \frac{a(T)}{\lambda + 1 - \beta \lambda} e_{B \oplus C}(t, t_0), \quad t \in I, \quad (2.49)$$

note that $u(t) \leq z(t)$, then

$$u(t) \leq \frac{a(T)}{\lambda + 1 - \beta \lambda} e_{\tilde{B} \oplus \tilde{C}}(t, t_0), \quad t \in I. \quad (2.50)$$

We got the desired inequality. $\square$

If we put $\lambda = \frac{1}{b(T)}$, then we can define the following corollary.

Corollary 2.3.2

$$\begin{aligned} u(t) &\leq a(t) + b(t) \int_{t_0}^t \left[v(t, s) u(s) + w(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau) u(\tau) \Delta \tau \right] \Delta s \\ &\quad + \frac{1}{b(T)} b(T) \int_{t_0}^T \left[v(T, s) u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau) u(\tau) \Delta \tau \right] \Delta s \\ &\leq a(t) + b(t) \int_{t_0}^t \left[v(t, s) u(s) + w(t, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau) u(\tau) \Delta \tau \right] \Delta s \\ &\quad + \int_{t_0}^T \left[v(T, s) u(s) + w(T, s) \int_{\alpha(t_0)}^{\alpha(s)} g(\tau) u(\tau) \Delta \tau \right] \Delta s, \quad t \in I. \end{aligned}$$

If

$$\beta = e_{\tilde{B} \oplus \tilde{C}}(T, t_0) < 1 + b(T),$$

then

$$\begin{aligned}
u(t) &\leq \frac{a(t)}{\frac{1}{b(T)} + 1 - \frac{\beta}{b(T)}} e_{\tilde{B} \oplus \tilde{C}}(t, t_0) \\
&\leq \frac{a(t)}{\frac{1+b(T)-\beta}{b(T)}} \\
&\leq \frac{a(T)b(T)}{1+b(T)-\beta} e_{\tilde{B} \oplus \tilde{C}}(t, t_0), \quad t \in I.
\end{aligned} \tag{2.51}$$

Theorem 2.3.3 Suppose that u, a, f, λ have the same properties as in theorem 2.3.1 We define $g(t, s)$ as in lemma 2.1.1 such that $g_t^\Delta(t, s) \geq 0$ and 2.1.1 holds. hypothesize that u satisfies !

$$\begin{aligned}
u(t) &\leq a(t) + \int_{t_0}^t f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s \\
&\quad + \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s, \quad t \in I.
\end{aligned} \tag{2.52}$$

If

$$\beta = \int_{t_0}^T f(s) e_A(s, t_0) \Delta s < \frac{1}{\lambda}, \tag{2.53}$$

then

$$u(t) \leq \frac{a(T)}{1 - \beta\lambda} \left[1 + \int_{t_0}^t f(s) e_A(s, t_0) \Delta s \right], t \in I. \tag{2.54}$$

Where

$$A(t) = f(t) + g(\sigma(t), t) + \int_{t_0}^t g_t^\Delta(t, \tau) \Delta \tau. \tag{2.55}$$

proof. Denote

$$\begin{aligned}
z(t) &= a(T) + \int_{t_0}^t f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s \\
&\quad + \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s, \quad t \in I.
\end{aligned} \tag{2.56}$$

Since $u, a, f, g : I \rightarrow \mathbb{R}_+$, are rd-continuous functions, and a is nondecreasing, then z is non-

decreasing on I ,

$$u(t) \leq z(t), \quad t \in I, \quad (2.57)$$

and

$$\begin{aligned} z(t_0) &= a(T) + \int_{t_0}^{t_0} f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s \\ &\quad + \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^t g(s, \tau) u(\tau) \Delta \tau \right] \Delta s \\ &= a(T) + \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^t g(s, \tau) u(\tau) \Delta \tau \right] \Delta s. \end{aligned} \quad (2.58)$$

From lemma 2.1, 2.56 and 2.57 we obtain

$$\begin{aligned} z^\Delta(t) &= f(t)u(t) + f(t) \int_{t_0}^t g(t, \tau) u(\tau) \Delta \tau \\ &= f(t) \left[u(t) + \int_{t_0}^t g(t, \tau) u(\tau) \Delta \tau \right] \\ &\leq f(t) \left[z(t) + \int_{t_0}^t g(t, \tau) z(\tau) \Delta \tau \right], \quad t \in I. \end{aligned} \quad (2.59)$$

Suppose that

$$V(t) = z(t) + \int_{t_0}^t g(t, \tau) z(\tau) \Delta \tau, \quad t \in I. \quad (2.60)$$

Then, clearly we'll obtain,

$$V(t_0) = z(t_0) \quad (2.61)$$

$$z(t_0) \leq V(t) \quad (2.62)$$

$$z^\Delta(t) \leq f(t)V(t) \quad (2.63)$$

From lemma 2.55, 2.60 we get

$$\begin{aligned}
 V^\Delta(t) &= z^\Delta(t) + g(\sigma(t), t)z(t) + \int_{t_0}^t g_t^\Delta(t, \tau)z(\tau)\Delta\tau \\
 &\leq f(t)V(t) + g(\sigma(t), t)V(t) + \int_{t_0}^t g_t^\Delta(t, \tau)V(\tau)\Delta\tau \\
 &\leq V(t) \left[f(t) + g(\sigma(t), t) + \int_{t_0}^t g_t^\Delta(t, \tau)\Delta\tau \right] \\
 &= A(t)V(t), \quad t \in I.
 \end{aligned}$$

Note that $A \in \mathbb{R}_+$. Hence, from lemma, 2.61 and the above inequality we have

$$\begin{aligned}
 V(t) &\leq V(t_0)e_A(t, t_0) \\
 &= z(t_0)e_A(t, t_0), \quad t \in I.
 \end{aligned} \tag{2.64}$$

By substituting 2.64 into 2.63 we get

$$z^\Delta(t) \leq f(t)z(t_0)e_A(t, t_0) \tag{2.65}$$

Now we first replace t by s in 2.65 to get the following inequality:

$$z^\Delta(s) \leq f(s)z(t_0)e_A(s, t_0) \tag{2.66}$$

Then, by integrating 2.66 from t_0 to t we get

$$\int_{t_0}^t z^\Delta(s)\Delta s \leq z(t_0) \int_{t_0}^t f(s)e_A(s, t_0)\Delta s. \tag{2.67}$$

We have

$$\int_{t_0}^t z^\Delta(s)\Delta s = z(t) - z(t_0), \tag{2.68}$$

we substitute now 2.67 into 2.68, we obtain

$$z(t) - z(t_0) \leq z(t_0) \int_{t_0}^t f(s)e_A(s, t_0)\Delta s, \tag{2.69}$$

this implies

$$z(t_0) \leq z(t_0) + z(t_0) \int_{t_0}^t f(s)e_A(s, t_0)\Delta s, \tag{2.70}$$

from 2.56 and 2.58 we have

$$(z(t_0) - a(T)) \frac{\lambda + 1}{\lambda} + a(T) = z(T). \quad (2.71)$$

Let's clarify this more, from 2.58 we have

$$z(t_0) - a(T) = \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right]. \quad (2.72)$$

Multiplying 2.72 by $\frac{\lambda + 1}{\lambda}$ and adding $a(T)$, we obtain:

$$\begin{aligned} & \frac{\lambda + 1}{\lambda} \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s + a(T) \\ &= \lambda + 1 \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s + a(T) \\ &= \lambda \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s \\ & \quad + \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s + a(T) \\ &= z(T) \end{aligned} \quad (2.73)$$

i.e.,

$$\frac{\lambda + 1}{\lambda} z(t_0) - \frac{1}{\lambda} a(T) = z(t). \quad (2.74)$$

From 2.70, 2.53, we have

$$\frac{\lambda + 1}{\lambda} z(t_0) - \frac{1}{\lambda} a(T) = z(T) \leq z(t_0) + z(t_0) \int_{t_0}^T f(s) e_A(s, t_0) \Delta s = z(t_0) + z(t_0) \beta. \quad (2.75)$$

From 2.27 we could obtain

$$z(t_0) \leq \frac{a(T)}{1 - \beta \lambda},$$

By replacing 2.3 into 2.70 we get

$$\begin{aligned} z(t) &\leq \frac{a(T)}{1 - \beta \lambda} + \frac{a(T)}{1 - \beta \lambda} \int_{t_0}^t f(s) e_A(s, t_0) \Delta s \\ &= \frac{a(T)}{1 - \beta \lambda} \left[1 + \int_{t_0}^t f(s) e_A(s, t_0) \Delta s \right]. \end{aligned} \quad (2.76)$$

Note that $u(t) \leq z(t)$, so we get the following inequality:

$$u(t) \leq \frac{a(T)}{1 - \beta\lambda} \left[1 + \int_{t_0}^t f(s) e_A(s, t_0) \Delta s \right], \quad t \in I. \quad (2.77)$$

This completes the proof. $\square$

Corollary 2.3.3 Suppose that u, a, f, g have the same properties as in theorem 2.3.3 suppose that u satisfies

$$\begin{aligned} u(t) &\leq a(t) + \int_{t_0}^t f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) u(\tau) \Delta \tau \right] \Delta s \\ &\quad + \int_{t_0}^T f(s) \left[u(s) + \int_{t_0}^s g(s, \tau) \Delta \tau \right] \Delta s, \quad t \in I. \end{aligned}$$

If

$$\beta = \int_{t_0}^t f(T) e_A(s, t_0) \Delta s < 1,$$

then

$$u(t) \leq \frac{a(T)}{1 - \beta} \left[1 + \int_{t_0}^t f(s) e_A(s, t_0) \Delta s \right], \quad t \in I.$$

Remark 2.3.2 If we put $a(t) \equiv u_0$, $g(s, t) = g(t)$ and $\lambda = 0$, then the theorem 2.3.3 reduces to [[6], Theorem.1]. If we take $\mathbb{T} = \mathbb{R}$, $a(t) \equiv u_0$, and $\lambda = 0$, then theorem 2.3.3 reduces to [[13], Theorem.2.1(a_1)], If we take $\mathbb{T} = \mathbb{Z}$, $a(t) = u_0$ and $\lambda = 0$, then theorem 2.3.3 reduces to [[13], Theorem.2.3(c_1)]

CHAPTER

3

ILLUSTRATIVE EXAMPLES AND NUMERICAL SIMULATION

This chapter will explore various examples and numerical simulation of Volterra-Fredholm inequalities on time scales.

3.1 Examples

Example 3.1.1 *Let's consider the following retarded Volterra-Fredholm type dynamic integral equation on time scales.*

$$\begin{aligned} u(t) = & a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} F \left(s, u(s), \int_{\alpha(t_0)}^s H(\tau, u(\tau)) \Delta \tau \right) \Delta s \\ & + \int_{\alpha(t_0)}^{\alpha(T)} \tilde{F} \left(s, u(s), \int_{\alpha(t_0)}^s \tilde{H}(\tau, u(\tau)) \Delta \tau \right) \Delta s, \quad t \in I. \end{aligned} \quad (3.1)$$

Where $u, a, b : I \rightarrow \mathbb{R}$ are defined as an rd-continuous functions, $|a|$ is nondecreasing, $b(T) \neq 0$, $\alpha \in \gamma$, $F, \tilde{F} : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $H, \tilde{H} : I \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

The following theorem gives an estimate for the solutions of Eq. [3.1](#).

Theorem 3.1.1 Let's assume that the functions $F, H, \tilde{F}$ and $\tilde{H}$ in [3.1](#) satisfy the following conditions:

$$|F(t, u, v)| \leq f_1(t)|u| + f_2(t)|v|, \quad t \in I, u, v \in \mathbb{R}, \quad (3.2)$$

$$|H(t, u)| \leq g(t)|u|, \quad t \in I, u \in \mathbb{R}, \quad (3.3)$$

$$|\tilde{F}(t, u, v)| \leq f_1(t)|u| + f_2(t)|v|, \quad t \in I, u, v \in \mathbb{R}, \quad (3.4)$$

$$|\tilde{H}(t, u)| \leq g(t)|u|, \quad t \in I, u \in \mathbb{R}. \quad (3.5)$$

Where $f_1, f_2, g : I \rightarrow \mathbb{R}_+$ are rd-continuous functions. If

$$\beta = e_{B \oplus C}(T, t_0) < 1 + |b(T)|,$$

then all the solutions of equation [3.1](#) satisfy

$$|u(t)| \leq \frac{|a(T)b(T)|}{1 + |b(t)| - \beta} e_{B \oplus C}(t, t_0), \quad t \in I, \quad (3.6)$$

such that

$$B(t) = \frac{A(t)}{1 - \mu(t)A(t)}, \quad (3.7)$$

$$C(t) = b(t) \left[f_1(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau) \Delta \tau \right] \alpha^\Delta(t). \quad (3.8)$$

Proof. According to [3.1](#) and [3.4](#) we have

$$\begin{aligned} |u(t)| &\leq |a(t)| + |b(t)| \int_{\alpha(t_0)}^{\alpha(t)} \left| F \left(s, u(s), \int_{\alpha(t_0)}^s H(\tau, u(\tau)) \Delta \tau \right) \right| \Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} \left| \tilde{F} \left(s, u(s), \int_{\alpha(t_0)}^s \tilde{H}(\tau, u(\tau)) \Delta \tau \right) \right| \Delta s \\ &\leq |a(t)| + |b(t)| \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)|u(s)| + f_2(s) \int_{\alpha(t_0)}^s g(\tau)|u(\tau)| \Delta \tau \right] \Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)|u(s)| + f_2(s) \int_{\alpha(t_0)}^s g(\tau)|u(\tau)| \Delta \tau \right] \Delta s, \quad t \in I. \end{aligned} \quad (3.9)$$

By using the corollary [2.3.1](#), we get the inequality [3.4](#). □

The next results will be investigated to deal with uniqueness of solution of equation [3.2](#).

Theorem 3.1.2 Assume that the functions $F, H, \tilde{F}$ and $\tilde{H}$ satisfy the conditions

$$|F(t, u_1, v_1) - F(t, u_2, v_2)| \leq f_1(t)|u_1 - u_2| + f_2(t)|v_1 - v_2|, \quad t \in I, u, v \in \mathbb{R}, \quad (3.10)$$

$$|H(t, u_1) - H(t, u_2)| \leq g(t)|u_1 - u_2|, \quad t \in I, u_1, u_2 \in \mathbb{R} \quad (3.11)$$

$$|\tilde{F}(t, u_1, v_1) - \tilde{F}(t, u_2, v_2)| \leq f_1(t)|u_1 - u_2| + f_2(t)|v_1 - v_2|, \quad t \in I, v, u \in \mathbb{R} \quad (3.12)$$

$$|\tilde{H}(t, u_1) - \tilde{H}(t, u_2)| \leq g(t)|u_1 - u_2|, \quad t \in I, u_1, u_2 \in \mathbb{R}, \quad (3.13)$$

where $f_1, f_2, g : I \rightarrow \mathbb{R}_+$ are considered as an rd-continuous functions. If

$$\beta = e_{B \oplus C}(T, t_0) < 1 + |b(T)|, \quad (3.14)$$

then the equation [3.1](#) has at most one solution on I , where $B(t)$ and $C(t)$ have the same definitions as in theorem [2.3.1](#),

Proof. Suppose that $u(t)$ and $v(t)$ are two solutions of equation [3.1](#) on I . From [3.1](#), [3.10](#) and [3.13](#) we find

$$\begin{aligned} & |u(t) - v(t)| \\ & \leq |b(t)| \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)|u(s) - v(s)| + f_2(s) \int_{\alpha(t_0)}^s g(\tau)|u(\tau) - v(\tau)| \Delta\tau \right] \Delta s \\ & + \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)|u(s) - v(s)| + f_2(s) \int_{\alpha(t_0)}^s g(\tau)|u(\tau) - v(\tau)| \Delta\tau \right] \Delta s, \quad t \in I. \end{aligned} \quad (3.15)$$

We applied corollary [2.3.1](#) on [3.15](#) to obtain the following correspondences

$$|a(t)| \equiv 0 \quad (3.16)$$

$$f_2(s) \equiv 0 \quad (3.17)$$

$$b(t) \equiv b(t) \quad (3.18)$$

$$g(\tau) \equiv g(\tau) \quad (3.19)$$

$$f_1(s) \equiv f_1(s) \quad (3.20)$$

Now, applying the conclusion of corollary [2.3.1](#) we get the following results:

Since $a(t) = 0$, then

$$|u(t) - v(t)| \leq \frac{0 \cdot B(t)}{1 - |b(t)| - \beta} C(t, t_0) \quad (3.21)$$

Then, we obtain

$$|u(t) - v(t)| \leq 0, \quad t \in I, \quad (3.22)$$

this implies

$$|u(t) - v(t)| = 0, \quad t \in I, \quad (3.23)$$

as a result, we conclude that $u(t) = v(t), t \in I$, which means that there is at most one solution of equation [3.1](#). $\square$

Example 3.1.2 Suppose that $\mathbb{T} = q^{\mathbb{N}_0} = \{1, q, q^2, \dots\}$ with $q > 1$ and consider the following Volterra-Fredholm type q -difference equation:

$$\begin{aligned} u(t) = & a(t) + b(t) \sum_{s=t_0}^t \left[f_1(s)u(s) + f_2(s) \sum_{\tau=t_0}^s g(\tau)u(\tau) \right] \\ & + \sum_{s=t_0}^t \left[f_1(s)u(s) + f_2(s) \sum_{\tau=t_0}^s g(\tau)u(\tau) \right], \quad t \in I. \end{aligned} \quad (3.24)$$

Such that $u, a, b, f_1, f_2, g : I \rightarrow \mathbb{R}_+$ are functions, $t_0, T \in q_0^{\mathbb{N}}, T > t_0$ and $I = [t_0, T] \cap q^{\mathbb{N}_0}$.

Theorem 3.1.3 Suppose that a is nondecreasing and $(q-1)tA(t) < 1$. If

$$\beta = \prod_{\tau=t_0}^{T/q} [1 + (q-1)\tau(B(\tau) + C(\tau) + (q-1)\tau B(\tau)C(\tau))] < 1 + b(T) \quad (3.25)$$

then the solution of equation [3.24](#) satisfies the following inequality

$$u(t) \leq \frac{a(T)b(T)}{1 + b(T) - \beta} \prod_{\tau=t_0}^{t/q} [1 + (q-1)\tau(B(\tau) + C(\tau) + (q-1)\tau B(\tau)C(\tau))], \quad t \in I, \quad (3.26)$$

where

$$A(t) = \frac{b(qt) - b(t)}{(q-1)t} \sum_{s=t_0}^{qt} \left[f_1(s) + f_2(s) \sum_{\tau=t_0}^s g(\tau) \right], \quad (3.27)$$

and

$$B(t) = \frac{A(t)}{1 - (q-1)tA(t)}, \quad (3.28)$$

and

$$C(t) = b(t) \left[f_1(t) + f_2(t) \sum_{\tau=t_0}^t g(\tau) \right]. \quad (3.29)$$

Proof. knowing that $\sigma(t) = qt$, $u(t) = (q-1)t$ for any $t \in q_0^{\mathbb{N}}$ and

$$e_p(t, t_0) = \prod_{\tau=t_0}^{t/q} [1 + (q-1)\tau p(\tau)], \quad (3.30)$$

for $t > t_0$, such that $t, t_0, \tau \in q_0^{\mathbb{N}}$. Suppose that $u(t)$ is a solution of equation 3.24 then we obtain

$$\begin{aligned} |u(t)| &\leq a(t) + b(t) \sum_{s=t_0}^t \left[f_1(s)|u(s)| + f_2(s) \sum_{\tau=t_0}^s g(\tau)|u(\tau)| \right] \\ &\quad + \sum_{s=t_0}^T \left[f_1(s)|u(s)| + f_2(s) \sum_{\tau=t_0}^s g(\tau)|u(\tau)| \right], \quad t \in I. \end{aligned} \quad (3.31)$$

Then a suitable application of Corollary 2.3.1 to 3.31 yields the desired result 3.30. $\square$

Example 3.1.3 Suppose that $\mathbb{T} = \mathbb{R}$ and consider the following retarded Volterra-Fredholm type integral equation:

$$\begin{aligned} u(t) &= a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)d\tau \right] ds \\ &\quad + \lambda \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau)d\tau \right] ds, \quad t \in I. \end{aligned} \quad (3.32)$$

Where $u, a, b, \alpha, f_1, f_2, g : I \rightarrow \mathbb{R}_+$ are continuous functions, $t_0, T \in \mathbb{R}$, $T > t_0$ and $I = [t_0, T]$.

Theorem 3.1.4 Suppose that a is nondecreasing, $\lambda \geq 0$ is a constant, $\alpha'(t) > 0$, $\alpha(t) \leq t$ and $b'(t) \geq 0$.

If

$$\beta = \exp \left(\int_{t_0}^T (A + C)(\tau)d\tau \right) < 1 + \frac{b(T)}{\lambda}, \quad (3.33)$$

then

$$u(t) \leq \frac{a(T)b(T)}{\lambda + b(T) - \beta\lambda} \exp \left(\int_{t_0}^t (A + C)(\tau) d\tau \right), \quad t \in I, \quad (3.34)$$

where we use the convention that $\frac{1}{0} = +\infty$,

$$A(t) = b'(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau) d\tau \right] ds, \quad (3.35)$$

and

$$C(t) = b(t) \left[f(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau) d\tau \right] \alpha'(t). \quad (3.36)$$

proof Denote that the time scale investigated for example 3.1.3 is $\mathbb{T} = \mathbb{R}$, this means that the graininess function $\mu(t) = 0$, the forward jump operator $\sigma(t) = t$, and

$$e_p(t, t_0) = \exp \left(\int_{t_0}^t p(\tau) d\tau \right), \quad (3.37)$$

for $t > t_0$. Suppose that $u(t)$ is a solution of Eq 3.32 we have

$$\begin{aligned} |u(t)| &\leq a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)|u(s)| + f_2(s) \int_{\alpha(t_0)}^s g(\tau)|u(\tau)| d\tau \right] ds \\ &\quad + \lambda \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)|u(s)| + f_2(s) \int_{\alpha(t_0)}^s g(\tau)|u(\tau)| d\tau \right] ds, \quad t \in I. \end{aligned} \quad (3.38)$$

Let's try to apply the theorem 2.3.1 to 3.38 to get the Eq 3.34.

The unknown function in theorem 2.3.1 is $u(t)$, which corresponds to $|u(t)|$ in Eq 3.38, in addition $a(t)$ in the while in 3.38 the overall coefficient corresponds to λ , then we effectively use $\lambda \equiv \frac{\lambda}{b(T)}$ for the application.

Since $\mu(t) = 0$, $B(t) = \frac{A(t)}{1 - \mu(t)A(t)}$, this leads us to conclude that $B(t) = A(t)$ and

$$A(t) = b'(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau) d\tau \right] ds, \quad (3.39)$$

and

$$C(t) = b(t) \left[f_1(\alpha(t)) + f_2(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} g(\tau) d\tau \right] \alpha'(t), \quad (3.40)$$

Now, applying the theorem 2.3.1 result, we find:

in theorem 2.3.1 we have

$$u(t) \leq \frac{a(T)}{\lambda + 1 - \beta\lambda} e_{B \oplus C}(t, t_0),$$

and For $\mathbb{T} = \mathbb{R}$, we have

1.

$$e_p(t, t_0) = \exp \left(\int_{t_0}^t p(\tau) d\tau \right) \quad (3.41)$$

2.

$$(B \oplus C)(t) = B(t) + C(t) + \mu(t)B(t)C(t) \quad (3.42)$$

due to $\mu(t) = 0$ we find that

$$(B \oplus C)(t) = B(t) + C(t), \quad (3.43)$$

substituting $B(t) = A(t)$, the exponential term becomes

$$\exp \left(\int_{t_0}^t (A(\tau) + C(\tau)) d\tau \right) \quad (3.44)$$

wich complete the proof. □

3.1 Numerical simulation

This section presents a concrete numerical simulation of Theorem 3.1.4 using MATLAB. The goal is to illustrate, through simulation, that the theoretical bound provided by the inequality is satisfied by an approximated solution computed numerically. This example was implemented using the MATLAB computing environment.

3.1.1 Problem Setting

We consider the following retarded Volterra–Fredholm type integral inequality on the real time scale $\mathbb{T} = \mathbb{R}$:

$$u(t) = a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau) d\tau \right] ds \\ + \lambda \int_{\alpha(t_0)}^{\alpha(T)} \left[f_1(s)u(s) + f_2(s) \int_{\alpha(t_0)}^s g(\tau)u(\tau) d\tau \right] ds$$

3.1.2 Choice of Functions and Verification of Hypotheses

We choose the following functions to satisfy the conditions of Theorem [3.1.4](#):

- $a(t) = 1 + 0.5t$: continuous and increasing,
- $b(t) = 0.2 + 0.1t$: continuous, nonnegative and increasing,
- $\alpha(t) = 0.5t$: strictly increasing, $\alpha(t) \leq t$,
- $f_1(s) = 1 + s^2$: positive and continuous,
- $f_2(s) = 2|\cos(s)|$: nonnegative and continuous on $[0, 1]$,
- $g(\tau) = e^{-\tau}$ positive and continuous,
- $\lambda = 0.2$: a positive constant.

Function	Definition	Condition (Thm 3.1.4)
$a(t)$	$1 + 0.5t$	Continuous, increasing
$b(t)$	$0.2 + 0.1t$	Continuous, $b'(t) \geq 0$
$\alpha(t)$	$0.5t$	Continuous $\alpha(t) \leq t$, $\alpha' > 0$
$f_1(t)$	$1 + t^2$	Continuous, ≥ 0
$f_2(t)$	$2 \cos(t) $	Continuous, ≥ 0
$g(t)$	e^{-t}	Continuous, ≥ 0

Table3.1 Verification of the conditions of Theorem 4.4

3.1.3 MATLAB Implementation

The simulation was implemented in MATLAB using the trapezoidal method to approximate the double integral structure with delay. A snapshot and the result are shown below.

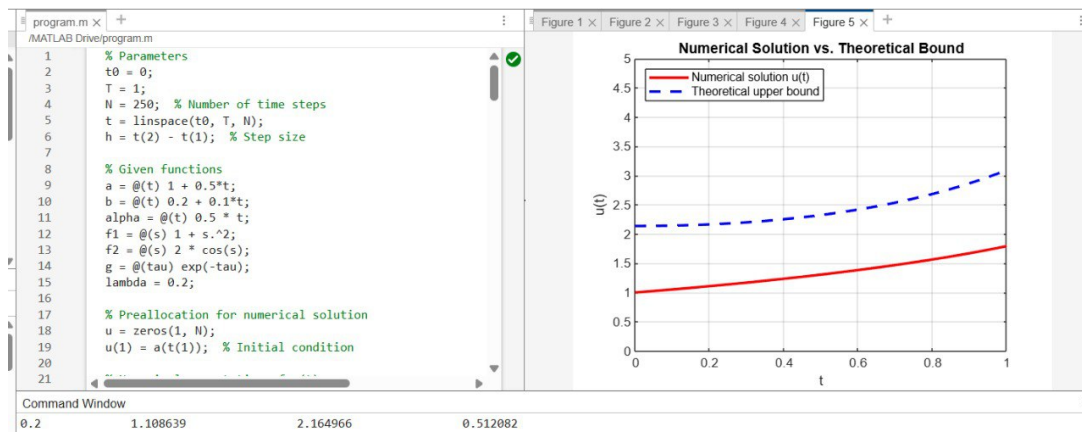


Fig2: Snapshot of the MATLAB code used for simulation

3.1.4 Results and Comparison

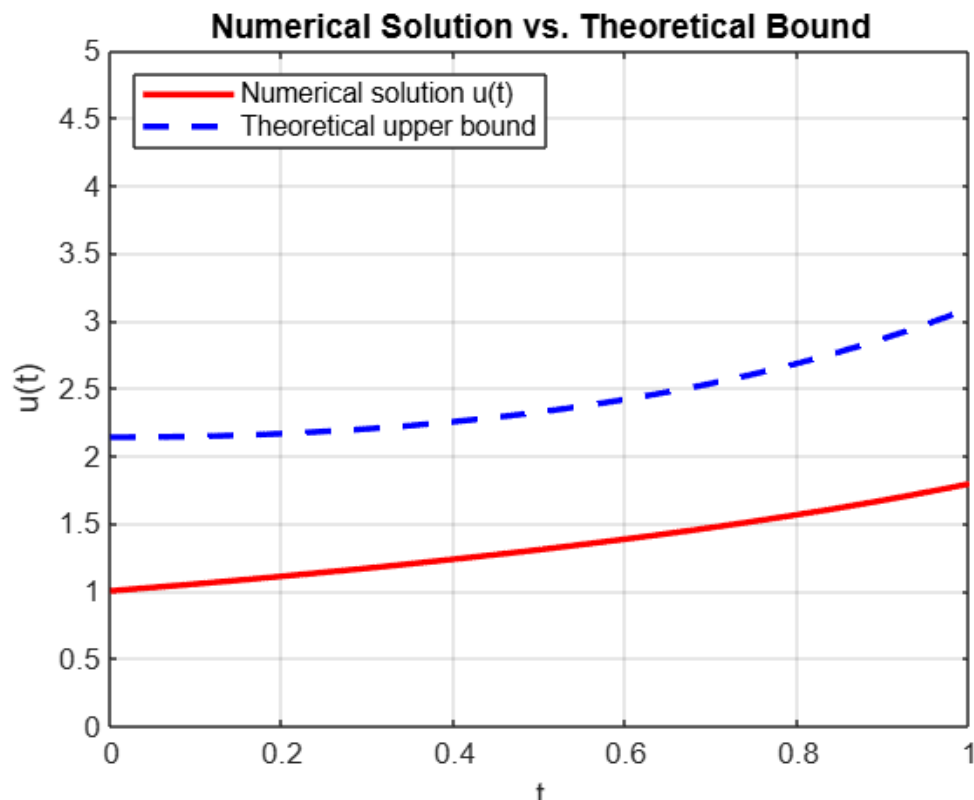


Fig2: Comparison between the approximated solution and the theoretical upper bound

The table below summarizes selected values:

t	$u(t)$	Bound(t)	Error
0.00	1.00000	2.137657	0.467802
0.20	1.108639	2.164966	0.512082
0.40	1.235650	2.254217	1.07538
0.60	1.382484	2.416882	0.572011
0.80	1.563953	2.684228	0.582645
1.00	1.793241	3.094142	0.579560

Table3.2 Comparison of the approximate solution and theoretical bound

3.1.5 Discussion

The results confirm that the approximate solution $u(t)$ remains below the theoretical bound on $[0, 1]$. This validates the theorem numerically and demonstrates that the inequality is useful in controlling the behavior of the solution.

CONCLUSION

This work has investigated the deep rôle of integral inequalities on various branches of mathematics, starting with the foundmental works of Bellman in 1919, and observe how they have been changed and has been developped right up to what were seeing today.

We fucused at the begining on the theory of time scales, the revolutionary concept wich was introduced by Stephan Hilger . This theory had effectively unify discret and continuous analyses, providing a powerfull framework for studying dynamic equations and for avoiding the repeatitive steps in mathematical proofs.

By delving into nonlinear delay integral inequalities of the Volterra-Fredholm type, we have not only summarized their historical development but also presented recent generalizations within the context of time scales. Through some examples and numerical simulation, this work aimed to demonstrate the practical utility and versatility of these inequalities in modern mathematical research. The insights gained here contribute to a deeper understanding of dynamic systems and offer valuable tools for future investigations in this ever-evolving field.

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