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Analytical Study and Numerical Test of a Heat Bresse Timoshenko system

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Dedicate

In the name of Allah, the Most Merciful

Gratitude fills this page before any word of knowledge does

To the purest soul, my beloved mother, whose boundless love and selfless sacrifices shaped every step of my journey. Though you are no longer by my side, your unwavering spirit lives within me, guiding and inspiring me always.

To my noble father, whose unwavering love and endless sacrifices have been the silent force behind every success in my life. Your strength carried me through my hardest days, and your belief in me gave me the courage to rise. I owe more to you than words can ever express.

To my dear siblings, Soumaya and Sara, Zahra, my lifelong companions. In every moment of joy or hardship. We have grown together, carried each other through storms, and shared dreams shaped by love and resilience. This achievement is not mine alone, it belongs to all of us.

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To everyone who has ever helped me in any way may Allah repay you with endless goodness and grace.

To all those who have ever supported me or extended a helping hand may Allah reward each of you with abundant goodness and blessings.

To myself, for every sleepless night, every moment of doubt overcome with courage, and every step taken with patience and determination.

Mahmoudi Chaima

Dedicate

In the name of Allah, Most Merciful I dedicate this work:

To my heaven on earth, my mother whose endless love, sacrifices, resilience, and encouragement have inspired me every step of the way.

To my hero, my lovely father your sacrifices and unwavering belief in me have carried me further than you know. Everything I am, I owe to your guidance and support.

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ABSTRACT IN ENGLISH

This study investigates a one-dimensional Bresse–Timoshenko beam model incorporating microtemperature effects and viscous damping on the transverse displacement of the structure. The global well-posedness of the model is established using the Faedo–Galerkin approximation method alongside rigorous analytical estimates. A Lyapunov functional constructed through the multiplier method enables the proof that the total energy of the system decays exponentially over time, irrespective of the wave propagation speeds or specific parameter values. To corroborate the theoretical findings, numerical simulations are conducted using an explicit Euler scheme for time discretization and the classical finite difference method for spatial discretization. Two numerical examples are provided to illustrate the accuracy and relevance of the theoretical results.

Key words : Bresse Timochenko system, microtemperature effect, exponential stability, Lyapunov functional, Faedo-Galerkin method, finite difference method.

ABSTRACT IN FRENCH

Cette étude porte sur un modèle unidimensionnel de poutre de Bresse–Timoshenko intégrant les effets de microtempérature ainsi qu'un amortissement visqueux appliqué au déplacement transversal de la structure. L'existence et unicité globale de la solution du system a été démontrée à l'aide de la méthode d'approximation de Faedo–Galerkin, en basant sur la méthode d'énergie qui nous permet de construire une fonctionnelle dite de Lyapunov, équivalente a l'énergie total du system en utilisant la méthode des multiplicateurs.cette fonctionnelle nous permet de montrer et prouver que l'énergie totale du système décroît de manière exponentielle et cela confirme le comprtement asymptotique de la solution de ce system, indépendamment des vitesses de propagation des ondes ou des valeurs spécifiques des paramètres. Afin de valider ces résultats théoriques, des simulations numériques sont effectuées à l'aide d'un schéma d'Euler explicite pour la discrétisation temporelle et de la méthode des différences finies pour la discrétisation spatiale. Deux exemples numériques sont présentés pour illustrer la précision et la portée des résultats obtenus.

Mots-clÃ©s : Système de Bresse–Timoshenko, effet de microtempérature, stabilité exponentielle, fonctionnelle de Lyapunov, méthode de Faedo–Galerkin, méthode des différences finies.

ملخص

تتناول هذه الدراسة نموذج عمود بريس–تيموشينكو أحادي البعد الذي يدمج تأثيرات درجة الحرارة المجهرية والتخامد اللزج على الإزاحة العرضية للهيكل. تم إثبات وجود ووحدانية الحل للنموذج باستخدام طريقة فاودو–غاليركين مع بعض التقديرات القبلية. وقد تم بناء دالة ليابونوف مناسبة مكافئة لطاقة النظام باستخدام طريقة المضاعِفات، مما مكّن من إثبات أن الطاقة الكلية للنظام تتناقص بشكل أُسّي مع مرور الزمن، مستقلة عن سرعات انتشار الموجات أو عن أي علاقة أخرى بين معاملات النظام. ومن أجل دعم النتائج النظرية المتحصّل عليها، أُجريت محاكاة عددية باستخدام مخطط أويلر الصريح للتجزئة الزمنية، وطريقة الفروق المنتهية لتجزئة الفضاء. كما اعتمدنا على طريقة النقطة الثابتة لحل النظام المجزئ وتم تقديم مثالين عدديين لتوضيح دقة وفعالية النتائج النظرية.

كلمات مفتاحية: نظام بريس–تيموشينكو، تأثير الحرارة المجهرية، الاستقرار الأُسّي، دالة ليابونوف، طريقة فاودو–غاليركين، طريقة الفروق المنتهية

NOTATIONS

- $H_0^1(0,1)$, $H_0^2(0,1)$: Sobolev spaces of order 1 and 2, respectively.
- $H_*^1(0,1)$: Space of functions with zero mean.
- $H_*^2(0,1)$: Space of Functions with vanishing derivative at the boundary..
- $C^k([0,T];X)$: Space of functions from $[0,T]$ to to a Banach space$X$ that are k-times continuously differentiable.
- $L^p(\Omega)$: Lebesgue space.
- $L^p_{loc}(\Omega)$: set of locally integrable functions.
- $W^{k,p}(\Omega)$: the Sobolev space.
- $< .,. >$: the scalar product.
- $\|.\|$: the norm.

GENERAL INTRODUCTION

In all these years, there have been many works dealing with the beam theories and this is due to his important applications in high technology of zexible structures. Historically, it is well known that the Euler-Bernoulli beam theory is one of the oldest beam theories which is a simplification of the linear isotropicbeams. It was first enunciated circa 1750, but was not applied on a large scale until the development of the Eiffel Tower and the Ferris Wheel in the late 19th century. Later on, other beam theories appeared and they considered as an improvement on the Euler-Bernoulli theory like the Rayleigh beam theory ([3]) as well as the Timoshenko beam theory ([16]).

In ([6]), Elishakoff et al. gave a brief description of the beam model in one-dimensional case for beam vibrations corresponding to the following Bresse-Timoshenoko system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0, \end{cases}$$

In this work, we consider the heat Bresse-Timoshenko model ([19]) where the temperature influence on the beam. The model in question is the following

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t = 0, & \text{in } (0,1)\times(0,+\infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\omega_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_3\omega_t - k_1\omega_{xx} + \lambda\omega + \gamma\psi_{tx} = 0, & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{0.0.1}$$

subject to the following initial data and boundary conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \varphi_{tt}(x,0)=\varphi_2(x) & x\in(0,1),\\ \varphi_{ttt}(x,0)=\varphi_3(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x), & x\in(0,1),\\ \omega(x,0)=\omega_0(x),\ \omega_t(x,0)=\omega_1(x), & x\in(0,1),\\ \varphi_x(0,t)=\varphi_x(1,t)=\psi(0,t)=\psi(1,t)=0, & t\in(0,+\infty),\\ \omega_x(0,t)=\omega_x(1,t)=0. & t\in(0,+\infty), \end{cases} \tag{0.0.2}$$

where $(x,t)\in(0,1)\times\mathbb{R}^+$. Here $\varphi=\varphi(x,t)$ denotes the beam deflection from its equilibrium position, $\psi=\psi(x,t)$ denotes the rotation of the cross-sections with the neglection of the shear deformation and $\omega=\omega(x,t)$ is the microtemperature which represents an internal thermal activity at a microscopic level.

The coefficients of the system can be explained as follows:

ρ_1,ρ_2,ρ_3 : these are density-like parameters (massper, unit length, unit volume) associated with the respective variables:

ρ_1: Inertia of φ.

ρ_2: High-order inertial coupling with φ_{ttx}.

ρ_3: Inertia of ω.

k: A stiffness coefficient.

μ_1: A damping coefficient.

b: A bending stiffness of rotational stiffness.

γ: A coupling parameter which can be describes thermoelastic coupling or a rotational-viscoelastic interaction.

k_1: A higher-order stiffness linked to thermal or flexural effects in ω.

λ: A linear restoring coefficient for ω, introducing a kind of potential term.

The first equation describes axial dynamics of the beam where:

φ_{tt}: Denotes the longitudinal acceleration.

$(\varphi_x+\psi)_x$: Represent the divergence of shear force.

φ_t: Denotes the viscous damping in the axial motion.

The second equation describes rotational (shear, bending) dynamic where:

φ_{ttx}: Denotes the inertial coupling with longitudinal motion.

ψ_{xx}: Represent the bending moment.

ω_x : Is the coupling with heat.

The third equation describes thermal evolution with microstructure where:

ω_t : Is the temperature evolution in microstructure (inertia).

ω_{xx}: Denotes the standard thermal diffusion.

ψ_{tx} : Represent how mechanical influences the temperature in microstructure.

In the first of this work, we establish the existence and uniqueness of global solutions using the Faedo-Galerkin method combined with appropriate a priori estimates. Then, based on the energy method we construct a suitable Lyapunov functional, by using the multipliers method we prove exponential stability of the solution for the system, independently of the wave propagation speeds or other stability number. Finally, we use an Euler scheme for time discretization and classical finite difference method for spatial discretization, then we introduce a point fixed algorithme to solve the discretized problem and we gave two numerical examples to illustrate the obtained theoretical result.

This dissertation is structured into form chapters as follows:

The first chapter, includes reminders of the classical preliminaries, notations and the mathematical tools which are necessary for the study of this work .

The second chapter, is devoted to the study of the existence and uniqueness of solution by using Faedo-Galerking method.

In the third chapter, we establish an exponential decay result of solution

In chapter four, we present some numerical examples which support the theoretical result .

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. In other words, it is known that for the problem determined by (0.0.1) we can always take solutions where φ and ω are constants, for this reason, we impose that

$$\int_0^1 \varphi_0 dx = \int_0^1 \varphi_1 dx = \int 0^1 \omega_0 dx = 0. \tag{0.0.3}$$

It is worth nothing that condition (0.0.3) is imposed to guarantee the solution decays. Thus, if we want to avoid this behavior we need to impose condition (0.0.3). Also, in order to be able to use Poincaré's inequality for φ and ω, we perform the following transformation: From $(0.0.1)_1$, $(0.0.1)_3$ and taking into account the boundary conditions, we get

$$\rho_1 \frac{d^2}{dt^2} \int_0^1 \varphi dx + \mu_1 \frac{d}{dt} \int_0^1 \varphi dx = 0,$$

and

$$\rho_3 \frac{d}{dt} \int_0^1 \omega dx + \lambda \int_0^1 \omega dx = 0.$$

So, $v_1(t) = \displaystyle\int_0^1 \varphi dx$ and $v_2(t) = \displaystyle\int_0^1 \omega dx$ are the solutions of the following initial value problems

$$\begin{cases} \rho_1 v_1''(t) + \mu_1 v_1'(t) = 0, \\ v_1(0) = \displaystyle\int_0^1 \varphi_0(x)\, dx, \; v_1'(0) = \int_0^1 \varphi_1(x)\, dx, \end{cases} \tag{0.0.4}$$

and

$$\begin{cases} \rho_3 v_2'(t) + \lambda v_2(t) = 0, \\ v_2(0) = \displaystyle\int_0^1 \omega_0(x)\, dx. \end{cases} \tag{0.0.5}$$

The solutions of (0.0.4) and (0.0.5) are given by

$$\begin{aligned} v_1(t) &= \int_0^1 \varphi(x,t)\, dx = \int_0^1 \varphi_0(x)\, dx + \frac{\rho_1}{\mu_1} \int_0^1 \varphi_1(x)\, dx \\ &\quad - \frac{\rho_1}{\mu_1} \left(\int_0^1 \varphi_1(x)\, dx \right) e^{-\frac{\mu_1}{\rho_1} t}, \; t \geq 0, \end{aligned}$$

and

$$v_2(t) = \int_0^1 \omega\,(x,t)\,dx = \left(\int_0^1 \omega_0\,(x)\,dx\right) e^{-\dfrac{\lambda}{\rho_3}t},\ t \geq 0.$$

If we take $\bar{\varphi}\,(x,t) = \varphi\,(x,t) - \displaystyle\int_0^1 \varphi\,(x,t)\,dx$ and $\bar{\omega}\,(x,t) = \omega\,(x,t) - \displaystyle\int_0^1 \omega\,(x,t)\,dx$, we obtain

$$\int_0^1 \bar{\varphi}\,(x,t)\,dx = \int_0^1 \bar{w}\,(x,t)\,dx = 0.$$

In it follows, we will work with $\bar{\varphi}$ instead of φ and $\bar{w}$ instead of w, but we write φ and w for the simplicity of the notation.

CHAPTER 1

Preliminary and basic concepts

In this chapter, we present some preliminary concepts and results that are essential for the proofs developed in this work. In particular, we briefly review fundamental notions related to Banach spaces, L^P spaces, Sobolev spaces, and other relevant theorems. Familiarity with these definitions and results is crucial for the analysis carried out in this study.

1.1 Functional Spaces

1.1.1 Inner product and norm

Definition 1.1.1 (Inner product). *([8]) An inner product on a vector space E over a field K (usually $\mathbb{R}$ or $\mathbb{C}$) is a function*

$$\langle \cdot, \cdot \rangle : V \times V \to K,$$

which satisfies, for all u, v, $w \in E$ and all scalars $\alpha \in K$, the following properties:

1.Linearity in the first argument (or sometimes the second depending on convention):

$$\langle v, w \rangle + \alpha \langle u, w \rangle = \langle v, w + \alpha u \rangle .$$

2.conjugate symmetry (for complex spaces reduces to symmetry for real spaces):

$$\overline{\langle u, v \rangle} = \langle v, u \rangle .$$

3.positive -definiteness

$$\langle v, v\rangle \geq 0 \text{ and } \langle v, v\rangle = 0 \text{ if and only if } v = 0.$$

The function $\langle \cdot, \cdot \rangle$ is called an inner product and the vector space E equipped with it is called an inner product space.

Definition 1.1.2 (Norm). *([8]) A norm on a vector space E over a field K (usually $\mathbb{R}$ or $\mathbb{C}$) is a function*

$$\|\cdot\| : E \to [0, +\infty),$$

which satisfies, for all, u, $v \in E$ and all scalars $\alpha \in K$, the following properties:

1 Positivity:

$$\|v\| \geq 0 \text{ and } \|v\| = 0 \text{ if and only if } v = 0.$$

2 Homogeneity (absolute scalability):

$$\|\alpha u\| = |\alpha| \, \|u\| \, .$$

3 Triangle inequality:

$$\|u + v\| \leq \|u\| + \|v\| \, .$$

The function $\|\cdot\|$ is called a norm, and the pair $(E, \|\cdot\|)$ is called a normed vector space.

1.1.2 Complete Space

Definition 1.1.3 (Cauchy Sequence). *([2]) A sequence $(x_n) \subset E$ is called a Cauchy sequence if*

$$\forall \varepsilon > 0, \; \exists N \in \mathbb{N} \text{ such that } \forall m, \; n \geq N, \; \|x_n - x_m\| < \varepsilon.$$

Definition 1.1.4 (Complete Space). *([2]) A normed space E is said to be complete if every Cauchy sequence in E converges to a limit in E.*

1.1.3 Banach Space

Definition 1.1.5 (Banach Space). *([2]) A Banach space is a normed vector space that is complete.*

Example 1.1.1. *1.Every finite-dimensional normed space is a Banach space.*

2.If E is a Banach space and $1 \leq p \leq +\infty$, then the space $L^p(E)$, equipped with the norm $\|\cdot\|_p$, is also a

Banach space.

3.Let X be a set and let E be a Banach space. The vector space $B(X,E)$ consisting of all bounded functions from X into E, equipped with the norm $\|\cdot\|_\infty$, is a Banach space. If,in addition, X is a topological space, then the space $C_b(X,E)$ consisting of all bounded continuous functions from X into E, equipped with the induced norm $\|\cdot\|_\infty$ is also a Banach space.

4. Let$1 \leq p \leq +\infty$, the vector space of functions

$$f : \mathbb{R}^n \to K,$$

that are continuous and satisfy

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx < +\infty,$$

can be equipped with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

This defines a normed space, and it is in fact a Banach space ,denoted by $L^p(\mathbb{R}^n, K)$.

Proposition 1.1.1 (Dual space)**.** *([2]) Dual space X' is the linear space off all continuous linear functional $f : X \to \mathbb{K}$, the space X' equipped by the norm $\|\cdot\|_{X'}$ define by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\},$$

is also a Banach space.

Definition 1.1.6 (Bidual Space)**.** *([2]) The bidual space X'' is the dual of X', is the linear space off all continuous linear functional $f : X' \to K$, the space X'' equipped by the norm $\|\cdot\|_{X''}$ define by*

$$\|f\|_{X''} = \sup\{|f(u)| : \|u\| \leq 1\},$$

is also a Banach space.

Definition 1.1.7. *([2]) Let X be a Banach space and let X' denote its dual space (the space of all continuous linear functionals on X). The bidual $X'' = (X')'$ is the dual of X' and there exists a canonical embedding*

$$J : X \to X'', \text{ defined by } J(x)(u) = u(x), \ \forall u \in X'.$$

Definition 1.1.8 (Reflexive Space)**.** *([2]) A Banach space X is reflexive if every continuous linear* *functional on* X' *is the form*

$$f(u) = u(x) \text{ for some } x \in X.$$

That is, every element of the bidual X'' *is an evaluation at some point in* X*, i.e.,* $J(X) = X''$.

1.1.4 Hilbert Space

Definition 1.1.9 (Hilbert Space). *A Hilbert space* H *is a vector space supplied with inner product* $\langle u, v \rangle$ *such that* $\|u\| = \sqrt{\langle u, u \rangle}$ *is the norm which let* H *complete.*

Example 1.1.2. *The standard inner product on* $\mathbb{C}^n$ *is given by*

$$\langle x, y \rangle = \sum_{j=1}^{n} \overline{x_j} y_j,$$

where $x = (x_1, ..., x_n)$ *and* $y = (y_1, ..., y_n)$ *, which* $x_j, y_j \in \mathbb{C}$*. This space is complete, and therefore it is the finite-dimensional Hilbert space.*

1.1.5 The $\mathcal{L}^p$ spaces

Definition 1.1.10 (The $\mathcal{L}^p$ spaces)**.** *([18]) Let* $p \in [1, +\infty[$*, the* $\mathcal{L}^p$ *space is defined as the set of all measurable functions* $f : \Omega \to \mathbb{R}$ *(or* $\mathbb{C}$*) such that*

$$\int_\Omega |f(x)|^p \, dx < +\infty,$$

this space is equipped with the norm

$$\|f\|_{\mathcal{L}^p} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{P}}.$$

1.1.6 The $L^p(\Omega)$ spaces

Definition 1.1.11 (The $L^p(\Omega)$ spaces)**.** *([18]) Let* $1 \leq P \leq \infty$*, and let* Ω *be an domain in* $\mathbb{R}^n$*,* $n \in \mathbb{N}$*. Define the standard Lebesgue space* $L^p(\Omega)$ *, by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

If $p=\infty$, *we have*

$$L^{p}(\Omega)=\{f:\Omega\rightarrow\mathbb{R}: f \text{ is measurable and there is exists a constant } C \text{ such that }, |f(x)|\leq C \text{ a.e in } \Omega\}.$$

Also, we denote by

$$\|f\|_{\infty}=\inf\{C>0,\ |f(x)|\leq C \text{ a.e in } \Omega\}.$$

Remark 1.1.1. *([18]) In particularly, where* $p=2$, $L^{p}(\Omega)$ *equipped with inner product*

$$\langle f,g\rangle_{L^{2}(\Omega)}=\int_{\Omega}f(x)\,g(x)\,dx,$$

is a Hilbert space.

1.1.7 The Sobolev space $W^{m,p}(\Omega)$

Proposition 1.1.2 (The Sobolev space $W^{m,p}(\Omega)$)**.** *([12]) Let* Ω *be an open domain in* $\mathbb{R}^{N}$. *Then, the distribution* $T\in D^{'}(\Omega)$ *is in* $L^{p}(\Omega)$ *if there exists a function* $f\in L^{p}(\Omega)$ *such that*

$$\langle T,\varphi\rangle=\int_{\Omega}f(x)\,\varphi(x)\,dx, \text{ for all } \varphi\in D(\Omega),$$

where $1\leq p\leq\infty$, *and it's well-known that* f *is unique.*

Definition 1.1.12. *([12]) Let* $m\in\mathbb{N}$ *and* $p\in[0,\infty]$. *The* $W^{m,p}(\Omega)$ *is the space of all* $f\in L^{p}(\Omega)$, *defined as*

$$W^{m,p}(\Omega)=\left\{\begin{array}{l} f\in L^{p}(\Omega), \text{ such that } \partial^{\alpha}f\in L^{p}(\Omega) \text{ for all } \alpha\in\mathbb{N}^{m} \text{ such that} \\ |\alpha|=\sum_{j=1}^{n}\alpha_{j}\leq m, \text{ where, } \partial^{\alpha}=\partial_{1}^{\alpha_{1}}\partial_{2}^{\alpha_{2}}..\partial_{n}^{\alpha_{n}} \end{array}\right\}.$$

Theorem 1.1.1. *([5])* $W^{m,p}(\Omega)$ *is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)}=\sum_{|\alpha|\leq m}\|\partial^{\alpha}f\|_{L^{p}}, 1\leq p<\infty, \text{ for all } f\in W^{m,p}(\Omega).$$

Definition 1.1.13. *([5]) Denote by* $W_{0}^{m,p}(\Omega)$ *the closure of* $D(\Omega)$ *in* $W^{m,p}(\Omega)$.

Definition 1.1.14. *([5]) When* $p=2$, *we denote by* $W^{m,2}(\Omega)=H^{m}(\Omega)$ *and* $W_{0}^{m,2}(\Omega)=H_{0}^{m}(\Omega)$

supplied the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ *a real Hilbert space with their usual scalar product*

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

1.1.8 The $L^P(0, T, X)$ spaces

Definition 1.1.15 (The $L^P(0, T, X)$ spaces)**.** *([2]) Let X be a Banach space, denote by $L^P(0, T, X)$ the space of measurable functions*

$$\begin{aligned} f :]0, T[&\to X \\ t &\to f(t) \end{aligned},$$

such that

$$\left(\int_0^T \|f(t)^p \, dt\|_X \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \text{ for } 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \, \|f(t)\|_X.$$

Theorem 1.1.2. *([14]) The space $L^P(0, T, X)$ is complete.*

Theorem 1.1.3. *([5]) $L^P(0, T, X)$ equipped with the norm $\|.\|_{L^P(0,T,X)}$, $1 \leq p \leq \infty$ is a Banach space.*

Proposition 1.1.3. *([5]) Let X, Y be to Banach spaces, $X \subset Y$ with continuous embedding, then we have $L^P(0, T, X) \subset L^P(0, T, Y)$ with continuous embedding.*

1.1.9 The L^1_{loc} spaces

Definition 1.1.16 (The L^1_{loc} space)**.** *([2]) Let $\Omega \subset \mathbb{R}^n$ be an open set . A function $f : \Omega \to \mathbb{R}$ is said to belong to the space $L^1_{loc}(\Omega)$ if:*

$$\int_K |f(x)| \, dx < \infty; \text{ for every compact subset } K \subset \Omega,$$

In other words

$$f \in L^1_{loc}(\Omega) \text{ if } f \text{ is integrable on every compact subset of } \Omega.$$

1.2 Separable space

Definition 1.2.1 (Separable space). *([17]) Let E be a normed vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$. We say that E is separable if there exists a subset $A \subset E$ such that $\overline{A} = E$ and A is at most countable.*

Example 1.2.1. *([17]) 1. $\mathbb{R}^n$ is separable space for all $n \in \mathbb{N}^*$, since the set $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable and dense in $\mathbb{R}^n$.*

2. Let $\Omega \subset \mathbb{R}^n$ be a measurable set with finite measure. Then the $L^p(\Omega)$ space for $1 \leq P < \infty$ is separable space.

3. Every finite-dimensional normed space is separable space.

1.3 Orthogonal and orthonormal

1.3.1 Orthogonal

Definition 1.3.1 (Orthogonal). *([10]) Let H be a space ($\mathbb{R}$ or $\mathbb{C}$).*

1. Let $u, v \in H$. We say that u and v are orthogonal (and we write $u \perp v$) if $(u \backslash v) = 0$.

2. Let $A \subset H$. The orthogonal complement of A is the set

$$A^{\perp} = \{u \in H; (u \backslash v) = 0 \text{ for all } v \in A\}.$$

1.3.2 Orthonormal

Definition 1.3.2 (Orthonormal). *([10]) Let $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space H is called an orthonormal system if:*

1. $\langle e_n, e_n \rangle = 1$ *for all* n.
2. $\langle e_n, e_m \rangle = 0$ *for all* $n \neq m$.

1.4 Hilbert basis

Definition 1.4.1 (Hilbert basis). *([17]) Let H be a Hilbert space over K, where $K = \mathbb{R}$ or $\mathbb{C}$, and let $B = \{e_i, i \in I\} \subset H$ be a family of elements in H (where I is an arbitrary index set). The family B is* *called a Hilbert basis of H if it satisfies the following two properties:*

1. $(e_i \backslash e_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j, \end{cases}$ *for all* $i, j \in I$.

2. $\overline{Vect\{e_i, i \in I\}} = H$. *That is,*

$$Vect\{e_i, i \in I\} = \left\{ \sum_{i \in J} \alpha_i e_i, J \subset I, card(J) < +\infty, (\alpha_i)_{i \in J} \subset K \right\}.$$

Corollary 1.4.1. *([17]) If* $B = \{e_i, i \in I\}$ *is a Hilbertian basis then every element* $u \in H$ *can be written as*

$$u = \sum_{i=1}^{\infty} \langle u, e_i \rangle e_i,$$

with

$$\|u\|^2 = \sum_{i=1}^{\infty} |\langle u, e_i \rangle|^2 .$$

Theorem 1.4.1. *([17]) Every separable Hilbert space admits a Hilbertian basis.*

1.5 Sobolev embedding

1.5.1 Continuous embedding

Definition 1.5.1 (Continuous embedding)**.** *([2]) Let V and W be two normed vector spaces. We say that V is continuously injected into W, and we write* $V \hookrightarrow W$, *if V is a subspace of W and the canonical injection*

$$\begin{aligned} j : V &\to W \\ x &\to j(x) = x \end{aligned},$$

is continuous, meaning that there exists a constant $c > 0$ *such that:*

$$\|x\|_W \leq c \|x\|_V, \ \forall x \in V.$$

1.5.2 Compact embedding

Definition 1.5.2 (Compact embedding). *([2]) Let V and W be two normed vector spaces. We say that V is compactly injected into W, and we write $V \hookrightarrow\hookrightarrow W$, if V is a subspace of W and the canonical injection*

$$\begin{array}{c} j : V \to W \\ x \to j(x) = x \end{array},$$

compact, meaning that every bounded subset of V is relatively compact in W.

1.6 Existence Methods

1.6.1 Carathéodory's Theorem

Theorem 1.6.1 (Carathéodory's Theorem (Existence Theorem for ODEs)). *([15], [9]) Let $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a function satisfying the following Caratheodory conditions:*

1 For each fixed $U \in \mathbb{R}^n$ the function $t \to F(t, U)$ is measurable on $[a, b]$.

2 For almost every $t \in [0, T]$, the function $U \to F(t, U)$ is continuous on $\mathbb{R}^n$.

3 There exists a function $m(t) \in L^1_{loc}([0, T])$ such that

$$\|F(t, U)\| \leq m(t)(1 + \|U\|), \text{ for all } U \in \mathbb{R}^n.$$

then, for every initial condition $U_0 \in \mathbb{R}^n$, there exists an absolutely continuous function $U(t)$ defined on a subinterval $[0, \delta] \subseteq [0, T]$, such that

$$U'(t) = F(t, U(t)), \text{ for almost every } t \in [0, \delta], \; U(0) = U_0.$$

1.6.2 The Aubin- Lions -Simon Theorem

Theorem 1.6.2 (The Aubin- Lions -Simon Theorem). *([4]) Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \leq p, r \leq \infty$. For $T > 0$, we define*

$$E_{p,r} = \left\{ v \in L^p(0, T, B_0), \; \frac{dv}{dt} \in L^r(0, T, B_2) \right\}.$$

i) If $p < +\infty$, the embedding of $E_{p,r}$ in $L^{p}(0,T,B_1)$ is compact.

ii) If $p = +\infty$,and $r > 1$,the embedding of $E_{p,r}$ in $C^{0}(0,T,B_1)$ is compact.

1.6.3 Gronwell's lemma

Lemma 1.6.1 (Gronwell's lemma). *([15]) Let $T > 0$ and let φ be a function such that, $\varphi \in L^1(0,T)$, $\varphi \geq 0$, almost everywhere and ϕ be a function such that, $\phi \in L^1(0,T)$, $\phi \geq 0$, almost everywhere and $\varphi\phi \in L^1[0,T]$, $C_1, C_2 \geq 0$. Suppose that*

$$\phi(t) \leq C_1 + C_2 \int_0^t \varphi(s)\phi(s)\,ds, \text{ for a.e. } t \in]0,T[,$$

then,

$$\phi(t) \leq C_1 \exp\left(C_2 \int_0^t \varphi(s)\,ds\right), \text{ for a.e..} t \in]0,T[,$$

Proof. Let

$$F(t) = C_1 + C_2 \int_0^t \varphi(s)\phi(s)\,ds, \text{for } t \in]0,T[, \tag{1.6.1}$$

we have,

$$\phi(t) \leq F(t),$$

from (1.6.1), we have

$$\begin{aligned} F'(t) &= C_2\varphi(t)\phi(t) \\ &\leq C_2\varphi(t)\lambda(t), \text{ for a e } t \in]0,T[, \end{aligned}$$

Consequently

$$\frac{d}{dt}\left\{F(t)\exp\left(-\int_0^t C_2\varphi(s)\,ds\right)\right\} \leq 0,$$

then

$$F(t) \leq C_1 \exp\left(C_2 \int_0^t \varphi(s)\,ds\right), \text{ for a.e } t \in]0,T[.$$

Since $\phi \leq F$, then our result holds.

In particle, if $C_1 = 0$, we have $\phi = 0$ for almost everywhere $t \in]0, T[$. $\square$

1.7 Sobolev-Bochner Theorem

Theorem 1.7.1 (Sobolev-Bochner Theorem)**.** *([13]) Let X be a Banach space reflexive and $p > 1$, such that*

$$W^{1,p}(0, T, X) \hookrightarrow C(0, T, X),$$

with the continuous embedding and the compact embedding in $[0, T$.

1.8 Some functional inequalities

1.8.1 Young's inequality

Theorem 1.8.1 (Young's inequality)**.** *([2]) Let a, $b \geq 0$ for any $\varepsilon > 0$ we have*

$$ab \leq \frac{a^2}{4\varepsilon} + \varepsilon b^2.$$

1.8.2 Holder's inequality

Theorem 1.8.2 (Holder's inequality)**.** *([2]) Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p, q \leq +\infty$, such that $\frac{1}{P} + \frac{1}{q} = 1$, then*

$$f, g \in L^1(\Omega) \text{ and } \int_\Omega |fg|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

1.8.3 Cauchy-Schwarz's inequality

Theorem 1.8.3 (Cauchy-Schwarz's inequality)**.** *([2]) We put $p = q = 2$ in Holder's inequality, we obtain the Cauchy-Schwarz's inequality:*

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

1.8.4 Poincaré's inequality

Proposition 1.8.1 (Poincaré's inequality). *([2]) Suppose that $1 \leq p < +\infty$ and Ω is bounded open set. Then there exists a constant C such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1.$$

Corollary 1.8.1. *([2]) Let Ω be an open set of $\mathbb{R}^n$ bounded in at least one space direction. Then the seminorm*

$$|v|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla v(x)|^2 \, dx\right)^{\frac{1}{2}},$$

is a norm over $H_0^1(\Omega)$ which is equivalent to the usual norm induced by that of $H^1(\Omega)$.

1.9 Green's Formula

Theorem 1.9.1 (Green's Formula). *([1]) Let Ω be an open bounded regular set of class C^1. If u and v are functions of $H^1(\Omega)$, they satisfy*

$$\int_\Omega u(x) \frac{\partial v}{\partial x_i}(x) \, dx = -\int_\Omega v(x) \frac{\partial u}{d x_i}(x) \, dx + \int_{\partial\Omega} u(x) v(x) \eta_i(x) \, ds,$$

where $\eta = (\eta_i)_{1 \leq i \leq n}$ is the outward unit normal to $\partial\Omega$.
Theorem 1.9.2 (Green's Formula). *([1]) Let Ω be an open bounded regular set of class C^2. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \triangle u(x) v(x) \, dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v(x) \, ds.$$

1.10 Operator basics

Definition 1.10.1. *([7]) A linear operator A with domain $D(A)$ in a Banach space X, i.e., $A : D(A) \subset X \to X$, is closed if the graph $G(A) = \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$.*
Definition 1.10.2. *([7]) A linear operator A with domain $D(A)$ in a Banach space X, i.e., $A : D(A) \subset X \to X$, is closed if for a sequence $(x_n)_{n \in N} \subset D(A)$ the limit $x_n \to x \in X$ and $Ax_n \to y \in X$ exist, then $x \in D(A)$ and $Ax = y$.*

1.11 Classification of Partial Differential Equations (PDEs)

There exist various types of partial differential equations (PDEs), which can be classified as follows:

1.11.1 Elliptic, Parabolic, and Hyperbolic PDEs

Consider the general form of a second-order PDE:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

where A, B, C, D, E, F, and G are constants for simplicity.

The type of the PDE depends on the sign of the discriminant:

$$\Delta=B^2-4AC.$$

If: $\Delta<0$, the PDE is called Elliptic.

If: $\Delta=0$, the PDE is called Parabolic.

If: $\Delta>0$, the PDE is called Hyperbolic.

Example 1.11.1. *1. Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, $(\Delta=-4)$ so the equation is Elliptic.*

2. Heat (diffusion) equation $\frac{\partial u}{\partial t}-d\frac{\partial^2 u}{\partial x^2}=0$, $(\Delta=0)$ so the equation is Parabolic.

3. Wave equation $\frac{\partial^2 u}{\partial y^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$, $\left(\Delta=4c^2\right)$ so the equation is Hyperbolic.

1.11.2 Order of a PDE

Definition 1.11.1. *The order of a PDE is the highest order of the derivatives appearing in the equation.*

Example 1.11.2. *1. First order: $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$.*

2. Second order: $\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial y^2}=0$.

1.11.3 Linear and Nonlinear PDEs

A PDE is called linear if it can be written in the form:

$$Lu = f.$$

where the operator L is linear with respect to u. Otherwise, it is called nonlinear.

Example 1.11.3. *1. Laplace equation:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *the equation is linear.*

2. Burgers equation $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ *the equation is nonlinear.*

1.12 Faedo-Galerkin Scheme for the Bresse-Timoshenko System with Microtemperature

We apply the Faedo-Galerkin method to construct approximate solutions in finite-dimensional spaces in order to prove the existence of a solution to the Bresse-Timoshenko system with microtemperature.

Step 1: Projection of the system from infinite-dimensional into finite-dimensional

Choice of Basis: Let $\{v_j\}_{j=1}^{\infty}$ *be a Hilbert space. We define the finite-dimensional space*

$$H_n = span\,\{v_1, v_2, ..., v_n\}\,.$$

Approximation of the unknowns solutions: We approximate each unknown solution of the system (displacement φ^n, rotation ψ^n, shear ω^n) by finite sums

$$\varphi^n\,(t,x) = \sum_{j=1}^{n} a_j^n\,(t)\,v_j\,(x)\,,\ \text{and similarly for the other variables.}$$

We substitute the approximate functions into the (PDEs) of the system and test against the basis $\{v_j\}$. *This yields a finite-dimensional system of ordinary differential equations (ODEs) in time for the coefficients* $\left(a_j^n\,(t)\,, b_j^n\,(t)\,, ..\right)$. *Existence of the Approximate Solution: The resulting ODE system satisfies the conditions of Carathéodory's theorem, which ensures the existence of a local solution.*

Step 2: A Priori estimates *By multiplying the equations suitably and integrating over time, we derive* *energy estimates that are uniform in n. These allow the extension of the solution to a global time interval.*

Step 3: Passage to the limit *Using compactness results, we extract a converging subsequence. Passing to the limit in the equations yields a strong solution to the original infinite-dimensional problem.*

1.13 Stability

Definition 1.13.1 (Equilibrium point). *([11]) A point x^* is said to be an equilibrium point if and only if $f(x^*) = 0$, that is x^* is a constant solution of the differential equation*

$$x'(t) = f(x(t)).$$

In other words, if the initial condition is $x(t_0) = x^*$ then $x(t) = x^*$, $\forall t \geq t_0$.

1.13.1 Stability and instability

Definition 1.13.2 (Stability of an equilibrium point). *([11]) The equilibrium point x^* of the differential equation $x'(t) = f(x(t))$ is:*
a. Stable: If, for every $\varepsilon > 0$, there exist $\delta > 0$ such that for every solution $x(t)$ of the differential equation, we have

$$\|x_0 - x^*\| \leq \delta \Longrightarrow \|x(t) - x^*\| < \varepsilon, \ \forall t \geq 0.$$

b. Instable: The equilibrium point is said to be unstable if it is not stable.

1.13.2 Types of stability

1.13.2.1 Asymptotic Stability

Definition 1.13.3 (Asymptotic Stability). *([11]) The equilibrium point x^* is said to be asymptotically stable if it is stable, and in addition, there exists a $r > 0$, such that for every solution $x(t)$ of $\dot{x}(t) = f(x(t))$, we have*

$$\|x_0 - x^*\| \leq r \Longrightarrow \lim_{t\to\infty} \|x(t) - x^*\| = 0.$$

Example 1.13.1. *([11]) We consider the system*

$$\begin{cases} \dot{x}(t) = \dfrac{-x}{1+t} \\ t \geq t_0,\ t_0 \in \mathbb{R}_+ \\ x(t_0) = x_0. \end{cases}$$

the system admits a solution given by the following formula:

$$\begin{aligned} X(t,t_0,x_0) &= x_0 \exp\left(\int_{t_0}^{t} \frac{-1}{1+s} ds\right), \\ &= x_0 \exp\left(-\left[\log(1+s)\right]_{t_0}^{t}\right), \\ &= x_0 \left(\frac{1+t_0}{1+t}\right). \end{aligned}$$

Note that the equilibrium $x_0 = 0$ is stationary and we have:

$$\lim_{t\to\infty} X(t,t_0,x_0) = 0.$$

So, the equilibrium point is asymptotically stable.

1.13.2.2 Exponential stability

Definition 1.13.4 (Exponential stability). *([11]) An equilibrium point x^* is exponentially stable for the equation if there exist two strictly positive scalars k and α such that :*

$$\|x(t) - x^*\| \leq k \|x_0 - x^*\| \exp(-\alpha t).$$

Example 1.13.2. *([11]) Consider the following system:*

$$\dot{x} = -\left(1 + \sin\left(x^2\right)\right) x, \textit{with } x(0) = x_0,$$

It is clear that $x^* = 0$ is an equilibrium point .

The solution of the system is given by the following formula:

$$x(t) = x(0)\exp\left(\int_0^t -\left(1+\sin(x^2)\right) ds\right), \forall t \geq 0,$$

and

$$|x(t)| \leq k\,|x_0|\exp(-t).$$

From this, we obtain exponential stability.

1.13.3 Lyapunov function

Definition 1.13.5 (Lyapunov functional). *([11]) Let U be a neighborhood of zero $V : \mathbb{R}_+ \times U \to \mathbb{R}$ a function that is continuous and differentiable on U.*

1. We say that V is a Lyapunov function in the broad sense at 0 if it satisfies the following two properties:

a. V is positive definite.

b. $V'(t,x) \leq 0$ for every $x \in U$.

2. We say that V is a strict Lyapunov function at 0 if it satisfies the following two properties:

a. V is positive definite.

b. $V'(t,x) < 0$ for every $x \in U \backslash 0$.

CHAPTER 2

Global Well-Posedness of the Bresse-Timoshenko System with Microtemperature via the Faedo-Galerkin Method

In this chapter, we prove the well-posedness of the system by using Faedo-Galerkin method. The analysis begins with the formulation of the approximate problem in a finite-dimensional subspace. We then derive uniform a priori estimates that are essential for the passage to the limit. Using weak convergence and compactness results, we rigorously justify the convergence of the approximate solutions to a global strong solution of the original system.

2.1 *Global well-posedness*

In this section, we give the existence and uniqueness result for problem (0.0.1)-(0.0.2) using the Faedo-Galerkin method.

Theorem 2.1.1. *Assume that the initial data*

$$\begin{cases} (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in H_*^2(0,1) \cap H_*^1(0,1) \times H_*^2(0,1) \cap H_*^1(0,1) \\ \times H^2(0,1) \times L^2(0,1), \\ (\psi_0, \psi_1) \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1), \\ (\omega_0, \omega_1) \in H^2(0,1) \cap H_*^1(0,1) \times L^2(0,1), \end{cases} \tag{2.1.1}$$

where

$$H_*^2(0,1) = \left\{ u \in H^2(0,1) : u_x(0) = u_x(1) = 0 \right\},$$

$$H_*^1(0,1) = \left\{ u \in H^1(0,1) : \int_0^1 u dx = 0 \right\},$$

$$H_0^1 = \left\{ u \in H^1 : u(0) = u(1) = 0 \right\},$$

$$H^1 = \left\{ u \in L^2 : u_x \in L^2 \right\},$$

$$H^2 = \left\{ u \in H^1 : u_{xx} \in L^2 \right\}.$$

Then, the problem (0.0.1)-(0.0.2) has a unique global strong solution satisfies

$$\begin{cases} \varphi \in C\left(\mathbb{R}_+, H_*^2(0,1) \cap H_*^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, H_*^1(0,1)\right), \\ \psi \in C\left(\mathbb{R}_+, L^2(0,1)\right) \cap C^2\left(\mathbb{R}_+, H_0^1(0,1)\right), \\ \omega \in C\left(\mathbb{R}_+, L^2(0,1)\right) \cap C^2\left(\mathbb{R}_+, H_*^1(0,1)\right). \end{cases} \tag{2.1.2}$$

In addition, the solution (φ, ψ, ω) depends continuously on the initial data.

Proof. **Faedo–Galerkin approximations.**

Step 1: Projection the system from infinite-dimensional into finite-dimensional

First of all, we give a Hilbert basis of $H^2(0,1) \cap H_0^1(0,1)$ and $L^2(0,1)$, which satisfied

$$e_j''(x) = \lambda e_j(x),$$

that is

$$e_j''(x) - \lambda e_j(x) = 0, \tag{2.1.3}$$

to solve (2.1.3), we consider the new variable

$$e_j(x) = e^{px}, \ p \in \mathbb{C},$$

then, we get

$$p^2 - \lambda = 0,$$

then

$$p^2 = \lambda,$$

we distinguish three cases

Case 01: If $\lambda = 0$ *we get* $p = 0$, *this implies*

$$e_j(x) = c_1 + c_2 x,$$

because $e_j \in H^2(0,1) \cap H_0^1(0,1)$, $\forall j \in \mathbb{N}^*$, *we have* $e_j(0) = c_1 = 0$ *and* $e_j(1) = c_2 = 0$, *which lead to*

$$e_j(x) = 0, \ \forall j \geq 1.$$

Case 02: If $\lambda > 0$ *we get* $p = |\alpha|$ *with* $\alpha \neq 0$, *this implies*

$$e_j(x) = c_1 e^{|\alpha| x} + c_2 e^{-|\alpha| x}, \ c_1, \ c_2 \in \mathbb{R},$$

since $e_j(0) = c_1 + c_2 = 0$ *that means* $c_1 = -c_2$, *then*

$$e_j(x) = c_1 \left(e^{|\alpha| x} - e^{-|\alpha| x} \right),$$

since $e_j(1) = 0$, *we get*

$$c_1 \left(e^{|\alpha|} - e^{-|\alpha|} \right) = 0,$$

then, $e^{|\alpha|} - e^{-|\alpha|} = 0$ *implies that* $\alpha = 0$ *which contradicts the fact that* $\alpha \neq 0$. *Hence,* $c_1 = c_2 = 0$, *which leads to*

$$e_j(x) = 0.$$

Case 03: If $\lambda < 0$ *we get* $p = i|\alpha|$ *with* $\alpha \neq 0$, *this implies*

$$e_j(x) = c_1 \cos(|\alpha| x) + c_2 \sin(|\alpha| x), \ c_1, \ c_2 \in \mathbb{R},$$

since $e_j(0) = 0$, *we get* $c_1 = 0$, *which implies*

$$e_j(x) = c_2 \sin(|\alpha| x),$$

because $e_j(1) = 0$, *we get* $|\alpha| = 2j\pi$, $j \geq 1$, *so*

$$e_j(x) = c_2 \sin(2j\pi x), \; j \geq 1 \text{ and } c_2 \neq 0.$$

By the same method, we give a Hilbert basis of $H_*^2(0,1) \cap H_*^1(0,1)$ *and* $L^2(0,1)$ *which satisfied*

$$\sigma_j''(x) = \lambda' \sigma_j(x),$$

implies

$$\sigma_j''(x) - \lambda' \sigma_j(x) = 0, \tag{2.1.4}$$

to solve (2.1.4), *we consider the new variable*

$$\sigma_j(x) = e^{px}, \; p \in \mathbb{C},$$

we obtain

$$p^2 = \lambda',$$

we distinguish three cases

Case 01: If $\lambda' = 0$ *we get* $p = 0$, *this implies*

$$\sigma_j(x) = c_1 + c_2 x,$$

because $\sigma_j \in H_*^2(0,1) \cap H_*^1(0,1), \forall j \in \mathbb{N}^*$, *we have* $\sigma_j(0) = c_1 = 0$ *and* $\sigma_j(1) = c_2 = 0$, *which lead to*

$$\sigma_j(x) = 0, \; \forall j \in \mathbb{N}^*.$$

Case 02: If $\lambda' > 0$ *we get* $p = |\alpha|$ *with* $\alpha \neq 0$, *this implies*

$$\sigma_j(x) = c_1 \cos(|\alpha| x) + c_2 \sin(|\alpha| x), \; c_1, \; c_2 \in \mathbb{R},$$

since $\sigma_j(0) = c_1 + c_2 = 0$ *that means* $c_1 = -c_2$, *then*

$$\sigma_j(x) = c_1 \left(e^{|\alpha| x} - e^{-|\alpha| x} \right),$$

since since $\sigma_j(1) = 0$, *we get*

$$c_1\left(e^{|\alpha|} - e^{-|\alpha|}\right) = 0,$$

then, $e^{|\alpha|} - e^{-|\alpha|} = 0$ *implies that* $\alpha = 0$ *which contradicts the fact that* $\alpha \neq 0$. *Hence,* $c_1 = c_2 = 0$, *which leads to*

$$\sigma_j(x) = 0.$$

Case 03: If $\lambda' < 0$ *we get* $p = i|\alpha|$ *with* $\alpha \neq 0$, *this implies*

$$\sigma_j(x) = c_1\cos(|\alpha|x) + c_2\sin(|\alpha|x),\ c_1,\ c_2 \in \mathbb{R},$$

since $\sigma_j(0) = c_2 = 0$, *which implies*

$$\sigma_j(x) = c_1\cos(|\alpha|x),$$

because $\sigma_j(1) = 0$, *we get* $|\alpha| = 2j\pi$, $j \geq 1$, *so*

$$\sigma_j(x) = c_1\cos(|\alpha|x),\ j \geq 1 \text{ and } c_1 \neq 0.$$

For given initial data satisfies (2.1.1), *we seek functions* $y_j^n,\ h_j^n,\ R_j^n \in C^2([0,T])$, *such that*

$$\begin{cases} \varphi^n(x,t) = \sum_{j=1}^{j=n} y_j^n(t)\sigma_j(x), \\ \psi^n(x,t) = \sum_{j=1}^{j=n} h_j^n(t)e_j(x), \\ \omega^n(x,t) = \sum_{j=1}^{j=n} R_j^n(t)\sigma_j(x), \end{cases} \tag{2.1.5}$$

check the following approximate problem

$$\begin{cases} \rho_1\varphi_{tt}^n - k(\varphi_x^n + \psi^n)_x + \mu_1\varphi_t^n = 0, \\ -\rho_2\varphi_{ttx}^n - b\psi_{xx}^n + k(\varphi_x^n + \psi^n) + \gamma\omega_x^n = 0, \\ \rho_3\omega_t^n - k_1\omega_{xx}^n + \lambda\omega^n + \gamma\psi_{tx}^n = 0, \end{cases} \tag{2.1.6}$$

with the initial data

$$\begin{cases} \varphi^n(x,0) = \varphi_0^n(x), \; \varphi_t^n(x,0) = \varphi_1^n(x), \; \varphi_{tt}^n(x,0) = \varphi_2^n(x), \\ \varphi_{ttt}^n(x,0) = \varphi_3^n(x), \; \psi^n(x,0) = \psi_0^n(x), \; \psi_t^n(x,0) = \psi_1^n(x), \\ \omega^n(x,0) = \omega_0^n(x), \; \omega_t^n(x,0) = \omega_1^n(x), \end{cases} \tag{2.1.7}$$

which satisfies

$$\begin{cases} \varphi_0^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \varphi_0 \sigma_i dx \right) \sigma_i \underset{n\to\infty}{\longrightarrow} \varphi_0 \text{ strongly in } H_*^2(0,1) \cap H_*^1(0,1), \\ \\ \varphi_1^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \varphi_1 \sigma_i dx \right) \sigma_i \underset{n\to\infty}{\longrightarrow} \varphi_1 \text{ strongly in } H_*^2(0,1) \cap H_*^1(0,1), \\ \varphi_2^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \varphi_2 \sigma_i dx \right) \sigma_i \underset{n\to\infty}{\longrightarrow} \varphi_2 \text{ strongly in } H^2(0,1), \\ \varphi_3^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \varphi_3 \sigma_i dx \right) \sigma_i \underset{n\to\infty}{\longrightarrow} \varphi_3 \text{ strongly in } L^2(0,1), \\ \psi_0^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \psi_0 e_i dx \right) e_i \underset{n\to\infty}{\longrightarrow} \psi_0 \text{ strongly in } H^2(0,1) \cap H_0^1(0,1), \\ \psi_1^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \psi_1 e_i dx \right) e_i \underset{n\to\infty}{\longrightarrow} \psi_1 \text{ strongly in } H_0^1(0,1), \\ \omega_0^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \omega_0 \sigma_i dx \right) \sigma_i \underset{n\to\infty}{\longrightarrow} \omega_0 \text{ strongly in } H_*^2(0,1) \cap H_*^1(0,1), \\ \qquad\qquad n\to\infty \\ \omega_1^n = \sum\limits_{i=0}^{n} \left(\int_0^1 \omega_1 \sigma_i dx \right) \sigma_i \underset{n\to\infty}{\longrightarrow} \omega_1 \text{ strongly in } L^2(0,1). \end{cases} \tag{2.1.8}$$

Now we substitute the approximate functions (2.1.5) into the system (2.1.6) this yields a finite-dimensional system of ordinary differential equations(ODEs).By multiplying $(2.1.6)_1$, $(2.1.6)_2$*and* $(2.1.6)_3$*, respectively by* $\sigma_j(x)$, $e_j(x)$ *and* $\sigma_j(x)$*, for every* $n \geq 1$*, and integrating over* $(0,1)$ *with respect to* x*, we* *get*

$$\begin{cases} \rho_1 \left\langle \varphi_{tt}^n, \sigma_j \right\rangle_{L^2(0,1)} - k \left\langle (\varphi_x^n + \psi^n)_x, \sigma_j \right\rangle_{L^2(0,1)} + \mu_1 \left\langle \varphi_t^n, \sigma_j \right\rangle_{L^2(0,1)} = 0, \\ \\ -\rho_2 \left\langle \varphi_{ttx}^n, e_j \right\rangle_{L^2(0,1)} - b \left\langle \psi_{xx}^n, e_j \right\rangle_{L^2(0,1)} + k \left\langle (\varphi_x^n + \psi^n), e_j \right\rangle_{L^2(0,1)} \\ +\gamma \left\langle \omega_x^n, \varphi_{ttx}^n \right\rangle_{L^2(0,1)} = 0, \\ \\ \rho_3 \left\langle \omega_t^n, \sigma_j \right\rangle_{L^2(0,1)} - k_1 \left\langle \omega_{xx}^n, \sigma_j \right\rangle_{L^2(0,1)} + \lambda \left\langle \omega^n, \sigma_j \right\rangle_{L^2(0,1)} + \gamma \left\langle \psi_{tx}^n, \sigma_j \right\rangle_{L^2(0,1)} = 0, \end{cases}$$

implies

$$\begin{cases} \rho_1 \int_0^1 \varphi_{tt}^n \sigma_j dx - k \int_0^1 (\varphi_x^n + \psi^n)_x \sigma_j dx + \mu_1 \int_0^1 \varphi_t^n \sigma_j dx = 0, \\ \\ -\rho_2 \int_0^1 \varphi_{ttx}^n e_j dx - b \int_0^1 \psi_{xx}^n e_j dx + k \int_0^1 (\varphi_x^n + \psi^n) e_j dx + \gamma \int_0^1 \omega_x^n e_j dx = 0, \\ \\ \rho_3 \int_0^1 \omega_t^n \sigma_j dx - k_1 \int_0^1 \omega_{xx}^n \sigma_j dx + \lambda \int_0^1 \omega^n \sigma_j dx + \gamma \int_0^1 \psi_{tx}^n \sigma_j dx = 0, \end{cases} \tag{2.1.9}$$

by using (2.1.5) and integration over $(0,1)$ *with respect to* x*, we obtain*

$$\rho_1 \int_0^1 \varphi_{tt}^n \sigma_j dx = \rho_1 \int_0^1 \left(\sum_{i=1}^{i=n} (y_i^n(t))'' \sigma_i(x) \right) \sigma_j dx = \rho_1 (y_j^n(t))'', \tag{2.1.10}$$

and

$$\begin{aligned} -k \int_0^1 \varphi_{xx}^n \sigma_j dx &= -k \int_0^1 \left(\sum_{i=1}^{i=n} y_i^n(t) \sigma_{ixx}(x) \right) \sigma_j dx \\ &= +k(2\pi i)^2 \sum_{i=1}^{i=n} y_i^n(t) \int_0^1 \cos(2\pi i x) \cos(2\pi j x) dx = 2k(\pi j)^2 y_j^n(t) \\ &= k_2 y_j^n(t), \text{ where } k_2 = 2k(\pi j)^2. \end{aligned} \tag{2.1.11}$$

because if $i \neq j$*, we get*

$$-k \int_0^1 \varphi_{xx}^n \sigma_j dx = k(2\pi i)^2 \sum_{i=1}^{i=n} y_i^n(t) \int_0^1 \left[\cos(2\pi(i+j)x) + \cos(2\pi(i-j)x) \right] dx$$

$$= \quad k(2\pi i)^2 \sum_{i=1}^{i=n} y_i^n(t) \left[\frac{1}{2\pi(i+j)} \sin(2\pi(i+j)x) + \frac{1}{2\pi(i-j)} \sin(2\pi(i-j)x)\right]_0^1 = 0,$$

if $i = j$, we get

$$\begin{aligned} -k\int_0^1 \varphi_{xx}^n \sigma_j dx &= \frac{k}{2}(2\pi i)^2 \sum_{i=1}^{i=n} y_i^n(t) \int_0^1 (\cos(4\pi i x) + 1)\, dx \\ &= \frac{k}{2}(2\pi i)^2 \sum_{i=1}^{i=n} y_i^n(t) \left[\frac{1}{4\pi i} \sin(4\pi i x) + x\right]_0^1 = 2k(\pi i)^2 y_i^n(t). \end{aligned}$$

and by the same method

$$-k\int_0^1 \psi_x^n \sigma_j dx = -k\int_0^1 \sum_{i=1}^{i=n} h_i^n(t)\, e_{ix}(x)\, \sigma_j dx = -k(2\pi j)\, h_j^n(t) = -k_3 h_j^n(t), \text{ where } k_3 = k(2\pi j). \tag{2.1.12}$$

and

$$\mu_1 \int_0^1 \varphi_t^n \sigma_j dx = \mu_1 \int_0^1 \sum_{i=1}^{i=n} (y_i^n(t))' \sigma_i(x)\, \sigma_j dx = \mu_1 (y_j^n(t))'. \tag{2.1.13}$$

$$\begin{aligned} -\rho_2 \int_0^1 \varphi_{ttx}^n e_j dx &= -\rho_2 \int_0^1 \sum_{i=1}^{i=n} (y_i^n(t))'' \sigma_{ixx}(x)\, e_j dx \\ &= \rho_2 (2\pi i) \sum_{i=1}^{i=n} (y_i^n(t))'' \int_0^1 \sin(2\pi i x) \sin(2\pi j x)\, dx = \rho_2(\pi i)(y_i^n(t))'' \\ &= \rho_4 (y_j^n(t))'', \text{ where } \rho_4 = \rho_2(\pi j). \end{aligned} \tag{2.1.14}$$

because if $i \neq j$, we get

$$\begin{aligned} -\rho_2 \int_0^1 \varphi_{ttx}^n e_j dx &= \rho_2(\pi i) \sum_{i=1}^{i=n} (y_i^n(t))'' \int_0^1 (cos(2\pi(i-j)x) - \cos(2\pi(i+j)x)\, dx \\ &= \rho_2(\pi i) \sum_{i=1}^{i=n} (y_i^n(t))'' \left[\frac{1}{2\pi(i-j)} \sin(2\pi(i-j)x) - \frac{1}{2\pi(i+j)} \sin(2\pi(i-j)x)\right]_0^1 = 0, \end{aligned}$$

if $i = j$, we get

$$-\rho_2 \int_0^1 \varphi_{ttx}^n e_j dx \quad = \quad \rho_2(\pi i) \sum_{i=1}^{i=n} (y_i^n(t))'' \int_0^1 (1 - \cos(4\pi i x))\, dx$$

$$= \rho_2(\pi i)\sum_{i=1}^{i=n}(y_i^n(t))''\left[x-\frac{1}{4\pi i}\sin(4\pi i x)\right]_0^1 = \rho_2(\pi i)(y_i^n(t))''.$$

by the same method, we get

$$-b\int_0^1 \psi_{xx}^n e_j dx = -b\int_0^1 \sum_{i=1}^{i=n} h_i^n(t)\, e_{ixx}(x)\, e_j dx = 2b(\pi j)^2 h_j^n(t) = b_2 h_j^n(t)\,,\ \text{where } b_2 = 2b(\pi j)^2\,. \tag{2.1.15}$$

again by the same method, we get

$$k\int_0^1 \varphi_x^n e_j dx = k\int_0^1 \sum_{i=1}^{i=n} y_i^n(t)\,\sigma_{jx}(x)\, e_j dx = -k(\pi j)y_j^n(t) = -k_4 y_j^n(t)\,,\ \text{where } k_4 = k(\pi j). \tag{2.1.16}$$

and

$$k\int_0^1 \psi^n e_j dx = k\int_0^1 \sum_{i=1}^{i=n} h_i^n(t)\, e_i(x)\, e_j dx = kh_j^n(t)\,. \tag{2.1.17}$$

and

$$\gamma\int_0^1 \omega_x^n e_j dx = \gamma\int_0^1 \sum_{i=1}^{i=n} R_i^n(t)\,\sigma_{ix}(x)\, e_j dx = -\gamma(\pi i)R_j^n(t) = -\gamma_1 R_j^n(t)\,,\ \text{where } \gamma_1 = \gamma(\pi j). \tag{2.1.18}$$

and

$$\rho_3\int_0^1 \omega_t^n \sigma_j dx = \rho_3\int_0^1 \sum_{i=1}^{i=n}(R_i^n(t))'\sigma_i(x)\,\sigma_j dx = \rho_3(R_j^n(t))'. \tag{2.1.19}$$

and by the same method, we obtain

$$-k_1\int_0^1 \omega_{xx}^n \sigma_j dx = -k_1\int_0^1 \sum_{i=1}^{i=n} R_i^n(t)\,\sigma_{ixx}(x)\,\sigma_j dx = 2k_1(\pi j)^2 R_j^n(t) = k_5 R_j^n(t)\,,\ \text{where } k_5 = 2k_1(\pi j)^2. \tag{2.1.20}$$

and

$$\lambda\int_0^1 \omega^n \sigma_j dx = \lambda\int_0^1 \sum_{i=1}^{i=n} R_i^n(t)\,\sigma_i(x)\,\sigma_j dx = \lambda R_j^n(t)\,. \tag{2.1.21}$$

again by the same method, we get

$$\gamma\int_0^1 \psi_{tx}^n \sigma_j dx = \gamma\int_0^1 \sum_{i=1}^{i=n}(h_i^n(t))'e_{ix}(x)\,\sigma_j dx = \gamma(\pi i)(h_i^n(t))' = \gamma_2(h_j^n(t))',\ \text{where } \gamma_2 = \gamma(\pi j). \tag{2.1.22}$$

and by substituting (2.1.10), (2.1.11), (2.1.12), (2.1.13), (2.1.14), (2.1.15), (2.1.16), (2.1.17), (2.1.18), (2.1.19), (2.1.20), (2.1.21) and (2.1.22) in (2.1.9) , we get

$$\begin{cases} \rho_1 (y_j^n(t))'' + k_2 y_j^n(t) - k_3 h_j^n(t) + \mu_1 (y_j^n(t))' = 0, \\ \rho_4 (y_j^n(t))'' + b_2 h_j^n(t) - k_4 y_j^n(t) - \gamma_1 R_j^n(t) = 0, \\ \rho_3 (R_j^n(t))' + k_5 R_j^n(t) + \lambda R_j^n(t) + \gamma_2 (h_j^n(t))' = 0, \end{cases}$$

implies

$$\begin{cases} \rho_1 (y_j^n(t))'' + \mu_1 (y_j^n(t))' + k_2 y_j^n(t) - k_3 h_j^n(t) = 0, \\ \rho_4 (y_j^n(t))'' - k_4 y_j^n(t) + b_2 h_j^n(t) - \gamma_1 R_j^n(t) = 0, \\ \rho_3 (R_j^n(t))' + \gamma_2 (h_j^n(t))' + \rho_5 R_j^n(t) = 0, \end{cases} \tag{2.1.23}$$

where $\rho_5 = \lambda + k_5$. According to Caratheodory existence theorem for standard theory of ordinary differential equations, the finite dimensional problem (2.1.23) had a local solution. To prove that, we rewrite the system (2.1.23) as a first-order system, we introduce the new variables

$$Y_1(t) = y_j^n(t), \; Y_2(t) = (y_j^n(t))', \; Z_1(t) = h_j^n(t) \;\; \{\text{and } \; R_1(t) = R_j^n(t),$$

which implies

$$Y_1'(t) = Y_2(t),$$

the system (2.1.23) becomes

$$\begin{cases} \rho_1 Y_2'(t) + \mu_1 Y_2(t) + k_2 Y_1(t) - k_3 Z_1(t) = 0, \\ \rho_4 Y_2'(t) - k_4 Y_1(t) + b_2 Z_1(t) - \gamma_1 R_1(t) = 0, \\ \rho_3 R_1'(t) + \gamma_2 Z_1'(t) + \rho_5 R_1(t) = 0, \end{cases} \tag{2.1.24}$$

we now solve the first two simultaneously to express $y_2'(t)$ and $Z_1(t)$, the system can be written in matrix form

$$\begin{bmatrix} \rho_1 & -k_3 \\ \rho_4 & b_2 \end{bmatrix} \begin{bmatrix} Y_2'(t) \\ Z_1(t) \end{bmatrix} = \begin{bmatrix} -\mu_1 Y_2(t) - k_2 Y_1(t) \\ k_4 Y_1(t) + \gamma_1 R_1(t) \end{bmatrix},$$

let $\Delta = \rho_1 b_2 + \rho_4 k_3 \neq 0$, and the matrix is invertible, and we obtain

$$Y_2'(t) = \frac{1}{\Delta} \left(-b_2 \mu_1 Y_2(t) - b_2 k_2 Y_1(t) + k_3 k_4 Y_1(t) + k_3 \gamma_1 R_1(t) \right),$$

and

$$Z_1(t) = \frac{1}{\Delta}\left(\rho_4\mu_1 Y_2(t) + \rho_4 k_2 Y_1(t) + \rho_1 k_4 Y_1(t) + \rho_1\gamma_1 R_1(t)\right), \tag{2.1.25}$$

differentiating $Z_1(t)$ with respect to t, we get

$$Z_1'(t) = \frac{1}{\Delta}\left(\rho_4\mu_1 Y_2'(t) + (\rho_4 k_2 + \rho_1 k_4) Y_2(t) + \rho_1\gamma_1 R_1'(t)\right). \tag{2.1.26}$$

From (2.1.24)$_3$, we get

$$R_1'(t) = -\frac{\gamma_2}{\rho_3} Z_1'(t) - \frac{\rho_5}{\rho_3} R_1(t),$$

substituting (2.1.25) and (2.1.26) in (2.1.24), we get

$$\begin{aligned} R_1'(t) &= -\frac{\gamma_2}{\rho_3\Delta}(\rho_4\mu_1 Y_2'(t) + (\rho_4 k_2 + \rho_1 k_4) Y_2(t) + \rho_1\gamma_1 R_1'(t)) - \frac{\rho_5}{\rho_3} R_1(t) \\ &= \frac{-1}{1 + \frac{\gamma_2\rho_1\gamma_1}{\rho_3\Delta}}\left(\frac{\gamma_2\rho_4\mu_1}{\rho_3\Delta} Y_2'(t) + \frac{\gamma_2(\rho_4 k_2 + \rho_1 k_4)}{\rho_3\Delta} Y_2(t) - \frac{\rho_5}{\rho_3} R_1(t)\right) \\ &= \frac{-1}{\rho_3\Delta + \gamma_2\rho_1\gamma_1}\left(\gamma_2\rho_4\mu_1 Y_2'(t) + \gamma_2(\rho_4 k_2 + \rho_1 k_4) Y_2(t)\right) + \frac{\Delta\rho_5}{\rho_3\Delta + \gamma_2\rho_1\gamma_1} R_1(t). \end{aligned}$$

Thus, the system can be written in the vectorial form

$$U(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ R_1(t) \end{bmatrix} \in \mathbb{R}^{3n},\ U'(t) = F(t, U(t)),$$

where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a function defined as

$$F(t, U(t)) \quad = \quad \begin{bmatrix} Y_2(t) \\ Y_2'(Y_2(t),\ Y_1(t),\ R_1(t)) \\ R_1'(Y_2(t),\ Y_1(t),\ R_1(t)) \end{bmatrix}$$

$$= \begin{bmatrix} Y_2(t) \\ \\ \frac{1}{\Delta}\left(-b_2\mu_1 Y_2(t) - b_2 k_2 Y_1(t) + k_3 k_4 Y_1(t) + k_3\gamma_1 R_1(t)\right) \\ \\ \frac{-1}{\rho_3\Delta + \gamma_2\rho_1\gamma_1}\left(\gamma_2\rho_4\mu_1 Y_2'(t) + \gamma_2(\rho_4 k_2 + \rho_1 k_4) Y_2(t)\right) \\ + \frac{\Delta\rho_5}{\rho_3\Delta + \gamma_2\rho_1\gamma_1} R_1(t)\left(Y_2(t),\ Y_1(t),\ R_1(t)\right) \end{bmatrix}.$$

The function $U \to F(t, U(t))$ is clearly continuous in $\mathbb{R}^3$, because $F(U)$ involves only algebraic combinations (linear and polynomial) of the components $Y_2(t), Y_1(t), R_1(t)$, with coefficients depending only on the physical constants of the model $(\text{e.g.}, \rho_4, , \mu_1, k_4, \text{ etc.})$.

The function $t \to F(t, U(t))$ is clearly measurable on $[0, T]$, because $F(t, U(t))$ does not depend explicitly on t (i.e., the system is autonomous). Therefore, $F(t, U(t)) = F(U(t))$ is constant in t.

The third condition of Carathéodory's theorem requires that there exists a function $m(t) \in L^1_{loc}$ such that

$$F(t, U(t)) \leq m(t)(1 + \|U\|), \text{ for all } U \in \mathbb{R}^3$$

as we see the function $F(U(t))$ contains only linear expressions in $Y_2(t), Y_1(t), R_1(t)$, with constant coefficients. So, it grows at most linearly with respect to $\|U\|$. This means we can take $m(t) = C$ with $C > 0$, such that

$$F(t, U(t)) \leq C(1 + \|U\|),$$

since $m(t)$ is constant, it clearly belongs to L^1_{loc}.

We conclude that the system (2.1.24) *admits at least one local solution $U(t)$ defined on interval $[0, T)$, for any given initial data $U(0) = U_0 \in \mathbb{R}^3$.* That means *Caratheodory 's theorem guarantees the existence of an absolutely continuous solution $\left(y_j^n, h_j^n, R_j^n\right) \in C^2([0, t_n))^3$ where $T_n > 0$ to the original system* (2.1.23).

Step 2: A priori estimates

A priori estimates allow that the extension of the solution to a global time interval $(t_n = \infty)$,by multiplying the equations suitably and integrating, we derive energy estimates that are uniform in n.

A priori estimate I:

For every $n \geq 1$, multiplying (2.1.23)$_1$, (2.1.23)$_2$ and (2.1.23)$_3$, respectively, by $\left(y_j^n\right)'$, $\left(h_j^n\right)'$ and $\left(R_j^n\right)'$, and integrating over $(0,1)$, we get

$$\begin{cases} \rho_1 \int_0^1 \varphi_{tt}^n \left(\sum_{j=1}^{j=n} \left(y_j^n\right)' \sigma_j \right) dx - k \int_0^1 \left(\varphi_x^n + \psi^n\right)_x \left(\sum_{j=1}^{j=n} \left(y_j^n\right)' \sigma_j \right) dx \\ \\ +\mu_1 \int_0^1 \varphi_t^n \left(\sum_{j=1}^{j=n} \left(y_j^n\right)' \sigma_j \right) dx = 0, \\ \\ -\rho_2 \int_0^1 \varphi_{ttx}^n \left(\sum_{j=1}^{j=n} \left(h_j^n\right)' e_j \right) dx - b \int_0^1 \psi_{xx}^n \left(\sum_{j=1}^{j=n} \left(h_j^n\right)' e_j \right) dx \\ \\ +k \int_0^1 \left(\varphi_x^n + \psi^n\right) \left(\sum_{j=1}^{j=n} \left(h_j^n\right)' e_j \right) dx + \gamma \int_0^1 \omega_x^n \left(\sum_{j=1}^{j=n} \left(h_j^n\right)' e_j \right) dx = 0, \\ \\ \rho_3 \int_0^1 \omega_t^n \left(\sum_{j=1}^{j=n} \left(R_j^n\right)' \sigma_j \right) dx - k_1 \int_0^1 \omega_{xx}^n \left(\sum_{j=1}^{j=n} \left(R_j^n\right)' \sigma_j \right) dx \\ \\ +\lambda \int_0^1 \omega^n \left(\sum_{j=1}^{j=n} \left(R_j^n\right)' \sigma_j \right) dx + \gamma \int_0^1 \psi_{tx}^n \left(\sum_{j=1}^{j=n} \left(R_j^n\right)' \sigma_j \right) dx = 0, \end{cases}$$

implies

$$\begin{cases} \rho_1 \int_0^1 \varphi_{tt}^n \varphi_t^n dx - k \int_0^1 \left(\varphi_x^n + \psi^n\right)_x \varphi_t^n dx + \mu_1 \int_0^1 \varphi_t^n \varphi_t^n dx = 0, \\ \\ -\rho_2 \int_0^1 \varphi_{ttx}^n \psi_t^n dx - b \int_0^1 \psi_{xx}^n \psi_t^n dx + k \int_0^1 \left(\varphi_x^n + \psi^n\right) \psi_t^n dx + \gamma \int_0^1 \omega_x^n \psi_t^n dx = 0, \\ \\ \rho_3 \int_0^1 \omega_t^n \omega^n dx - k_1 \int_0^1 \omega_{xx}^n \omega^n dx + \lambda \int_0^1 \omega^n \omega^n dx + \gamma \int_0^1 \psi_{tx}^n \omega^n dx = 0, \end{cases}$$

by using Integrating by parts with respect to x, we get

$$\begin{cases} \rho_1 \dfrac{d}{2dt} \displaystyle\int_0^1 (\varphi_t^n)^2 dx + k \int_0^1 (\varphi_x^n + \psi^n)\, \varphi_{tx}^n dx + \mu_1 \int_0^1 (\varphi_t^n)^2 dx = 0, \\ \rho_2 \displaystyle\int_0^1 \varphi_{tt}^n \psi_{tx}^n dx + b \int_0^1 \psi_x^n \psi_{tx}^n dx + k \int_0^1 (\varphi_x^n + \psi^n)\, \psi_t^n dx + \gamma \int_0^1 \omega_x^n \psi_t^n dx = 0, \\ \rho_3 \dfrac{d}{2dt} \displaystyle\int_0^1 (\omega^n)^2 dx + k_1 \int_0^1 (\omega_x^n)^2 dx + \lambda \int_0^1 (\omega^n)^2 dx - \gamma \int_0^1 \psi_t^n \omega_x^n dx = 0, \end{cases} \tag{2.1.27}$$

summing (2.1.27)$_1$, (2.1.27)$_2$ and (2.1.27)$_3$, we get

$$\rho_1 \frac{d}{2dt} \int_0^1 (\varphi_t^n)^2\, dx + k \frac{d}{2dt} \int_0^1 (\varphi_x^n + \psi^n)^2\, dx + \mu_1 \int_0^1 (\varphi_t^n)^2\, dx$$
$$+\rho_2 \int_0^1 \varphi_{tt}^n \psi_{tx}^n dx + \frac{bd}{2dt} \int_0^1 (\psi_x^n)^2\, dx + \rho_3 \frac{d}{2dt} \int_0^1 (\omega^n)^2\, dx$$
$$+k_1 \int_0^1 (\omega_x^n)^2\, dx + \lambda \int_0^1 (\omega^n)^2\, dx = 0.$$

$$\begin{aligned} &\rho_1 \frac{d}{2dt} \int_0^1 (\varphi_t^n)^2\, dx + k \frac{d}{2dt} \int_0^1 (\varphi_x^n + \psi^n)^2\, dx + \mu_1 \int_0^1 (\varphi_t^n)^2\, dx \\ &+\rho_2 \int_0^1 \varphi_{tt}^n \psi_{tx}^n dx + \frac{bd}{2dt} \int_0^1 (\psi_x^n)^2\, dx + \rho_3 \frac{d}{2dt} \int_0^1 (\omega^n)^2\, dx \\ &+k_1 \int_0^1 (\omega_x^n)^2\, dx + \lambda \int_0^1 (\omega^n)^2\, dx = 0. \end{aligned} \tag{2.1.28}$$

By (2.1.6)$_1$, we have

$$\psi_{xt}^n = \frac{\rho_1}{k} \varphi_{ttt}^n - \varphi_{xxt}^n + \frac{\mu_1}{k} \varphi_{tt}^n, \tag{2.1.29}$$

replacing (2.1.29) in (2.1.28), we arrive at

$$\rho_1 \frac{d}{2dt} \int_0^1 (\varphi_t^n)^2\, dx + k \frac{d}{2dt} \int_0^1 (\varphi_x^n + \psi^n)^2\, dx + \mu_1 \int_0^1 (\varphi_t^n)^2\, dx$$

$$+\rho_2 \int_0^1 \varphi_{tt}^n \left(\frac{\rho_1}{k}\varphi_{ttt}^n - \varphi_{xxt}^n + \frac{\mu_1}{k}\varphi_{tt}^n\right) dx + \frac{bd}{2dt}\int_0^1 (\psi_x^n)^2\, dx + \rho_3 \frac{d}{2dt}\int_0^1 (\omega^n)^2\, dx$$

$$+k_1 \int_0^1 (\omega_x^n)^2\, dx + \lambda \int_0^1 (\omega^n)^2\, dx \;=\; 0,$$

again by using integrating by parts, we get

$$\rho_1 \frac{d}{2dt}\int_0^1 (\varphi_t^n)^2\, dx + k\frac{d}{2dt}\int_0^1 (\varphi_x^n + \psi^n)^2\, dx + \mu_1 \int_0^1 (\varphi_t^n)^2\, dx$$

$$\frac{\rho_2\rho_1}{k}\frac{d}{2dt}\int_0^1 (\varphi_{tt}^n)^2\, dx + \rho_2 \frac{d}{2dt}\int_0^1 (\varphi_{tx}^n)^2\, dx + \frac{\rho_2\mu_1}{k}\int_0^1 (\varphi_{tt}^n)^2\, dx$$

$$+\frac{bd}{2dt}\int_0^1 (\psi_x^n)^2\, dx + \rho_3 \frac{d}{2dt}\int_0^1 (\omega^n)^2\, dx + k_1 \int_0^1 (\omega_x^n)^2\, dx + \lambda \int_0^1 (\omega^n)^2\, dx \;=\; 0,$$

which implies

$$\frac{d}{2dt}\int_0^1 (\rho_1 (\varphi_t^n)^2 + k(\varphi_x^n + \psi^n)^2 + \frac{\rho_2\rho_1}{k}(\varphi_{tt}^n)^2 + \rho_2 (\varphi_{tx}^n)^2 + b(\psi_x^n)^2$$

$$+\rho_3 (\omega^n)^2) dx$$

$$= \; -\mu_1 \int_0^1 (\varphi_t^n)^2\, dx - \frac{\rho_2\mu_1}{k}\int_0^1 (\varphi_{tt}^n)^2\, dx - k_1 \int_0^1 (\omega_x^n)^2\, dx - \lambda \int_0^1 (\omega^n)^2\, dx.$$

Then, we obtain

$$\frac{dE^n(t)}{dt} \le 0,$$

an integration over $(0,1)$ *, we get*

$$0 \le E^n(t) \le E^n(0), \tag{2.1.30}$$

where

$$\begin{aligned} E^n(t) \;&=\; \frac{1}{2}\int_0^1 \Big(\rho_1 (\varphi_t^n)^2 + k(\varphi_x^n + \psi^n)^2 + \frac{\rho_2\rho_1}{k}(\varphi_{tt}^n)^2 + \rho_2 (\varphi_{tx}^n)^2 + b(\psi_x^n)^2 \\ &\quad +\rho_3 (\omega^n)^2\Big)\, dx. \end{aligned}$$

Now, through-out (2.1.8) and (2.1.30), we can find a positive constant C independent of n such that

$$E^{n}(t) \leq C,\ \forall t \geq 0. \tag{2.1.31}$$

Therefore, from (2.1.31), we get $t_n = T$, for all $T > 0$.

A priori estimate II:

Differentiating the first, second and third equations of (2.1.6) with respect to t and multiplying by φ_{tt}^{n}, ψ_{tt}^{n} and ω_{t}^{n} respectively. By integrating over $(0,1)$, we get

$$\begin{cases} \rho_1 \int_0^1 \varphi_{ttt}^{n}\varphi_{tt}^{n}dx - k\int_0^1 \left(\varphi_x^n + \psi^n\right)_{xt}\varphi_{tt}^{n}dx + \mu_1\int_0^1 \varphi_{tt}^{n}\varphi_{tt}^{n}dx = 0, \\ -\rho_2\int_0^1 \varphi_{tttx}^{n}\psi_{tt}^{n}dx - b\int_0^1 \psi_{xxt}^{n}\psi_{tt}^{n}dx + k\int_0^1 \left(\varphi_x^n + \psi^n\right)_t \psi_{tt}^{n}dx \\ +\gamma\int_0^1 \omega_{xt}^{n}\psi_{tt}^{n}dx = 0, \\ \rho_3\int_0^1 \omega_{tt}^{n}\omega_{t}^{n}dx - k_1\int_0^1 \omega_{xxt}^{n}\omega_{t}^{n}dx + \lambda\int_0^1 \omega_{t}^{n}\omega_{t}^{n}dx + \gamma\int_0^1 \psi_{ttx}^{n}\omega_{t}^{n}dx = 0, \end{cases}$$

by using integrating by parts with respect to x, we obtain

$$\begin{cases} \rho_1\frac{d}{2dt}\int_0^1 (\varphi_{tt}^{n})^2dx + k\int_0^1 \left(\varphi_x^n + \psi^n\right)_t \varphi_{ttx}^{n}dx + \mu_1\int_0^1 \left(\varphi_{tt}^{n}\right)^2 dx = 0, \\ +\rho_2\int_0^1 \varphi_{ttt}^{n}\psi_{ttx}^{n}dx + b\frac{d}{2dt}\int_0^1 \left(\psi_{xt}^{n}\right)^2 dx + k\int_0^1 \left(\varphi_x^n + \psi^n\right)_t \psi_{tt}^{n}dx \\ +\gamma\int_0^1 \omega_{xt}^{n}\psi_{tt}^{n}dx = 0, \\ \rho_3\frac{d}{2dt}\int_0^1 (\omega_{t}^{n})dx + k_1\int_0^1 \left(\omega_{xt}^{n}\right)^2 dx + \lambda\int_0^1 \left(\omega_{t}^{n}\right)^2 dx - \gamma\int_0^1 \psi_{tt}^{n}\omega_{xt}^{n}dx = 0. \end{cases} \tag{2.1.32}$$

summing $(2.1.32)_1$, $(2.1.32)_2$ and $(2.1.32)_3$, we get

$$\rho_1\frac{d}{2dt}\int_0^1 (\varphi_{tt}^{n})^2dx + k\frac{d}{2dt}\int_0^1 \left(\varphi_x^n + \psi^n\right)_t^2 dx + \mu_1\int_0^1 \left(\varphi_{tt}^{n}\right)^2 dx \tag{2.1.33}$$

$$+\rho_2\int_0^1 \varphi^n_{ttt}\psi^n_{ttx}dx+b\frac{d}{2dt}\int_0^1 (\psi^n_{xt})^2\,dx+\rho_3\frac{d}{2dt}\int_0^1(\omega^n_t)dx$$
$$+k_1\int_0^1 (\omega^n_{xt})^2\,dx+\lambda\int_0^1 (\omega^n_t)^2\,dx \;=\; 0,$$

By (2.1.6)$_1$*, we have*

$$\psi^n_{ttx}=\frac{\rho_1}{k}\varphi^n_{tttt}-\varphi^n_{xxtt}+\frac{\mu_1}{k}\varphi^n_{ttt}, \tag{2.1.34}$$

and by substituting (2.1.34) *in* (2.1.33)*, we arrive at*

$$\rho_1\frac{d}{2dt}\int_0^1(\varphi^n_{tt})^2dx+k\frac{d}{2dt}\int_0^1 (\varphi^n_x+\psi^n)^2_t\,dx+\mu_1\int_0^1 (\varphi^n_{tt})^2\,dx$$
$$+\rho_2\int_0^1 \varphi^n_{ttt}\left(\frac{\rho_1}{k}\varphi^n_{tttt}-\varphi^n_{xxtt}+\frac{\mu_1}{k}\varphi^n_{ttt}\right)dx+b\frac{d}{2dt}\int_0^1 (\psi^n_{xt})^2\,dx$$
$$+\rho_3\frac{d}{2dt}\int_0^1(\omega^n_t)dx+k_1\int_0^1 (\omega^n_{xt})^2\,dx+\lambda\int_0^1 (\omega^n_t)^2\,dx \;=\; 0,$$

again by using integrating by parts with respect to x, we get

$$\rho_1\frac{d}{2dt}\int_0^1(\varphi^n_{tt})^2dx+k\frac{d}{2dt}\int_0^1 (\varphi^n_x+\psi^n)^2_t\,dx+\mu_1\int_0^1 (\varphi^n_{tt})^2\,dx$$
$$\frac{\rho_2\rho_1}{k}\frac{d}{2dt}\int_0^1 (\varphi^n_{ttt})^2\,dx+\rho_2\frac{d}{2dt}\int_0^1 (\varphi^n_{xtt})^2\,dx+\frac{\rho_2\mu_1}{k}\int_0^1 (\varphi^n_{ttt})^2\,dx$$
$$+b\frac{d}{2dt}\int_0^1 (\psi^n_{xt})^2\,dx+\rho_3\frac{d}{2dt}\int_0^1(\omega^n_t)dx+k_1\int_0^1 (\omega^n_{xt})^2\,dx$$
$$+\lambda\int_0^1 (\omega^n_t)^2\,dx \;=\; 0,$$

which implies

$$\frac{d}{2dt}\int_0^1(\rho_1\,(\varphi^n_{tt})^2+k\,(\varphi^n_x+\psi^n)^2_t+\frac{\rho_2\rho_1}{k}\,(\varphi^n_{ttt})^2+\rho_2\,(\varphi^n_{xtt})^2+b\,(\psi^n_{xt})^2$$
$$+\,\rho_3\,(\omega^n_t)^2)dx$$

$$= -\mu_1 \int_0^1 (\varphi_{tt}^n)^2 \, dx - \frac{\rho_2 \mu_1}{k} \int_0^1 (\varphi_{ttt}^n)^2 \, dx - k_1 \int_0^1 (\omega_{xt}^n)^2 \, dx - \lambda \int_0^1 (\omega_t^n)^2 \, dx.$$

Then, we obtain

$$\frac{dE_1^n(t)}{dt} \leq 0,$$

an integration over $(0,1)$ *, we get*

$$0 \leq E_1^n(t) \leq E_1^n(0), \; \forall t \geq 0, \tag{2.1.35}$$

where

$$E_1^n(t) = \frac{1}{2} \int_0^1 \Big(\rho_1 (\varphi_{tt}^n)^2 + k (\varphi_x^n + \psi^n)_t^2 + \frac{\rho_2 \rho_1}{k} (\varphi_{ttt}^n)^2 + \rho_2 (\varphi_{xtt}^n)^2 + b (\psi_{xt}^n)^2$$
$$+ \rho_3 (\omega_t^n)^2 \Big) \, dx.$$

Again, by (2.1.8) and (2.1.35), we can find a positive constant C_1 *independent of* n *such that*

$$E_1^n(t) \leq C_1, \; \forall t \geq 0. \tag{2.1.36}$$

A priori estimate III:

Multiplying $(2.1.6)_1$, $(2.1.6)_2$, $(2.1.6)_3$*, respectively, by* $-\varphi_{txx}^n$, $-\psi_{txx}^n$, $-\omega_{xx}^n$*, and integrating over* $(0,1)$ *, we get*

$$\begin{cases} \rho_1 \int_0^1 \varphi_{tt}^n (-\varphi_{txx}^n) \, dx - k \int_0^1 (\varphi_x^n + \psi^n)_x (-\varphi_{txx}^n) \, dx + \mu_1 \int_0^1 \varphi_t^n (-\varphi_{txx}^n) \, dx = 0, \\ -\rho_2 \int_0^1 \varphi_{ttx}^n (-\psi_{txx}^n) \, dx - b \int_0^1 \psi_{xx}^n (-\psi_{txx}^n) \, dx + k \int_0^1 (\varphi_x^n + \psi^n)(-\psi_{txx}^n) \, dx \\ + \gamma \int_0^1 \omega_x^n (-\psi_{txx}^n) \, dx = 0, \\ \rho_3 \int_0^1 \omega_t^n (-\omega_{xx}^n) \, dx - k_1 \int_0^1 \omega_{xx}^n (-\omega_{xx}^n) \, dx + \lambda \int_0^1 \omega^n (-\omega_{xx}^n) \, dx \\ + \gamma \int_0^1 \psi_{tx}^n (-\omega_{xx}^n) \, dx = 0, \end{cases}$$

by using integrating by parts with respect to x, we obtain

$$
\begin{cases}
\rho_1 \dfrac{d}{2dt} \displaystyle\int_0^1 (\varphi_{tx}^n)^2 dx + k \int_0^1 (\varphi_x^n + \psi^n)_x \, \varphi_{txx}^n dx + \mu_1 \int_0^1 (\varphi_{tx}^n)^2 \, dx = 0, \\
\rho_2 \displaystyle\int_0^1 \varphi_{ttx}^n \psi_{txx}^n dx + b \dfrac{d}{2dt} \int_0^1 (\psi_{xx}^n)^2 dx + k \int_0^1 (\varphi_x^n + \psi^n)_x \, \psi_{tx}^n dx \\
+\gamma \displaystyle\int_0^1 \omega_{xx}^n \psi_{tx}^n dx = 0, \\
\rho_3 \dfrac{d}{2dt} \displaystyle\int_0^1 (\omega_x^n) dx + k_1 \int_0^1 (\omega_{xx}^n)^2 \, dx + \lambda \int_0^1 (\omega_x^n)^2 \, dx - \gamma \int_0^1 \psi_{tx}^n \omega_{xx}^n dx = 0.
\end{cases}
\tag{2.1.37}
$$

summing (2.1.39)$_1$, (2.1.39)$_2$ and (2.1.39)$_3$, we get

$$
\begin{aligned}
&\rho_1 \frac{d}{2dt} \int_0^1 (\varphi_{tx}^n)^2 dx + k \frac{d}{2dt} \int_0^1 (\varphi_{xx}^n + \psi_x^n)^2 \, dx + \mu_1 \int_0^1 (\varphi_{tx}^n)^2 \, dx \\
&+\rho_2 \int_0^1 \varphi_{ttx}^n \psi_{txx}^n dx + b \frac{d}{2dt} \int_0^1 (\psi_{xx}^n)^2 dx + \rho_3 \frac{d}{2dt} \int_0^1 (\omega_x^n)^2 dx \\
&+k_1 \int_0^1 (\omega_{xx}^n)^2 \, dx + \lambda \int_0^1 (\omega_x^n)^2 \, dx \; = \; 0,
\end{aligned}
\tag{2.1.38}
$$

By differentiating the equation (2.1.6)$_1$ with respect to t and x, we get

$$
\psi_{txx}^n = \frac{\rho_1}{k} \varphi_{tttx}^n - \varphi_{txxx}^n + \frac{\mu_1}{k} \varphi_{ttx}^n,
\tag{2.1.39}
$$

and by inserting (2.1.39) in (2.1.38), we get

$$
\begin{aligned}
&\rho_1 \frac{d}{2dt} \int_0^1 (\varphi_{tx}^n)^2 dx + k \frac{d}{2dt} \int_0^1 (\varphi_{xx}^n + \psi_x^n)^2 \, dx + \mu_1 \int_0^1 (\varphi_{tx}^n)^2 \, dx \\
&+\rho_2 \int_0^1 \varphi_{ttx}^n \left(\frac{\rho_1}{k} \varphi_{tttx}^n - \varphi_{txxx}^n + \frac{\mu_1}{k} \varphi_{ttx}^n \right) dx + b \frac{d}{2dt} \int_0^1 (\psi_{xx}^n)^2 dx \\
&+\rho_3 \frac{d}{2dt} \int_0^1 (\omega_x^n)^2 dx + k_1 \int_0^1 (\omega_{xx}^n)^2 \, dx + \lambda \int_0^1 (\omega_x^n)^2 \, dx \; = \; 0,
\end{aligned}
$$

again by using integrating by parts with respect to x*, we obtain*

$$\rho_1 \frac{d}{2dt}\int_0^1 (\varphi_{tx}^n)^2 dx + k\frac{d}{2dt}\int_0^1 (\varphi_{xx}^n + \psi_x^n)^2\, dx + \mu_1 \int_0^1 (\varphi_{tx}^n)^2\, dx$$

$$\frac{\rho_2\rho_1}{k}\frac{d}{2dt}\int_0^1 (\varphi_{ttx}^n)^2\, dx + \rho_2\frac{d}{2dt} + \frac{\rho_2\mu_1}{k}\int_0^1 (\varphi_{ttx}^n)^2\, dx$$

$$+b\frac{d}{2dt}\int_0^1 (\psi_{xx}^n)^2 dx + \rho_3\frac{d}{2dt}\int_0^1 (\omega_x^n)^2 dx + k_1\int_0^1 (\omega_{xx}^n)^2\, dx$$

$$+\lambda\int_0^1 (\omega_x^n)^2\, dx \;=\; 0,$$

which implies

$$\begin{aligned}
&\frac{d}{2dt}\int_0^1 (\rho_1\,(\varphi_{tx}^n)^2 + k\,(\varphi_{xx}^n + \psi_x^n)^2 + \frac{\rho_2\rho_1}{k}\,(\varphi_{ttx}^n)^2 + \rho_2\,(\varphi_{txx}^n)^2 + b\,(\psi_{xx}^n)^2 \\
&+\rho_3\,(\omega_x^n)^2)dx \\
&= \; -\mu_1\int_0^1 (\varphi_{tx}^n)^2\, dx - \frac{\rho_2\mu_1}{k}\int_0^1 (\varphi_{ttx}^n)^2\, dx - k_1\int_0^1 (\omega_{xx}^n)^2\, dx \\
&-\lambda\int_0^1 (\omega_x^n)^2\, dx.
\end{aligned} \tag{2.1.40}$$

then, we obtain

$$\frac{dE_2^n(t)}{dt} \leq 0,$$

an integration over $(0,1)$ *, we get*

$$0 \leq E_2^n(t) \leq E_2^n(0)\,,\; \forall t \geq 0. \tag{2.1.41}$$

where

$$E_2^n(t) = \frac{1}{2}\int_0^1 \Big(\rho_1 (\varphi_{tx}^n)^2 + k(\varphi_{xx}^n + \psi_x^n)^2 + \frac{\rho_2\rho_1}{k}(\varphi_{ttx}^n)^2 + \rho_2(\varphi_{txx}^n)^2 + b(\psi_{xx}^n)^2$$

$$+\rho_3(\omega_x^n)^2\Big)\,dx.$$

Similarly to a priori estimates I and II, there exists a positive constant C_2 independent on n such that

$$E_2^n(t) \leq C_2,\ \forall t \geq 0. \tag{2.1.42}$$

A priori estimate IV:

Differentiating the first, second and third equations of (2.1.6) with respect to x, we obtain

$$\begin{cases} \rho_1\varphi_{ttx}^n - k(\varphi_x^n + \psi^n)_{xx} + \mu_1\varphi_{tx}^n = 0, \\ -\rho_2\varphi_{ttxx}^n - b\psi_{xxx}^n + k(\varphi_x^n + \psi^n)_x + \gamma\omega_{xx}^n = 0, \\ \rho_3\omega_{xt}^n - k_1\omega_{xxx}^n + \lambda\omega_x^n + \gamma\psi_{txx}^n = 0. \end{cases} \tag{2.1.43}$$

Now, multiplying $(2.1.43)_1$ by $-\varphi_{txxx}^n$, $(2.1.43)_2$ by $-\psi_{txxx}^n$ and $(2.1.43)_3$ by $-\omega_{xxx}^n$ and integrating over $(0,1)$, we get

$$\begin{cases} \rho_1\int_0^1 \varphi_{ttx}^n(-\varphi_{txxx}^n)\,dx - k\int_0^1 (\varphi_x^n + \psi^n)_{xx}(-\varphi_{txxx}^n)\,dx + \mu_1\int_0^1 \varphi_{tx}^n(-\varphi_{txxx}^n)\,dx = 0, \\ \\ -\rho_2\int_0^1 \varphi_{ttxx}^n(-\psi_{txxx}^n)\,dx - b\int_0^1 \psi_{xxx}^n(-\psi_{txxx}^n)\,dx + k\int_0^1 (\varphi_x^n + \psi^n)_x(-\psi_{txxx}^n)\,dx \\ \\ +\gamma\int_0^1 \omega_{xx}^n(-\psi_{txxx}^n)\,dx = 0, \\ \\ \rho_3\int_0^1 \omega_{xt}^n(-\omega_{xxx}^n)\,dx - k_1\int_0^1 \omega_{xxx}^n(-\omega_{xxx}^n)\,dx + \lambda\int_0^1 \omega_x^n(-\omega_{xxx}^n)\,dx \\ \\ +\gamma\int_0^1 \psi_{txx}^n(-\omega_{xxx}^n)\,dx = 0, \end{cases}$$

by using integrating by parts with respect to x, we obtain

$$\begin{cases} \rho_1 \dfrac{d}{2dt}\displaystyle\int_0^1 (\varphi^n_{txx})^2 dx + k\int_0^1 (\varphi^n_x + \psi^n)_{xx}\, \varphi^n_{txxx} dx + \mu_1 \int_0^1 (\varphi^n_{txx})^2 dx = 0, \\ \rho_2 \displaystyle\int_0^1 \varphi^n_{ttxx}\psi^n_{txxx} dx + b\dfrac{d}{2dt}\int_0^1 (\psi^n_{xxx})^2 dx + k\int_0^1 (\varphi^n_x + \psi^n)_{xx}\, \psi^n_{txx} dx \\ +\gamma \displaystyle\int_0^1 \omega^n_{xxx}\psi^n_{txx} dx = 0, \\ \rho_3 \dfrac{d}{2dt}\displaystyle\int_0^1 (\omega^n_{xx})^2 dx + k_1 \int_0^1 (\omega^n_{xxx})^2 dx + \lambda \int_0^1 (\omega^n_{xx})^2 dx \\ -\gamma \displaystyle\int_0^1 \psi^n_{txx}\omega^n_{xxx} dx = 0. \end{cases} \tag{2.1.44}$$

summing (2.1.44)$_1$, (2.1.44)$_2$ and (2.1.44)$_3$, we get

$$\rho_1 \frac{d}{2dt}\int_0^1 (\varphi^n_{txx})^2 dx + k\frac{d}{2dt}\int_0^1 (\varphi^n_x + \psi^n)^2_{xx}\, dx + \mu_1 \int_0^1 (\varphi^n_{txx})^2 dx \tag{2.1.45}$$

$$\rho_2 \int_0^1 \varphi^n_{ttxx}\psi^n_{txxx} dx + b\frac{d}{2dt}\int_0^1 (\psi^n_{xxx})^2 dx + \gamma\int_0^1 \omega^n_{xxx}\psi^n_{txx} dx$$

$$+\rho_3 \frac{d}{2dt}\int_0^1 (\omega^n_{xx})^2 dx + k_1\int_0^1 (\omega^n_{xxx})^2 dx + \lambda\int_0^1 (\omega^n_{xx})^2 dx - \gamma\int_0^1 \psi^n_{txx}\omega^n_{xxx} dx \;=\; 0,$$

By differentiating the equation (2.1.43)$_1$ with respect to t and x, we obtain

$$\psi^n_{txxx} = \frac{\rho_1}{k}\varphi^n_{tttxx} - \varphi^n_{txxxx} + \frac{\mu_1}{k}\varphi^n_{ttxx}, \tag{2.1.46}$$

and by substituting (2.1.46) in (2.1.45), we get

$$\rho_1 \frac{d}{2dt}\int_0^1 (\varphi^n_{txx})^2 dx + k\frac{d}{2dt}\int_0^1 (\varphi^n_x + \psi^n)^2_{xx}\, dx + \mu_1 \int_0^1 (\varphi^n_{txx})^2 dx$$

$$\rho_2 \int_0^1 \varphi^n_{ttxx}\left(\frac{\rho_1}{k}\varphi^n_{tttxx} - \varphi^n_{txxxx} + \frac{\mu_1}{k}\varphi^n_{ttxx}\right) dx + b\frac{d}{2dt}\int_0^1 (\psi^n_{xxx})^2 dx$$

$$+\rho_3 \frac{d}{2dt}\int_0^1 (\omega_{xx}^n)^2 dx + k_1 \int_0^1 (\omega_{xxx}^n)^2 dx + \lambda \int_0^1 (\omega_{xx}^n)^2 dx \;=\; 0,$$

again by using integrating by parts with respect to x, we get

$$\rho_1 \frac{d}{2dt}\int_0^1 (\varphi_{txx}^n)^2 dx + k\frac{d}{2dt}\int_0^1 (\varphi_x^n + \psi^n)_{xx}^2\, dx + \mu_1 \int_0^1 (\varphi_{txx}^n)^2 dx$$

$$+\frac{\rho_2\rho_1}{k}\left(\varphi_{ttxx}^n\right)^2 dx + \rho_2 \frac{d}{2dt}\int_0^1 \left(\varphi_{txxx}^n\right)^2 dx + \frac{\mu_1\rho_2}{k}\int_0^1 \left(\varphi_{ttxx}^n\right)^2 dx$$

$$+b\frac{d}{2dt}\int_0^1 (\psi_{xxx}^n)^2 dx + \rho_3 \frac{d}{2dt}\int_0^1 (\omega_{xx}^n)^2 dx + k_1 \int_0^1 (\omega_{xxx}^n)^2 dx$$

$$+\lambda \int_0^1 (\omega_{xx}^n)^2 dx \;=\; 0,$$

which implies

$$\frac{d}{2dt}\int_0^1 (\rho_1 \left(\varphi_{txx}^n\right)^2 + k\left(\varphi_x^n + \psi^n\right)_{xx}^2 + \frac{\rho_2\rho_1}{k}\left(\varphi_{ttxx}^n\right)^2 + \rho_2\left(\varphi_{txxx}^n\right)^2 + b\left(\psi_{xxx}^n\right)^2$$

$$+\rho_3 \left(\omega_{xx}^n\right)^2) dx \tag{2.1.47}$$

$$= -\mu_1 \int_0^1 \left(\varphi_{txx}^n\right)^2 dx - \frac{\mu_1\rho_2}{k}\int_0^1 \left(\varphi_{ttxx}^n\right)^2 dx - k_1 \int_0^1 \left(\omega_{xxx}^n\right)^2 dx - \lambda \int_0^1 \left(\omega_{xx}^n\right)^2 dx.$$

Since

$$e_{jxx} = \lambda e_j,$$
$$\sigma_{jxx} = \zeta \sigma_j,$$

then, we get

$$\int_0^1 \left(\varphi_{txxx}^n\right)^2 dx = \zeta^2 \int_0^1 \left(\varphi_{tx}^n\right)^2 dx, \tag{2.1.48}$$

and

$$\int_0^1 \left(\psi_{xxx}^n\right)^2 dx = \lambda^2 \int_0^1 \left(\psi_x^n\right)^2 dx. \tag{2.1.49}$$

Substituting (2.1.48) and (2.1.49) in (2.1.47), we obtain

$$\frac{d}{2dt}\int_0^1 (\rho_1 (\varphi_{txx}^n)^2 + k(\zeta\varphi_x^n + \psi_{xx}^n)^2 + \frac{\rho_2\rho_1}{k}(\varphi_{ttxx}^n)^2 + \rho_2\zeta^2(\varphi_{tx}^n)^2 + \lambda^2(\psi_x^n)^2$$

$$+\rho_3(\omega_{xx}^n)^2)dx \tag{2.1.50}$$

$$= -\mu_1\int_0^1 (\varphi_{txx}^n)^2\,dx - \frac{\mu_1\rho_2}{k}\int_0^1 (\varphi_{ttxx}^n)^2\,dx - k_1\int_0^1 (\omega_{xxx}^n)^2\,dx - \lambda\int_0^1 (\omega_{xx}^n)^2\,dx.$$

then, we obtain

$$\frac{dE_3^n(t)}{dt} \leq 0,$$

an integration over $(0,1)$ *, we get*

$$0 \leq E_3^n(t) \leq E_3^n(0)\,,\ \forall t \geq 0, \tag{2.1.51}$$

where

$$E_3^n(t) = \frac{1}{2}\int_0^1 \Big(\rho_1(\varphi_{txx}^n)^2 + k(\lambda\varphi_x^n + \psi_{xx}^n)^2 + \frac{\rho_2\rho_1}{k}(\varphi_{ttxx}^n)^2 + \rho_2\lambda^2(\varphi_{tx}^n)^2$$

$$+\zeta^2(\psi_x^n)^2 + \rho_3(\omega_{xx}^n)^2\Big)\,dx.$$

Finally, using (2.1.8) and (2.1.51), we deduce that there exists a positive constant C_3 *independent on* n *such that*

$$E_3^n(t) \leq C_3,\ \forall t \geq 0. \tag{2.1.52}$$

Step 3: Passage to the limit:

By using compactness results, we extract a converging subsequence and by passing to the limit in the equations yields a strong solution to the original infinite-dimensional problem (0.0.1).

From (2.1.31), (2.1.36), (2.1.42), (2.1.52) and Poincaré inequality, we conclude that

$$\begin{aligned}
&(\varphi^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right),\\
&(\varphi_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T;L^2(0,1)\right),\\
&(\psi^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T;H^2(0,1)\cap H_0^1(0,1)\right),
\end{aligned}$$

$$\begin{aligned} &(\psi_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T;L^2(0,1)\right), \\ &(\omega^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T;\ H_*^2(0,1)\cap H_*^1(0,1)\right), \\ &(\omega_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T;\ L^2(0,1)\right), \end{aligned} \tag{2.1.53}$$

By using Aubin–Lions–Simon theorem, since

The embedding of $H_*^1(0,1)$ in $L^2(0,1)$ is continuous.

The embedding of $H_*^2(0,1)\cap H_*^1(0,1)$ in $H_*^1(0,1)$ is compact.

The embedding of $H_0^1(0,1)$ in $L^2(0,1)$ is continuous.

The embedding of $H^2(0,1)\cap H_0^1(0,1)$ in $H_0^1(0,1)$ is compact.

Then, we get the embedding of $E_{\infty,\infty}$ *in* $C(0,T;H_*^1(0,1))$ *is compact where*

$$\begin{aligned} E_{\infty,\infty} = \Big\{\varphi^n/\ \varphi^n \in L^\infty\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right) \\ \text{and } \varphi_t^n = \frac{d\varphi^n}{dt} \in L^\infty\left(0,T;L_*^2(0,1)\right)\Big\}, \end{aligned}$$

also, the embedding of $\tilde{E}_{\infty,\infty}$ *in* $C(0,T;\ H_0^1(0,1))$ *is compact where*

$$\begin{aligned} \tilde{E}_{\infty,\infty} = \Big\{\psi^n/\psi^n \in L^\infty\left(0,T;H^2(0,1)\cap H_0^1(0,1)\right) \\ \text{and } \psi_t^n = \frac{d\psi^n}{dt} \in L^\infty\left(0,T;L^2(0,1)\right)\Big\}, \end{aligned}$$

by (2.1.53), we have $(\psi^n)_{n\in\mathbb{N}^*}$ *bounded in* $\tilde{E}_{\infty,\infty}$ *and* $(\omega^n)_{n\in\mathbb{N}^*}$, $(\varphi^n)_{n\in\mathbb{N}^*}$ *bounded in* $E_{\infty,\infty}$. *Then, there exist* $(\psi^m)_{m\geq1}$ *subsequence of* $(\psi^n)_{n\geq1}$ *and* $(\varphi^m)_{m\geq1}$, $(\omega^m)_{m\geq1}$ *subsequences of* $(\varphi^n)_{n\geq1}$ *and* $(\omega^n)_{n\geq1}$, *respectively, such that*

$$\begin{cases} \varphi^m \overset{m\to\infty}{\longrightarrow} \varphi \text{ strongly in } C(0,T;H_*^1(0,1)),\ \forall T>0, \\ \psi^m \overset{m\to\infty}{\longrightarrow} \psi \text{ strongly in } C(0,T;H_0^1(0,1)),\ \forall T>0, \\ \omega^m \overset{m\to\infty}{\longrightarrow} \omega \text{ strongly in } C(0,T;H_*^1(0,1)).\ \forall T>0, \end{cases} \tag{2.1.54}$$

Through (2.1.5), because $\left(y_j^n, h_j^n, R_j^n\right)_{j=1,..,n} \in \left[C^2(0,T)\right]^3$, $\forall T > 0$, *and*

$$\begin{aligned} (e_j)_{j\geq 1} &\subset C(0,L), \\ (\sigma_j)_{j\geq 1} &\subset C(0,L), \end{aligned}$$

then

$$\begin{cases} \varphi^n \in C^2\left(0,T;\ H_*^2(0,1) \cap H_*^1(0,1)\right), \ \forall n \in \mathbb{N}^*, \\ \psi^n \in C^2\left(0,T;\ H^2(0,1) \cap H_0^1(0,1)\right), \forall n \in \mathbb{N}^*, \\ \omega^n \in C^2\left(0,T;\ H_*^2(0,1) \cap H_*^1(0,1)\right). \forall n \in \mathbb{N}^*, \end{cases} \tag{2.1.55}$$

As $H_*^1(0,1) \hookrightarrow C(0,1)$, $H_0^1(0,1) \hookrightarrow C(0,1)$, *where* $(\hookrightarrow)$ *represent the continuous embedding, then we have the convergence in* $H_*^1(0,1)$ *and* $H_0^1(0,1)$ *implies the uniform convergence in* $C(0,1)$, *hence we have*

$$\begin{cases} |\varphi^m(x,t) - \varphi(x,t)| \leq \sup\limits_{x\in(0,1)} |\varphi^m(x,t) - \varphi(x,t)| \overset{m\to\infty}{\to} 0, \forall t \in [0,T], \\ |\psi^m(x,t) - \psi(x,t)| \leq \sup\limits_{x\in(0,1)} |\psi^m(x,t) - \psi(x,t)| \overset{m\to\infty}{\to} 0, \forall t \in [0,T], \\ |\omega^m(x,t) - \omega(x,t)| \leq \sup\limits_{x\in(0,1)} |\omega^m(x,t) - \omega(x,t)| \overset{m\to\infty}{\to} 0, \forall t \in [0,T], \end{cases} \tag{2.1.56}$$

Through (2.1.56) we obtain $\{\varphi^m\}_{m\geq 1}$ *simply converges to* φ *and* $\{\psi^m\}_{m\geq 1}$ *simply converges to* ψ, $\{\omega^m\}_{m\geq 1}$ *simply converges to* ω *for any* $t \in [0,T]$. *Also by have previous simply converges and dominated convergence theorem, we obtain for any* $t \in [0,T]$ *and* $\forall k \in \mathbb{N}^*$

$$\begin{cases} \lim\limits_{m\to\infty} \left\|\varphi_t^m(x,t) - \varphi_t^{m+k}(x,t)\right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left|\varphi_t^m(x,t) - \varphi_t^{m+k}(x,t)\right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left|\varphi_t^m(x,t) - \varphi_t^{m+k}(x,t)\right|^2 dx = 0, \\ \\ \lim\limits_{m\to\infty} \left\|\psi_t^m(x,t) - \psi_t^{m+k}(x,t)\right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left|\psi_t^m(x,t) - \psi_t^{m+k}(x,t)\right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left|\psi_t^m(x,t) - \psi_t^{m+k}(x,t)\right|^2 dx = 0, \\ \\ \lim\limits_{m\to\infty} \left\|\omega_t^m(x,t) - \omega_t^{m+k}(x,t)\right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left|\omega_t^m(x,t) - \omega_t^{m+k}(x,t)\right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left|\omega_t^m(x,t) - \omega_t^{m+k}(x,t)\right|^2 dx = 0, \end{cases} \tag{2.1.57}$$

also

$$\begin{cases} \lim\limits_{m\to\infty} \left\| \varphi_{tx}^{m}(x,t) - \varphi_{tx}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left| \varphi_{tx}^{m}(x,t) - \varphi_{tx}^{m+k}(x,t) \right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left| \varphi_{tx}^{m}(x,t) - \varphi_{tx}^{m+k}(x,t) \right|^2 dx = 0, \\ \\ \lim\limits_{m\to\infty} \left\| \psi_{tx}^{m}(x,t) - \psi_{tx}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left| \psi_{tx}^{m}(x,t) - \psi_{tx}^{m+k}(x,t) \right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left| \psi_{tx}^{m}(x,t) - \psi_{tx}^{m+k}(x,t) \right|^2 dx = 0, \\ \\ \lim\limits_{m\to\infty} \left\| \omega_{tx}^{m}(x,t) - \omega_{tx}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left| \omega_{tx}^{m}(x,t) - \omega_{tx}^{m+k}(x,t) \right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left| \omega_{tx}^{m}(x,t) - \omega_{tx}^{m+k}(x,t) \right|^2 dx = 0, \end{cases} \tag{2.1.58}$$

Using (2.1.57), (2.1.58), *we get*

$$\begin{cases} \lim\limits_{m\to\infty} \sup\limits_{t\in[0,1]} \left\| \varphi_t^{m}(x,t) - \varphi_t^{m+k}(x,t) \right\|_{H_*^1(0,1)}^2 = 0, \\ \lim\limits_{m\to\infty} \sup\limits_{t\in[0,1]} \left\| \psi_t^{m}(x,t) - \psi_t^{m+k}(x,t) \right\|_{H_0^1(0,1)}^2 = 0, \\ \lim\limits_{m\to\infty} \sup\limits_{t\in[0,1]} \left\| \omega_t^{m}(x,t) - \omega_t^{m+k}(x,t) \right\|_{H_*^1(0,1)}^2 = 0, \end{cases}$$

i.e $(\varphi_t^n)_{n\geq1}$ *and* $(\psi_t^n)_{n\geq1}$, $(\omega_t^n)_{n\geq1}$ *Cauchy sequences in* $X = C\left(0,T;H_*^1(0,1)\right)$, $Y = C(0,T;H_0^1(0,1))$ *and* $X = C\left(0,T;H_*^1(0,1)\right)$ *by the same order, implies*

$$\begin{cases} \|\varphi\|_X = \sup\limits_{t\in[0,1]} \|\varphi(x,t)\|_{H_*^1(0,1)}, \\ \|\psi\|_Y = \sup\limits_{t\in[0,1]} \|\psi(x,t)\|_{H_0^1(0,1)}, \\ \|\omega\|_X = \sup\limits_{t\in[0,1]} \|\omega(x,t)\|_{H_*^1(0,1)}, \end{cases}$$

as $(X,\|\cdot\|_X)$ *and* $(Y,\|\cdot\|_Y)$ *is a Banach spaces, then there exists a unique* g *and* h *in* X *and a unique* f *in*

Y, means that

$$\begin{cases} \varphi_t^m \overset{m\to\infty}{\to} g \text{ strongly in } X = C\left(0,T;H_*^1(0,1)\right), \\ \psi_t^m \overset{m\to\infty}{\to} f \text{ strongly in } Y = C\left(0,T;H_0^1(0,1)\right), \\ \omega_t^m \overset{m\to\infty}{\to} h \text{ strongly in } X = C\left(0,T;H_*^1(0,1)\right), \end{cases} \tag{2.1.59}$$

We prove that $g = \varphi_t$ and $\omega_t = h$, as the operator A define as follows

$$A : D(A) = C^1\left(0,T;H_*^1(0,1)\right) \subset C\left(0,T;H_*^1(0,1)\right) \to C\left(0,T;H_*^1(0,1)\right),$$
$$\varphi \to \varphi_t$$

is closed i.e if $(\varphi^m)_{m\geq 1} \subset D(A)$ converges strongly to $\varphi \in C\left(0,T;H_*^1(0,1)\right)$ and $(\varphi_t^m)_{m\geq 1} = A\varphi^m$ converges strongly to $g \in C\left(0,T;H_*^1(0,1)\right)$, then we get $\varphi \in C^1\left(0,T;H_*^1(0,1)\right)$ and $g = A\varphi = \varphi_t$, using (2.1.54) and (2.1.59), we directly obtain.

$$\varphi_t^m \overset{m\to\infty}{\longrightarrow} \varphi_t \text{ strongly in } X = C(0,T;H_*^1(0,1)),$$

and by using same method, we conclude that

$$\omega_t^m \overset{m\to\infty}{\longrightarrow} \omega_t \text{ strongly in } X = C(0,T;H_*^1(0,1)),$$

Also, we define the operator B as follows to prove $f = \psi_t$

$$B : D(B) = C^1\left(0,T;H_0^1(0,1)\right) \subset C\left(0,T;H_0^1(0,1)\right) \to C\left(0,T;H_0^1(0,1)\right),$$
$$\psi \to \psi_t$$

is closed i.e if $(\psi^m)_{m\geq 1} \subset D(B)$ converges strongly to $\psi \in C\left(0,T;H_0^1(0,1)\right)$ and $(\psi_t^m)_{m\geq 1} = A\psi^m$ converges strongly to $f \in C\left(0,T;H_0^1(0,1)\right)$, then we get $\psi \in C^1\left(0,T;H_0^1(0,1)\right)$ and $g = A\psi = \psi_t$ using (2.1.54) and (2.1.59), We directly obtain

$$\psi_t^m \overset{m\to\infty}{\longrightarrow} \psi_t \text{ strongly in } Y = C(0,T;H_0^1(0,1)),$$

we named

$$\begin{cases} \varphi_t^m \overset{m\to\infty}{\longrightarrow} \varphi_t \text{ strongly in } X = C(0,T;H_*^1(0,1)), \\ \psi_t^m \overset{m\to\infty}{\longrightarrow} \psi_t \text{ strongly in } Y = C(0,T;H_0^1(0,1)), \\ \omega_t^m \overset{m\to\infty}{\longrightarrow} \omega_t \text{ strongly in } X = C(0,T;H_*^1(0,1)), \end{cases} \tag{2.1.60}$$

Again, by using the previous simply converges and dominated convergence theorem and (2.1.55), (2.1.60), we get

$$\begin{cases} \lim\limits_{m\to\infty} \left\| \varphi_{tt}^m(x,t) - \varphi_{tt}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left| \varphi_{tt}^m(x,t) - \varphi_{tt}^{m+k}(x,t) \right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left| \varphi_{tt}^m(x,t) - \varphi_{tt}^{m+k}(x,t) \right|^2 dx = 0, \\ \\ \lim\limits_{m\to\infty} \left\| \psi_{tt}^m(x,t) - \psi_{tt}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left| \psi_{tt}^m(x,t) - \psi_{tt}^{m+k}(x,t) \right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left| \psi_{tt}^m(x,t) - \psi_{tt}^{m+k}(x,t) \right|^2 dx = 0, \\ \\ \lim\limits_{m\to\infty} \left\| \omega_{tt}^m(x,t) - \omega_{tt}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = \lim\limits_{m\to\infty} \int_0^1 \left| \omega_{tt}^m(x,t) - \omega_{tt}^{m+k}(x,t) \right|^2 dx \\ = \int_0^1 \lim\limits_{m\to\infty} \left| \omega_{tt}^m(x,t) - \omega_{tt}^{m+k}(x,t) \right|^2 dx = 0, \end{cases} \tag{2.1.61}$$

(2.1.61) implies

$$\begin{cases} \lim\limits_{m\to\infty} \sup\limits_{t\in[0,1]} \left\| \varphi_{tt}^m(x,t) - \varphi_{tt}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = 0, \\ \lim\limits_{m\to\infty} \sup\limits_{t\in[0,1]} \left\| \psi_{tt}^m(x,t) - \psi_{tt}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = 0, \\ \lim\limits_{m\to\infty} \sup\limits_{t\in[0,1]} \left\| \omega_{tt}^m(x,t) - \omega_{tt}^{m+k}(x,t) \right\|_{L^2(0,1)}^2 = 0, \end{cases}$$

i.e $(\varphi_{tt}^n)_{n\geq 1}$ *and* $(\psi_{tt}^n)_{n\geq 1}$, $(\omega_{tt}^n)_{n\geq 1}$ *Cauchy sequences in* $Z = C(0,T;L^2(0,1))$, *equipped by this norm*

$$\begin{cases} \|\varphi\|_Z = \sup\limits_{t\in[0,T]} \|\varphi(x,t)\|_{L^2(0,1)}, \\ \|\psi\|_Z = \sup\limits_{t\in[0,T]} \|\psi(x,t)\|_{L^2(0,1)}, \\ \|\omega\|_Z = \sup\limits_{t\in[0,T]} \|\omega(x,t)\|_{L^2(0,1)}, \end{cases}$$

As $((Z = C(0,T;L^2(0,1)), \|\cdot\|_Z)$ *is a Banach space, then there exists a unique* J, L *and* $S \in C(0,T;L^2(0,1))$

such that

$$\begin{cases} \varphi_{tt}^{m} \overset{m\to\infty}{\longrightarrow} J \text{ strongly in } C(0,T;L^{2}(0,1)), \\ \psi_{tt}^{m} \overset{m\to\infty}{\longrightarrow} L \text{ strongly in } C(0,T;L^{2}(0,1)), \\ \omega_{tt}^{m} \overset{m\to\infty}{\longrightarrow} S \text{ strongly in } C(0,T;L^{2}(0,1)). \end{cases} \tag{2.1.62}$$

Notice through (2.1.54) and (2.1.60), we get

$$\begin{cases} \varphi^{m} \overset{m\to\infty}{\longrightarrow} \varphi \text{ strongly in } C^{1}(0,T;H_{*}^{1}(0,1)), \\ \psi^{m} \overset{m\to\infty}{\longrightarrow} \psi \text{ strongly in } C^{1}(0,T;H_{0}^{1}(0,1)), \\ \omega^{m} \overset{m\to\infty}{\longrightarrow} \omega \text{ strongly in } C^{1}(0,T;H_{*}^{1}(0,1)), \end{cases} \tag{2.1.63}$$

As the the operator G define as follows

$$G : D(G) = C^{2}\left(0,T;L^{2}(0,1)\right) \subset C^{1}\left(0,T;L^{2}(0,1)\right) \to C\left(0,T;L^{2}(0,1)\right),$$
$$U \to U_{tt}$$

is closed, then using (2.1.62) and (2.1.63), we directly obtain

$$\begin{aligned} J &= G\varphi = \varphi_{tt}, \\ L &= G\psi = \psi_{tt}, \\ S &= G\omega = \omega_{tt}, \end{aligned} \tag{2.1.64}$$

(2.1.64) implies

$$\begin{cases} \varphi_{tt}^{m} \overset{m\to\infty}{\longrightarrow} \varphi_{tt} \text{ strongly in } C(0,T;H_{*}^{1}(0,1)), \\ \psi_{tt}^{m} \overset{m\to\infty}{\longrightarrow} \psi_{tt} \text{ strongly in } C(0,T;H_{0}^{1}(0,1)), \\ \omega_{tt}^{m} \overset{m\to\infty}{\longrightarrow} \omega_{tt} \text{ strongly in } C(0,T;H_{*}^{1}(0,1)). \end{cases}$$

From the estimates $E^{n}(t)$ and $E_{3}^{n}(t)$, we have

$$\varphi_{t}^{n} \in L^{\infty}\left(0,T;H_{*}^{2}(0,1) \cap H_{*}^{1}(0,1)\right),$$

and because $\varphi^{n} \in L^{\infty}\left(0,T;H_{*}^{2}(0,1) \cap H_{*}^{1}(0,1)\right)$, then

$$\varphi^{n} \in W^{1,\infty},$$

since, the embedding of $W^{1,\infty}$ *in* $C\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right)$ *is compact ([13]). Then, there exists* $(\varphi^m)_{m\geq 1}$ *subsequence of* $(\varphi^n)_{n\geq 1}$, *such that*

$$\varphi^m \to \varphi \text{ strongly in } C\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right).$$

Since, $H^2(0,1)\cap H_0^1(0,1)\subset L^2(0,1)$, *then*

$$\psi^n \in L^\infty\left(0,T;L^2(0,1)\right),$$

and because $\psi_t^n \in L^\infty\left(0,T;L^2(0,1)\right)$, *therefore*

$$\psi^n \in W^{1,\infty},$$

by using ([13]), there exists a subsequence $(\psi^m)_{m\geq 1}$ *of* $(\psi^n)_{n\geq 1}$, *such that*

$$\psi^m \to \psi \text{ strongly in } C\left(0,T;L^2(0,1)\right).$$

Since, $H_*^2(0,1)\cap H_*^1(0,1)\subset L^2(0,1)$, *then*

$$\omega^n \in L^\infty\left(0,T;L^2(0,1)\right),$$

and because $\omega_t^n \in L^\infty\left(0,T;L^2(0,1)\right)$, *therefore*

$$\omega^n \in W^{1,\infty},$$

by using ([13]), *there exists a subsequence* $(\omega^m)_{m\geq 1}$ *of* $(\omega^n)_{n\geq 1}$, *such that*

$$\omega^m \to \omega \text{ strongly in } C\left(0,T;L^2(0,1)\right).$$

By passing to the limit in (2.1.6)-(2.1.7), then, we conclude that the problem (0.0.1)-(0.0.2) admits a global strong solution satisfies (2.1.1). □

Continuous dependence and uniqueness

Uniqueness of solution

Let us assume that $\left(\Lambda^1, \Upsilon^1, \Gamma^1\right)$ *and* $\left(\Lambda^2, \Upsilon^2, \Gamma^2\right)$ *are two global solutions of system-data. Then,* $(\chi, \Xi, \Theta) = \left(\Lambda^1 - \Lambda^2,\ \Upsilon^1 - \Upsilon^2,\ \Gamma^1 - \Gamma^2\right)$ *satisfies*

$$\begin{cases} \rho_1 \chi_{tt} - k\left(\chi_x + \Xi\right)_x + \mu_1 \chi_t = 0, & \text{in } (0,1) \times (0,+\infty), \\ -\rho_2 \chi_{ttx} - b\Xi_{xx} + k\left(\chi_x + \Xi\right) + \gamma \Theta_x = 0, & \text{in } (0,1) \times (0,+\infty), \\ \rho_3 \Theta_t - k_1 \Theta_{xx} + \lambda \Theta + \gamma \Xi_{tx} = 0, & \text{in } (0,1) \times (0,+\infty), \end{cases} \tag{2.1.65}$$

with the following initial and boundary conditions

$$\begin{cases} \chi(x,0) = \chi_t(x,0) = \chi_{tt}(x,0) = \chi_{ttt}(x,0) = 0, & x \in (0,1), \\ \Xi(x,0) = \Xi_t(x,0) = 0, & x \in (0,1), \\ \Theta(x,0) = \Theta_t(x,0) = 0, & x \in (0,1), \\ \chi_x(0,t) = \chi_x(1,t) = \Xi(0,t) = \Xi(1,t) = 0, & t \in (0,+\infty), \\ \Theta_x(0,t) = \Theta_x(1,t) = 0, & t \in (0,+\infty). \end{cases} \tag{2.1.66}$$

Multiplying $(2.1.65)_1$ *by* χ_t*,* $(2.1.65)_2$ *by* Ξ_t *and* $(2.1.65)_3$ *by* Θ*, and integrating over* $(0,1)$*, we get*

$$\begin{cases} \rho_1 \int_0^1 \chi_{tt}\chi_t dx - k \int_0^1 \left(\chi_x + \Xi\right)_x \chi_t dx + \mu_1 \int_0^1 \chi_t^2 dx = 0, \\ -\rho_2 \int_0^1 \chi_{ttx}\Xi_t dx - b \int_0^1 \Xi_{xx}\Xi_t dx + k \int_0^1 \left(\chi_x + \Xi\right)\Xi_t dx \\ +\gamma \int_0^1 \Theta_x \Xi_t dx = 0, \\ \rho_3 \int_0^1 \Theta_t \Theta dx - k_1 \int_0^1 \Theta_{xx}\Theta dx + \lambda \int_0^1 \Theta^2 dx + \gamma \int_0^1 \Xi_{tx}\Theta dx = 0, \end{cases}$$

by using integrating by parts with respect to x, we get

$$\begin{cases} \rho_1 \dfrac{d}{2dt} \displaystyle\int_0^1 \chi_t^2 dx - k \int_0^1 (\chi_x + \Xi)_x \chi_t dx + \mu_1 \int_0^1 \chi_t^2 dx = 0, \\ \rho_2 \displaystyle\int_0^1 \chi_{tt} \Xi_{tx} dx - b \int_0^1 \Xi_{xx} \Xi_t dx + k \int_0^1 (\chi_x + \Xi) \Xi_t dx \\ + \gamma \displaystyle\int_0^1 \Theta_x \Xi_t dx = 0, \\ \rho_3 \dfrac{d}{2dt} \displaystyle\int_0^1 \Theta^2 dx + k_1 \int_0^1 \Theta_x^2 dx + \lambda \int_0^1 \Theta^2 dx - \gamma \int_0^1 \Theta_x \Xi_t dx = 0, \end{cases} \tag{2.1.67}$$

by differentiating the equation (2.1.66)$_1$ with respect to t, we obtain

$$\Xi_{tx} = \frac{\rho_1}{k} \chi_{ttt} - \chi_{txx} + \frac{\mu_1}{k} \chi_{tt}. \tag{2.1.68}$$

And by substituting (2.1.68) in (2.1.67), we get

$$\begin{cases} \rho_1 \dfrac{d}{2dt} \displaystyle\int_0^1 \chi_t^2 dx - k \int_0^1 (\chi_x + \Xi)_x \chi_t dx + \mu_1 \int_0^1 \chi_t^2 dx = 0, \\ \rho_2 \displaystyle\int_0^1 \chi_{tt} \left(\frac{\rho_1}{k} \chi_{ttt} - \chi_{txx} + \frac{\mu_1}{k} \chi_{tt} \right) dx - b \int_0^1 \Xi_{xx} \Xi_t dx \\ + k \displaystyle\int_0^1 (\chi_x + \Xi) \Xi_t dx + \gamma \int_0^1 \Theta_x \Xi_t dx = 0, \\ \rho_3 \dfrac{d}{2dt} \displaystyle\int_0^1 \Theta^2 dx + k_1 \int_0^1 \Theta_x^2 dx + \lambda \int_0^1 \Theta^2 dx - \gamma \int_0^1 \Theta_x \Xi_t dx = 0, \end{cases}$$

again by using integrating by parts with respect to x, we get

$$\begin{cases} \rho_1 \frac{d}{2dt}\int_0^1 \chi_t^2 dx - k\int_0^1 (\chi_x+\Xi)_x \chi_t dx + \mu_1 \int_0^1 \chi_t^2 dx \\ \frac{\rho_1\rho_2}{k}\frac{d}{2dt}\int_0^1 \chi_{tt}^2 dx + \rho_2 \frac{d}{2dt}\int_0^1 (\chi_{xt})^2 dx + \frac{\rho_2\mu_1}{k}\int_0^1 \chi_{tt}^2 dx = 0, \\ +\frac{d}{2dt} b\int_0^1 \Xi_x^2 dx + k\int_0^1 (\chi_x+\Xi)\,\Xi_t dx + \gamma\int_0^1 \Theta_x \Xi_t dx = 0, \\ \rho_3\frac{d}{2dt}\int_0^1 \Theta^2 dx + k_1\int_0^1 \Theta_x^2 dx + \lambda\int_0^1 \Theta^2 dx - \gamma\int_0^1 \Theta_x\Xi_t dx = 0, \end{cases} \tag{2.1.69}$$

summing (2.1.69)$_1$, (2.1.69)$_2$ and (2.1.69)$_3$, we obtain

$$\begin{aligned} &\rho_1 \frac{d}{2dt}\int_0^1 \chi_t^2 dx + k\frac{d}{2dt}\int_0^1 (\chi_x+\Xi)^2 dx + \mu_1\int_0^1 \chi_t^2 dx + \frac{\rho_1\rho_2}{k}\frac{d}{2dt}\int_0^1 \chi_{tt}^2 dx \\ &+\rho_2\frac{d}{2dt}\int_0^1 (\chi_{xt})^2 dx + \frac{\rho_2\mu_1}{k}\int_0^1 \chi_{tt}^2 dx + \frac{d}{2dt} b\int_0^1 \Xi_x^2 dx + \\ &\rho_3\frac{d}{2dt}\int_0^1 \Theta^2 dx + k_1\int_0^1 \Theta_x^2 dx + \lambda\int_0^1 \Theta^2 dx \;=\; 0, \end{aligned}$$

implies

$$\begin{aligned} &\frac{d}{2dt}\int_0^1 \Big(\rho_1\chi_t^2 + k(\chi_x+\Xi)^2 + \frac{\rho_1\rho_2}{k}\chi_{tt}^2 + \rho_2(\chi_{xt})^2 + b\Xi_x^2 \\ &+\rho_3\Theta^2\Big)\,dx \;=\; -\mu_1\int_0^1 \chi_t^2 dx - \frac{\rho_2\mu_1}{k}\int_0^1 \chi_{tt}^2 dx - k_1\int_0^1 \Theta_x^2 dx \\ &-\lambda\int_0^1 \Theta^2 dx. \end{aligned} \tag{2.1.70}$$

Then, we get

$$\frac{1}{2}\int_0^1 \Big(\rho_1\chi_t^2 + k(\chi_x+\Xi)^2 + \frac{\rho_1\rho_2}{k}\chi_{tt}^2 + \rho_2(\chi_{xt})^2 + b\Xi_x^2 + \rho_3\Theta^2\Big)\,dx \le 0,$$

which implies that $(\chi,\Xi,\Theta) = (0,0,0)$. Therefore, the problem system has a unique global solution.

Continuous dependence

Let (φ, ψ, ω) *be a global solution of* (0.0.1)-(0.0.2), *we have*

$$E\left(t\right) \geq 0, \text{ for all } t \geq 0.$$

Assume from the a priori estimates that the energy derivative satisfies

$$\frac{d}{dt}E\left(t\right) \leq E\left(t\right),$$

A simple integration over $(0,1)$, *we get*

$$\int_0^t \frac{d}{dt}E\left(s\right)dx \leq \int_0^t E\left(s\right)dx,$$

implies

$$E\left(t\right) \leq E\left(0\right) + \int_0^t E(s)ds.$$

Applying Gronwall's inequality, we obtain

$$\forall t \geq 0,\ E\left(t\right) \leq E\left(0\right)e^t.$$

This shows that the solution of problem (0.0.1)-(0.0.2) *depends continuously on the initial data.*

CHAPTER 3

Exponential stability for a Bresse-Timoshenko system

In this chapter, we establish an exponential stability result of solutions of the considered problem by using the energy method which based on the multiplication method to construct a suitable Lyapunov functional which is equivalent to the energy of this system. Furthermore, we present the energy decay and some technical lemmas that are necessary to achieve our goal.

3.1 Energy decay

In this section, we define an energy functional associated with the Bresse-Timoshenko system and show that it decreases over time t. For that, we need the following lemma to achieve our goal.

Lemma 3.1.1. *Let (φ, ψ, ω) be the solution of system (0.0.1)-(0.0.2). Then, the energy associated to the system (0.0.1)-(0.0.2) defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1 \varphi_t^2 + k(\varphi_x + \psi)^2 + \frac{\rho_2 \rho_1}{k}\varphi_{tt}^2 + \rho_2 \varphi_{tx}^2 + b\psi_x^2 + \rho_3 \omega^2\right) dx, \tag{3.1.1}$$

satisfies

$$\frac{d}{dt}E(t) = -k_1 \int_0^1 \omega_x^2 dx - \lambda \int_0^1 \omega^2 dx - \frac{\rho_2 \mu_1}{k}\int_0^1 \varphi_{tt}^2 dx - \mu_1 \int_0^1 \varphi_t^2 dx \leq 0. \tag{3.1.2}$$

Proof. Multiplying (0.0.1)$_1$, (0.0.1)$_2$, (0.0.1)$_3$, *respectively, by φ_t, ψ_t, ω* and integrating over $(0,1)$, *we* *get*

$$\begin{cases} \rho_1 \dfrac{d}{dt} \displaystyle\int_0^1 \varphi_{tt}\varphi dx - k \int_0^1 (\varphi_x + \psi)_x \varphi_t dx + \mu_1 \int_0^1 \varphi_t^2 dx = 0, \\ -\rho_2 \displaystyle\int_0^1 \varphi_{ttx}\psi_t dx - b \dfrac{d}{dt} \int_0^1 \psi_{xx}\psi_t dx + k \int_0^1 (\varphi_x + \psi)\psi_t dx + \gamma \int_0^1 \omega_x \psi_t dx = 0, \\ -\rho_3 \dfrac{d}{dt} \displaystyle\int_0^1 \omega_t \omega dx - k_1 \int_0^1 \omega_{xx}\omega dx + \lambda \int_0^1 \omega^2 dx + \gamma \int_0^1 \psi_{tx}\omega dx = 0. \end{cases}$$

by using integration by parts, we obtain

$$\begin{cases} \dfrac{\rho_1}{2} \dfrac{d}{dt} \displaystyle\int_0^1 \varphi_t^2 dx + k \int_0^1 (\varphi_x + \psi) \varphi_{tx} dx + \mu_1 \int_0^1 \varphi_t^2 dx = 0, \\ \rho_2 \displaystyle\int_0^1 \varphi_{tt}\psi_{tx} dx + \dfrac{b}{2} \dfrac{d}{dt} \int_0^1 \psi_x^2 dx + k \int_0^1 (\varphi_x + \psi)\ \psi_t dx + \gamma \int_0^1 \omega_x \psi_t dx = 0, \\ \dfrac{\rho_3}{2} \dfrac{d}{dt} \displaystyle\int_0^1 \omega^2 dx + k_1 \int_0^1 \omega_x^2 dx + \lambda \int_0^1 \omega^2 dx - \gamma \int_0^1 \psi_t \omega_x dx = 0. \end{cases} \tag{3.1.3}$$

By differentiating (0.0.1)$_1$ *with respect to* t*, we obtain*

$$\psi_{tx} = \frac{\rho_1}{k} (\varphi_{tt})_t - (\varphi_{xx})_t + \frac{\mu_1}{k} \varphi_{tt}. \tag{3.1.4}$$

Replacing (3.1.4) *in* (3.1.3)$_2$*, we arrive at*

$$\begin{cases} \dfrac{\rho_1}{2} \dfrac{d}{dt} \displaystyle\int_0^1 \varphi_t^2 dx + k \int_0^1 (\varphi_x + \psi) \varphi_{tx} dx + \mu_1 \int_0^1 \varphi_t^2 dx = 0, \\ \dfrac{\rho_2 \rho_1}{2k} \dfrac{d}{dt} \displaystyle\int_0^1 \varphi_{tt}^2 dx + \dfrac{\rho_2}{2} \dfrac{d}{dt} \int_0^1 \varphi_{tx}^2 dx + \dfrac{\rho_2 \mu_1}{k} \int_0^1 \varphi_{tt}^2 dx \\ + \dfrac{b}{2} \dfrac{d}{dt} \displaystyle\int_0^1 \psi_x^2 dx + k \int_0^1 (\varphi_x + \psi)\ \psi_t dx + \gamma \int_0^1 \omega_x \psi_t dx \ = \ 0, \\ \dfrac{\rho_3}{2} \dfrac{d}{dt} \displaystyle\int_0^1 \omega^2 dx + k_1 \int_0^1 \omega_x^2 dx + \lambda \int_0^1 \omega^2 dx - \gamma \int_0^1 \psi_t \omega_x dx = 0. \end{cases} \tag{3.1.5}$$

Summing (3.1.5)$_1$, (3.1.5)$_2$ *and* (3.1.5)$_3$, *we get*

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \frac{k}{2}\frac{d}{dt}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{\rho_2\rho_1}{2k}\frac{d}{dt}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \varphi_{tx}^2 dx$$

$$+\frac{b}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx + \frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \omega^2 dx$$

$$= -k_1\int_0^1 \omega_x^2 dx - \lambda\int_0^1 \omega^2 dx - \frac{\rho_2\mu_1}{k}\int_0^1 \varphi_{tt}^2 dx - \mu_1\int_0^1 \varphi_t^2 dx,$$

implies

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \frac{\rho_2\rho_1}{k}\varphi_{tt}^2 + \rho_2\varphi_{tx}^2 + b\psi_x^2 + \rho_3\omega^2\right)dx\right)$$

$$= \quad -k_1\int_0^1 \omega_x^2 dx - \lambda\int_0^1 \omega^2 dx - \frac{\rho_2\mu_1}{k}\int_0^1 \varphi_{tt}^2 dx - \mu_1\int_0^1 \varphi_t^2 dx,$$

Hence, the energy functional $E(t)$ *is given by*

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \frac{\rho_2\rho_1}{k}\varphi_{tt}^2 + \rho_2\varphi_{tx}^2 + b\psi_x^2 + \rho_3\omega^2\right)dx.$$

and

$$E'(t) = -k_1\int_0^1 \omega_x^2 dx - \lambda\int_0^1 \omega^2 dx - \frac{\rho_2\mu_1}{k}\int_0^1 \varphi_{tt}^2 dx - \mu_1\int_0^1 \varphi_t^2 dx \le 0.$$

□

3.2 Lyapunov functional

In this section, we construct suitable Lyapunov functionals that strengthen our energy-decay. For that, we need the following lemmas.

Lemma 3.2.1. *let* (φ,ψ,ω) *be the solution of the system* (0.0.1)-(0.0.2) *then, the functional*

$$F_1(t) = -\frac{\mu_1}{2}\int_0^1 \varphi_t^2 dx - k\int_0^1 \varphi_{tx}\varphi_x dx,$$

satisfies

$$F_1'(t) = \rho_1 \int_0^1 \varphi_{tt}^2 dx - k \int_0^1 \psi_x \varphi_{tt} dx - k \int_0^1 \varphi_{tx}^2 dx. \tag{3.2.6}$$

Proof. *Multiplying* (0.0.1)$_1$, *by* φ_{tt} *and integrating over* $(0,1)$, *we get*

$$-\rho_1 \int_0^1 \varphi_{tt}^2 dx + k \int_0^1 (\varphi_x + \psi)_x \varphi_{tt} dx - \mu_1 \int_0^1 \varphi_t \varphi_{tt} dx = 0,$$

Then

$$-\rho_1 \int_0^1 \varphi_{tt}^2 dx + k \int_0^1 \varphi_{xx} \varphi_{tt} dx + k \int_0^1 \psi_x \varphi_{tt} dx - \mu_1 \int_0^1 \varphi_t \varphi_{tt} dx = 0,$$

Satisfies

$$-\mu_1 \int_0^1 \varphi_t \varphi_{tt} dx - k \int_0^1 \varphi_x \varphi_{ttx} dx = -k \int_0^1 \psi_x \varphi_{tt} dx + \rho_1 \int_0^1 \varphi_{tt}^2 dx,$$

Then

$$-\mu_1 \int_0^1 \varphi_t \varphi_{tt} dx - k \int_0^1 \varphi_x \varphi_{ttx} dx - k \int_0^1 \varphi_{tx}^2 dx + k \int_0^1 \varphi_{tx}^2 dx = -k \int_0^1 \psi_x \varphi_{tt} dx + \rho_1 \int_0^1 \varphi_{tt}^2 dx,$$

by using integrating by part, we get

$$\frac{-\mu_1}{2} \frac{d}{dt} \int_0^1 \varphi_t^2 dx - k \frac{d}{dt} \int_0^1 \varphi_{tx} \varphi_x dx = \rho_1 \int_0^1 \varphi_{tt}^2 dx - k \int_0^1 \psi_x \varphi_{tt} dx - k \int_0^1 \varphi_{tx}^2 dx,$$

implies

$$\frac{d}{dt} \left(\frac{-\mu_1}{2} \int_0^1 \varphi_t^2 dx - k \int_0^1 \varphi_{tx} \varphi_x dx \right) = \rho_1 \int_0^1 \varphi_{tt}^2 dx - k \int_0^1 \psi_x \varphi_{tt} dx - k \int_0^1 \varphi_{tx}^2 dx.$$

Hence, the functional $F_1(t)$ *is given by*

$$F_1(t) = -\frac{\mu_1}{2} \int_0^1 \varphi_t^2 dx - k \int_0^1 \varphi_{tx} \varphi_x dx.$$

and

$$F_1'(t) = \rho_1 \int_0^1 \varphi_{tt}^2 dx - k \int_0^1 \psi_x \varphi_{tt} dx - k \int_0^1 \varphi_{tx}^2 dx.$$

□

Lemma 3.2.2. *Let (φ, ψ, ω) be the solution of (0.0.1)-(0.0.2). Then, the functional*

$$F_1(t) = -\frac{\mu_1}{2}\int_0^1 \varphi_t^2 dx - k\int_0^1 \varphi_{tx}\varphi_x dx, \tag{3.2.7}$$

satisfies, for any $\varepsilon_1 > 0$,

$$F_1'(t) \leq -k\int_0^1 \varphi_{tx}^2 dx + \varepsilon_1 k\int_0^1 \psi_x^2 dx + \left(\rho_1 + \frac{k}{4\varepsilon_1}\right)\int_0^1 \varphi_{tt}^2 dx. \tag{3.2.8}$$

Proof. *By differentiating (3.2.7), we obtain*

$$F_1'(t) = -\mu_1\int_0^1 \varphi_t\varphi_{tt} dx - k\int_0^1 \varphi_{ttx}\varphi_x dx - k\int_0^1 \varphi_{tx}^2 dx. \tag{3.2.9}$$

From $(0.0.1)_1$*, we have*

$$-\mu_1\varphi_t = \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x, \tag{3.2.10}$$

by substituting (3.2.10) in (3.2.9), we get

$$\begin{aligned} F_1'(t) &= \int_0^1 (\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x)\varphi_{tt} dx - k\int_0^1 \varphi_{ttx}\varphi_x dx \\ &\quad - k\int_0^1 \varphi_{tx}^2 dx \\ &= -k\int_0^1 \varphi_{tt}(\varphi_x + \psi)_x dx + \int_0^1 \rho_1(\varphi_{tt})^2 dx - k\int_0^1 \varphi_{ttx}\varphi_x dx \\ &\quad - k\int_0^1 \varphi_{tx}^2 dx, \end{aligned}$$

using integrating by parts, we obtain

$$\begin{aligned} F_1'(t) &= k\int_0^1 \varphi_{ttx}\varphi_x dx + k\int_0^1 \varphi_{ttx}\psi dx + \int_0^1 \rho_1(\varphi_{tt})^2 dx \\ &\quad - k\int_0^1 \varphi_{ttx}\varphi_x dx - k\int_0^1 \varphi_{tx}^2 dx \\ &= \int_0^1 \rho_1(\varphi_{tt})^2 dx - k\int_0^1 \varphi_{tt}\psi_x dx - k\int_0^1 \varphi_{tx}^2 dx, \end{aligned} \tag{3.2.11}$$

Now using Young's inequality, for any $\varepsilon_1 > 0$

$$-k\int_0^1 \varphi_{tt}\psi_x dx \leq \frac{k}{4\varepsilon_1}\int_0^1 (\varphi_{tt})^2 dx + \varepsilon_1 k\int_0^1 (\psi_x)^2 dx. \tag{3.2.12}$$

Replacing (3.2.12) in (3.2.11), we get

$$F_1'(t) \leq \int_0^1 \rho_1\varphi_{tt}^2 dx + \frac{k}{4\varepsilon_1}\int_0^1 \varphi_{tt}^2 dx + \varepsilon_1 k\int_0^1 \psi_x^2 dx - k\int_0^1 \varphi_{tx}^2 dx,$$

implies

$$F_1'(t) \leq -k\int_0^1 \varphi_{tx}^2 dx + \varepsilon_1 k\int_0^1 \psi_x^2 dx + \left(\rho_1 + \frac{k}{4\varepsilon_1}\right)\int_0^1 \varphi_{tt}^2 dx.$$

□

Lemma 3.2.3. *Let (φ, ψ, ω) be the solution of (0.0.1)-(0.0.2). Then, the functional*

$$F_2(t) = -\rho_2\int_0^1 \varphi_{tx}\psi dx + \rho_1\int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_t\omega dx,$$

satisfies

$$\begin{aligned} F_2'(t) &= \rho_1\int_0^1 \varphi_t^2 dx - \frac{k_1\rho_2}{\gamma}\int_0^1 \varphi_{xt}\omega_x dx - \frac{\lambda\rho_2}{\gamma}\int_0^1 \varphi_t\omega dx - k\int_0^1 (\varphi_x+\psi)^2 dx \\ &\quad -b\int_0^1 \psi_x^2 dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx + \gamma\int_0^1 \psi_x\omega dx. \end{aligned}$$

Proof. *Multiplying $(0.0.1)_1$, $(0.0.1)_2$, $(0.0.1)_3$, respectively, by φ, ψ, $\frac{\rho_3}{\gamma}\varphi_t$ and integrating over $(0,1)$, we get*

$$\begin{cases} \rho_1\int_0^1 \varphi_{tt}\varphi dx - k\int_0^1 (\varphi_x+\psi)_x \varphi dx + \mu_1\int_0^1 \varphi_t\varphi dx = 0, \\ -\rho_2\int_0^1 \varphi_{ttx}\psi dx - b\int_0^1 \psi_{xx}\psi dx + k\int_0^1 (\varphi_x+\psi)\psi dx + \gamma\int_0^1 \omega_x\psi dx = 0, \\ \frac{\rho_3\rho_2}{\gamma}\int_0^1 \omega_t\varphi_t dx - \frac{k_1\rho_2}{\gamma}\int_0^1 \omega_{xx}\varphi_t dx + \frac{\lambda\rho_2}{\gamma}\int_0^1 \omega\varphi_t dx + \rho_2\int_0^1 \psi_{tx}\varphi_t dx = 0, \end{cases}$$

by using integrating by part, we get

$$\begin{cases} -\rho_1 \int_0^1 \varphi_t^2 dx - k \int_0^1 (\varphi_x + \psi)_x \varphi dx + \frac{\mu_1}{2} \frac{d}{dt} \int_0^1 \varphi^2 dx = 0, \\ -\rho_2 \frac{d}{dt} \int_0^1 \varphi_{tx} \psi dx + b \int_0^1 \psi_x^2 dx + k \int_0^1 (\varphi_x + \psi) \psi dx - \gamma \int_0^1 \psi_x \omega dx = 0, \\ \frac{\rho_2 \rho_3}{\gamma} \frac{d}{dt} \int_0^1 \varphi_t \omega dx - \frac{k_1 \rho_2}{\gamma} \int_0^1 \omega_{xx} \varphi_t dx + \frac{\lambda \rho_2}{\gamma} \int_0^1 \omega \varphi_t dx + \rho_2 \int_0^1 \psi_{tx} \varphi_t dx = 0, \end{cases} \tag{3.2.13}$$

summing (3.2.13)$_1$, (3.2.13)$_2$ *and* (3.2.13)$_2$, *we get*

$$\begin{aligned} & -\rho_2 \frac{d}{dt} \int_0^1 \varphi_{tx} \psi dx + \rho_1 \frac{d}{dt} \int_0^1 \varphi \varphi_t dx + \frac{\mu_1}{2} \frac{d}{dt} \int_0^1 \varphi^2 dx + \frac{\rho_2 \rho_3}{\gamma} \frac{d}{dt} \int_0^1 \varphi_t \omega dx \\ = \; & \rho_1 \int_0^1 \varphi_t^2 dx - \frac{k_1 \rho_2}{\gamma} \int_0^1 \varphi_{xt} \omega_x dx - \frac{\lambda \rho_2}{\gamma} \int_0^1 \varphi_t \omega dx - k \int_0^1 (\varphi_x + \psi)^2 dx \\ & -b \int_0^1 \psi_x^2 dx + \frac{\rho_2 \rho_3}{\gamma} \int_0^1 \varphi_{tt} \omega dx + \gamma \int_0^1 \psi_x \omega dx, \end{aligned}$$

implies

$$\begin{aligned} & \frac{d}{dt} \left(-\rho_2 \int_0^1 \varphi_{tx} \psi dx + \rho_1 \int_0^1 \varphi \varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx + \frac{\rho_2 \rho_3}{\gamma} \int_0^1 \varphi_t \omega dx \right) \\ = \; & \rho_1 \int_0^1 \varphi_t^2 dx - \frac{k_1 \rho_2}{\gamma} \int_0^1 \varphi_{xt} \omega_x dx - \frac{\lambda \rho_2}{\gamma} \int_0^1 \varphi_t \omega dx - k \int_0^1 (\varphi_x + \psi)^2 dx \\ & -b \int_0^1 \psi_x^2 dx + \frac{\rho_2 \rho_3}{\gamma} \int_0^1 \varphi_{tt} \omega dx + \gamma \int_0^1 \psi_x \omega dx, \end{aligned}$$

Hence, the functional $F_2(t)$ *is given by*

$$F_2(t) = -\rho_2 \int_0^1 \varphi_{tx} \psi dx + \rho_1 \int_0^1 \varphi \varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx + \frac{\rho_2 \rho_3}{\gamma} \int_0^1 \varphi_t \omega dx.$$

and

$$F_2'(t) = \rho_1 \int_0^1 \varphi_t^2 dx - \frac{k_1\rho_2}{\gamma}\int_0^1 \varphi_{xt}\omega_x dx - \frac{\lambda\rho_2}{\gamma}\int_0^1 \varphi_t\omega dx - k\int_0^1 (\varphi_x+\psi)^2 dx$$
$$-b\int_0^1 \psi_x^2 dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx + \gamma\int_0^1 \psi_x\omega dx.$$

□

Lemma 3.2.4. *Let (φ,ψ,ω) be the solution of (0.0.1)-(0.0.2). Then, the functional*

$$F_2(t) = -\rho_2\int_0^1 \varphi_{tx}\psi dx + \rho_1\int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_t\omega dx, \tag{3.2.14}$$

satisfies

$$F_2'(t) \le -k\int_0^1 (\varphi_x+\psi)^2 dx - \frac{b}{2}\int_0^1 \psi_x^2 dx + \left(\frac{\lambda\rho_2}{2\gamma}+\rho_1\right)\int_0^1 \varphi_t^2 dx$$
$$+\frac{\rho_2\rho_3}{2\gamma}\int_0^1 \varphi_{tt}^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \varphi_{xt}^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \omega_x^2 dx \tag{3.2.15}$$
$$+\left(\frac{\lambda\rho_2}{2\gamma}+\frac{\rho_2\rho_3}{2\gamma}+\frac{\gamma^2}{2b}\right)\int_0^1 \omega^2 dx.$$

Proof. By differentiating $F_2(t)$, we get

$$F_2'(t) = -\rho_2\int_0^1 \varphi_{ttx}\psi dx - \rho_2\int_0^1 \varphi_{tx}\psi_t dx + \rho_1\int_0^1 \varphi_t^2 dx + \rho_1\int_0^1 \varphi\varphi_{tt} dx \tag{3.2.16}$$
$$+\mu_1\int_0^1 \varphi\varphi_t dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_t\omega_t dx.$$

From $(0.0.1)_1$ and $(0.0.1)_2$, we have

$$-\rho_1\varphi_{tt} = k(\varphi_x+\psi)_x - \mu_1\varphi_t, \tag{3.2.17}$$

and

$$-\rho_2\varphi_{ttx} = b\psi_{xx} - k(\varphi_x+\psi) - \gamma\omega_x, \tag{3.2.18}$$

Exploiting (3.2.17), (3.2.18) *in* (3.2.16), *we obtain*

$$\begin{aligned}
F_2'(t) &= \int_0^1 (b\psi_{xx} - k(\varphi_x + \psi) - \gamma\omega_x)\psi dx - \rho_2 \int_0^1 \varphi_{tx}\psi_t dx + \rho_1 \int_0^1 \varphi_t^2 dx \\
&\quad - \int_0^1 \varphi\left(-k(\varphi_x + \psi)_x - \mu_1\varphi_t\right) dx + \mu_1 \int_0^1 \varphi\varphi_t dx \\
&\quad + \frac{\rho_2\rho_3}{\gamma} \int_0^1 \varphi_{tt}\omega dx + \frac{\rho_2\rho_3}{\gamma} \int_0^1 \varphi_t\omega_t dx \\
&= b\int_0^1 \psi_{xx}\psi dx - k\int_0^1 (\varphi_x + \psi)\psi dx - \gamma\int_0^1 \omega_x\psi dx - \rho_2 \int_0^1 \varphi_{tx}\psi_t dx \\
&\quad + \rho_1 \int_0^1 \varphi_t^2 dx + k\int_0^1 (\varphi_x + \psi)_x \varphi dx + \frac{\rho_2\rho_3}{\gamma} \int_0^1 \varphi_{tt}\omega dx \\
&\quad + \frac{\rho_2\rho_3}{\gamma} \int_0^1 \varphi_t\omega_t dx,
\end{aligned}$$

using integrating by parts, we get

$$\begin{aligned}
F_2'(t) &= \rho_1 \int_0^1 \varphi_t^2 dx - b\int_0^1 \psi_x^2 dx - k\int_0^1 \psi(\varphi_x + \psi) dx \\
&\quad + \gamma\int_0^1 \psi_x\omega dx + \rho_2\left(\int_0^1 \frac{\varphi_t(\gamma\psi_{xt} + \rho_3\omega_t)}{\gamma} dx\right) \\
&\quad - k\int_0^1 \varphi_x(\varphi_x + \psi) dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx.
\end{aligned} \tag{3.2.19}$$

From (0.0.1)$_3$, *we have*

$$\gamma\psi_{xt} + \rho_3\omega_t = k_1\omega_{xx} - \lambda\omega. \tag{3.2.20}$$

By substituting (3.2.20) *in* (3.2.19), *we get*

$$F_2'(t) = \rho_1 \int_0^1 \varphi_t^2 dx - b\int_0^1 \psi_x^2 dx - k\int_0^1 \psi(\varphi_x + \psi) dx + \gamma\int_0^1 \psi_x\omega dx$$

$$+\rho_2(\int_0^1 \frac{\varphi_t\left(k_1\omega_{xx}-\lambda\omega\right)}{\gamma}dx) - k\int_0^1 \varphi_x\left(\varphi_x+\psi\right)dx$$

$$+\frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx,$$

and by using integrating by parts, we obtain

$$\begin{aligned} F_2'(t) &= \rho_1\int_0^1 \varphi_t^2 dx - \frac{k_1\rho_2}{\gamma}\int_0^1 \varphi_{xt}\omega_x dx - \frac{\lambda\rho_2}{\gamma}\int_0^1 \varphi_t\omega dx \\ &\quad -k\int_0^1 \left(\varphi_x+\psi\right)^2 dx - b\int_0^1 \psi_x^2 dx + \frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx \\ &\quad +\gamma\int_0^1 \psi_x\omega dx. \end{aligned} \tag{3.2.21}$$

Now using Young's inequality, we have

$$-\frac{k_1\rho_2}{\gamma}\int_0^1 \varphi_{xt}\omega_x dx \le \frac{k_1\rho_2}{2\gamma}\int_0^1 \varphi_{xt}^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \omega_x^2 dx. \tag{3.2.22}$$

$$-\frac{\lambda\rho_2}{\gamma}\int_0^1 \varphi_t\omega dx \le \frac{\lambda\rho_2}{2\gamma}\int_0^1 \varphi_t^2 dx + \frac{\lambda\rho_2}{2\gamma}\int_0^1 \omega^2 dx. \tag{3.2.23}$$

$$\frac{\rho_2\rho_3}{\gamma}\int_0^1 \varphi_{tt}\omega dx \le \frac{\rho_2\rho_3}{2\gamma}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_2\rho_3}{2\gamma}\int_0^1 \omega^2 dx. \tag{3.2.24}$$

$$\gamma\int_0^1 \psi_x\omega dx = \int_0^1 \left(\sqrt{b}\psi_x\right)\left(\frac{\gamma}{\sqrt{b}}\omega\right)dx \le \frac{b}{2}\int_0^1 \psi_x^2 dx + \frac{\gamma}{2b}\int_0^1 \omega^2 dx. \tag{3.2.25}$$

by substituting (3.2.22), (3.2.23), (3.2.24), *and* (3.2.25) *in* (3.2.21), *we get*

$$\begin{aligned} F_2'(t) &\le \rho_1\int_0^1 \varphi_t^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \varphi_{xt}^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \omega_x^2 dx + \frac{\lambda\rho_2}{2\gamma}\int_0^1 \varphi_t^2 dx + \frac{\lambda\rho_2}{2\gamma}\int_0^1 \omega^2 dx \\ &\quad -k\int_0^1 \left(\varphi_x+\psi\right)^2 dx - b\int_0^1 \psi_x^2 dx + \frac{\rho_2\rho_3}{2\gamma}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_2\rho_3}{2\gamma}\int_0^1 \omega^2 dx \\ &\quad +\frac{b}{2}\int_0^1 \psi_x^2 dx + \frac{\gamma}{2b}\int_0^1 \omega^2 dx, \end{aligned}$$

implies

$$\begin{aligned}
F_2'(t) \leq & -k\int_0^1 (\varphi_x+\psi)^2\,dx - \frac{b}{2}\int_0^1 \psi_x^2 dx + \left(\frac{\lambda\rho_2}{2\gamma}+\rho_1\right)\int_0^1 \varphi_t^2 dx \\
& + \frac{\rho_2\rho_3}{2\gamma}\int_0^1 \varphi_{tt}^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \varphi_{xt}^2 dx + \frac{k_1\rho_2}{2\gamma}\int_0^1 \omega_x^2 dx \\
& + \left(\frac{\lambda\rho_2}{2\gamma}+\frac{\rho_2\rho_3}{2\gamma}+\frac{\gamma^2}{2b}\right)\int_0^1 \omega^2 dx.
\end{aligned}$$

□

Now, we define a Lyapunov functional L by

$$L(t) = NE(t) + N_1F_1(t) + N_2F_2(t)\,, \tag{3.2.26}$$

where N, N_1, N_2 are positive constants.

3.3 Properties of Lyapunov functional

In this section, we study the Lyapunov functional associated to the system (0.0.1)-(0.0.2). We show that this functional is equivalent to the system's total energy. For that, we state and prove the following lemma.

Lemma 3.3.1. *Let (φ, ψ, ω) be the solution of (0.0.1)-(0.0.2). Then, there exist two positive constants c_1 and c_2, such that the Lyapunov functional (3.2.26) satisfies*

$$c_1E(t) \leq L(t) \leq c_2E(t)\,,\ \forall t \geq 0, \tag{3.3.27}$$

and

$$L'(t) \leq -\alpha E(t)\,,\ \forall t \geq 0, \tag{3.3.28}$$

where α is a positive constant.

Proof. *From (3.2.26), we can write*

$$\begin{aligned}
|L(t) - NE(t)| &= |N_1 F_1(t) + N_2 F_2(t)| \qquad (3.3.29)\\
&\leq N_1 |F_1(t)| + N_2 |F_2(t)| \\
&\leq N_1 \frac{\mu_1}{2} \int_0^1 \varphi_t^2 dx + N_1 k \left| \int_0^1 \varphi_{tx} \varphi_x dx \right| + N_2 \rho_2 \left| \int_0^1 \varphi_{tx} \psi dx \right| \\
&\quad + N_2 \rho_1 \left| \int_0^1 \varphi \varphi_t dx \right| + N_2 \frac{\mu_1}{2} \int_0^1 \varphi^2 dx + N_2 \frac{\rho_2 \rho_3}{\gamma} \left| \int_0^1 \varphi_t \omega dx \right| .
\end{aligned}$$

Integrating by parts the second term in the right hand side of (3.3.29), we get

$$\int_0^1 \varphi_{tx} \varphi_x dx = - \int_0^1 \varphi_t \varphi_{xx} dx, \qquad (3.3.30)$$

From $(0.0.1)_1$, *we get*

$$\varphi_{xx} = \frac{\rho_1}{k} \varphi_{tt} - \psi_x + \frac{\mu_1}{k} \varphi_t, \qquad (3.3.31)$$

by substituting (3.3.31) in (3.3.30), we get

$$\int_0^1 \varphi_{tx} \varphi_x dx = -\frac{\rho_1}{k} \int_0^1 \varphi_t \varphi_{tt} dx + \int_0^1 \varphi_t \psi_x dx - \frac{\mu_1}{k} \int_0^1 \varphi_t^2 dx. \qquad (3.3.32)$$

Inserting (3.3.32) in (3.3.29), we obtain

$$\begin{aligned}
|L(t) - NE(t)| &\leq N_1 \frac{\mu_1}{2} \int_0^1 \varphi_t^2 dx + N_1 \rho_1 \int_0^1 |\varphi_t \varphi_{tt}| dx + N_1 k \int_0^1 |\varphi_t \psi_x| dx \qquad (3.3.33)\\
&\quad + N_1 \mu_1 \int_0^1 \varphi_t^2 dx + N_2 \rho_2 \left| \int_0^1 \varphi_{tx} \psi dx \right| + N_2 \rho_1 \left| \int_0^1 \varphi \varphi_t dx \right| \\
&\quad + N_2 \frac{\mu_1}{2} \int_0^1 \varphi^2 dx + N_2 \frac{\rho_2 \rho_3}{\gamma} \left| \int_0^1 \varphi_t \omega dx \right| .
\end{aligned}$$

By applying Young's, Poincaré's inequalities, we have

$$\int_0^1 |\varphi_t \varphi_{tt}| dx \leq \frac{1}{2} \int_0^1 \varphi_t^2 dx + \frac{1}{2} \int_0^1 \varphi_{tt}^2 dx, \qquad (3.3.34)$$

$$\int_0^1 |\varphi_t \psi_x| \, dx \leq \frac{1}{2} \int_0^1 \varphi_t^2 dx + \frac{1}{2} \int_0^1 \psi_x^2 dx, \tag{3.3.35}$$

$$\left| \int_0^1 \varphi_{tx} \psi dx \right| \leq \frac{1}{2} \int_0^1 \varphi_{tx}^2 dx + \frac{1}{2} \int_0^1 \psi^2 dx, \tag{3.3.36}$$

$$\left| \int_0^1 \varphi \varphi_t dx \right| \leq \frac{1}{2} \int_0^1 \varphi^2 dx + \frac{1}{2} \int_0^1 \varphi_t^2 dx, \tag{3.3.37}$$

$$\left| \int_0^1 \varphi_t \omega dx \right| \leq \frac{1}{2} \int_0^1 \varphi_t^2 dx + \frac{1}{2} \int_0^1 \omega^2 dx, \tag{3.3.38}$$

and using the following inequality

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 \, dx + 2 \int_0^1 \psi_x^2 dx, \tag{3.3.39}$$

by substituting (3.3.34), (3.3.35), (3.3.36), (3.3.37), (3.3.38) *and* (3.3.39), *we directly obtain*

$$|L(t) - NE(t)| \leq \eta E(t), \text{ with } \eta \geq 0.$$

Therefore

$$(N - \eta) E(t) \leq L(t) \leq (N + \eta) E(t).$$

By choosing N *sufficiently large and depending of* N_1, N_2, *such that*

$$c_1 E(t) \leq L(t) \leq c_2 E(t),$$

with $c_2 = \eta + N, \quad c_1 = -\eta + N.$

Now, by differentiating $L(t)$, *exploiting* (3.1.2), (3.2.8), (3.2.15) *and setting* $\varepsilon_1 = \dfrac{1}{N_1}$, *we get*

$$\begin{aligned} L'(t) &= NE'(t) + N_1 F_1'(t) + N_2 F_2'(t) \\ &\leq N \left(-k_1 \int_0^1 \omega_x^2 dx - \lambda \int_0^1 \omega^2 dx - \frac{\rho_2 \mu_1}{k} \int_0^1 \varphi_{tt}^2 dx - \mu_1 \int_0^1 \varphi_t^2 dx \right) \\ &\quad + N_1 \left(-k \int_0^1 \varphi_{tx}^2 dx + \frac{1}{N_1} k \int_0^1 \psi_x^2 dx + \left(\rho_1 + \frac{k}{4} N_1 \right) \int_0^1 \varphi_{tt}^2 dx \right) \end{aligned}$$

$$
\begin{aligned}
& +N_2\left(-k\int_0^1(\varphi_x+\psi)^2\,dx-\frac{b}{2}\int_0^1\psi_x^2dx+\left(\frac{\lambda\rho_2}{2\gamma}+\rho_1\right)\int_0^1\varphi_t^2dx+\frac{\rho_2\rho_3}{2\gamma}\int_0^1\varphi_{tt}^2dx\right.\\
& \left.+\frac{k_1\rho_2}{2\gamma}\int_0^1\varphi_{xt}^2dx+\frac{k_1\rho_2}{2\gamma}\int_0^1\omega_x^2dx+\left(\frac{\lambda\rho_2}{2\gamma}+\frac{\rho_2\rho_3}{2\gamma}+\frac{\gamma^2}{2b}\right)\int_0^1\omega^2dx\right)\\
\leq\ & -\left(\mu_1 N-N_2\left(\frac{\lambda\rho_2}{2\gamma}+\rho_1\right)\right)\int_0^1\varphi_t^2dx-\left(N_1k-N_2\frac{k_1\rho_2}{2\gamma}\right)\int_0^1\varphi_{tx}^2dx\\
& -N_2k\int_0^1(\varphi_x+\psi)^2\,dx-\left(N\frac{\rho_2\mu_1}{k}-N_1\left(\rho_1+\frac{k}{4}N_1\right)-N_2\frac{\rho_2\rho_3}{2\gamma}\right)\int_0^1\varphi_{tt}^2dx\\
& -\left(N_2\frac{b}{2}-k\right)\int_0^1\psi_x^2dx-\left(Nk_1-N_2\frac{k_1\rho_2}{2\gamma}\right)\int_0^1\omega_x^2dx\\
& -\left(N\lambda-N_2\left(\frac{\lambda\rho_2}{2\gamma}+\frac{\rho_2\rho_3}{2\gamma}+\frac{\gamma^2}{2b}\right)\right)\int_0^1\omega^2dx.
\end{aligned}
$$

First, we choose N_2 large enough such that

$$N_2\frac{b}{2}-k>0.$$

We take N_1 large enough so that

$$N_1k-N_2\frac{k_1\rho_2}{2\gamma}>0.$$

Finally, we pick N large enough (even larger so that (3.3.27) remains valid) such that

$$
\begin{cases}
\mu_1 N-N_2\left(\dfrac{\lambda\rho_2}{2\gamma}+\rho_1\right)>0,\\
N\dfrac{\rho_2\mu_1}{k}-N_1\left(\rho_1+\dfrac{k}{4}N_1\right)-N_2\dfrac{\rho_2\rho_3}{2\gamma}>0,\\
Nk_1-N_2\dfrac{k_1\rho_2}{2\gamma}>0.\\
N\lambda-N_2\left(\dfrac{\lambda\rho_2}{2\gamma}+\dfrac{\rho_2\rho_3}{2\gamma}+\dfrac{\gamma^2}{2b}\right)>0.
\end{cases}
$$

All these choices leads to the desired result. □

We are now ready to state and prove the following exponential stability result.

3.4 Main stability result

In this section, we state and prove the theorem needed for the proof of stability result.

Theorem 3.4.1. *Let (φ, ψ, ω) be the solution of (0.0.1)-(0.0.2). Then, there exist two positive constants λ_0 and λ_1, such that*

$$E(t) \leq \lambda_0 e^{-\lambda_1 t}, \ \forall t \geq 0. \tag{3.4.40}$$

Proof. *By using the estimation (3.3.28) and having in mind the equivalence of $E(t)$ and $L(t)$, we conclude that*

$$L^{'}(t) \leq -\alpha L(t), \ \forall t \geq 0,$$

And by using (3.3.27), we get

$$L^{'}(t) \leq -\nu_1 L(t), \ \forall t \geq 0, \tag{3.4.41}$$

where $\nu_1 = \dfrac{\alpha}{c_2} > 0$. A simple integration of (3.4.41) over $(0, t)$, we get

$$\int_0^t \frac{L^{'}(t)}{L(t)} dt \leq -\int_0^t \nu_1 dt, \ \forall t \geq 0,$$

implies

$$\ln(L(t)) - \ln(L(0)) \leq -\nu_1 t, \ \forall t \geq 0,$$

we obtain

$$L(t) \leq L(0) e^{-\nu_1 t}, \ \forall t \geq 0,$$

we get

$$E(t) \leq \lambda_0 e^{-\lambda_1 t}, \ \forall t \geq 0.$$

where $\lambda_0 = \dfrac{E(0)}{c_2}$.

By using the other side of the equivalence relation (3.3.27) and using (3.3.28) again, we get

$$L^{'}(t) \leq -\nu_2 L(t), \ \forall t \geq 0, \tag{3.4.42}$$

where $\nu_2 = \dfrac{\alpha}{c_1}$. A simple integration of (3.4.42) over $(0, 1)$, we get

$$L(t) \leq L(0) e^{-\nu_2 t}, \ \forall t \geq 0,$$

implies

$$E\left(t\right) \leq \lambda_0 e^{-\lambda_2 t},\ \forall t \geq 0.$$

where $\lambda_0 = \dfrac{E\left(0\right)}{c_1}$. *The proof is complete.* $\square$

CHAPTER 4

Numerical simulation

In this section, we will solve numerically the system (0.0.1) $-$ (0.0.2) in the one-dimension domain. For that, we used the Euler scheme for discretization of temporal variable and the classic finite difference method for discretization of spatial variable. Furthermore, in order to verify the asymptotic behavior of the solution of discretize problem, we give some examples in which the numerical experiments show that the discrete energy E^n decays exponentially for different choices of the system parameters.

4.1 Discretization of the continuous problem

Let us introduce the functions $\hat{\varphi} = \varphi_t$, $\hat{\psi} = \psi_t$ and for any $M, N \in \mathbb{N}$, we introduce the nets

$$\Omega_N = \left\{ x_i = ih,\ i = 0, ..., N+1 \text{ where } h = \frac{1}{N+1} \right\},$$

$$\Gamma_M = \left\{ t_n = n\Delta t,\ n = 0, ..., M+1 \text{ where } \Delta t = \frac{T}{M+1} \right\}.$$

Taking a backward Euler scheme in time and finite differences in space, our problem consists to find $(\hat{\varphi}, \psi, \omega)$ satisfying the following numerical scheme

$$\left\{ \begin{array}{l} \frac{\rho_1}{\Delta t}\left(\hat{\varphi}_i^n - \hat{\varphi}_i^{n-1}\right) = \frac{k}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right) + \frac{k}{2h}\left(\psi_{i+1}^n - \psi_{i-1}^n\right) - \mu_1 \hat{\varphi}_i^n, \\ \frac{-\rho_2}{2h\Delta t}\left(\hat{\varphi}_{i+1}^n - \hat{\varphi}_{i-1}^n\right) = \frac{-\rho_2}{2h\Delta t}\left(\hat{\varphi}_{i+1}^{n-1} - \hat{\varphi}_{i-1}^{n-1}\right) + \frac{b}{h^2}\left(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n\right) \\ -\frac{k}{2h}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) - k\psi_i^n - \frac{\gamma}{2h}\left(\omega_{i+1}^n - \omega_{i-1}^n\right), \\ \frac{\rho_3}{\Delta t}\left(\omega_i^n - \omega_i^{n-1}\right) = \frac{k_1}{h^2}\left(\omega_{i+1}^n - 2\omega_i^n + \omega_{i-1}^n\right) - \lambda\omega_i^n - \frac{\gamma}{2h}\left(\hat{\psi}_{i+1}^n - \hat{\psi}_{i-1}^n\right), \end{array} \right. \tag{4.1.1}$$

where $\varphi_i^n = \varphi(x_i, t_n)$, $\hat{\varphi}_i^n = \varphi_t(x_i, t_n)$, $\psi_i^n = \psi(x_i, t_n)$, $\hat{\psi}_i^n = \psi_t(x_i, t_n)$, $\omega_i^n = \omega(x_i, t_n)$, for all $i = 1, ..., N$ and $n = 1, ..., M$. To simplicity our numerical calculations in our scheme, we consider the discrete boundary conditions given by

$$\begin{cases} \psi_0^n = \psi_{N+1}^n = 0, \\ \varphi_{N+1}^n = \varphi_N^n,\ \varphi_1^n = \varphi_0^n, \\ \omega_{N+1}^n = \omega_N^n,\ \omega_1^n = \omega_0^n, \end{cases} \tag{4.1.2}$$

and initial conditions

$$\begin{cases} \psi_i^0 = \psi_0(x_i),\ \hat{\psi}_i^0 = \psi_1(x_i),\ \varphi_i^0 = \varphi_0(x_i),\ \hat{\varphi}_i^0 = \varphi_1(x_i), \\ \omega_i^0 = \omega_0(x_i), \end{cases} \tag{4.1.3}$$

where

$$\varphi_i^n = \varphi_i^{n-1} + \Delta t \hat{\varphi}_i^n,\ \hat{\psi}_i^n = \frac{\psi_i^n - \psi_i^{n-1}}{\Delta t},$$

for all $i = 1, ..., N$ and $n = 1, ..., M$.

Note that to find $(\hat{\varphi}, \psi, \omega)$. we need to solve three coupled systems of algebraic equations. So, to solve the problem (4.1.1) $-$ (4.1.3) at each time step, we propose to consider the following fixed-point algorithm

$$\begin{cases} \hat{\varphi}_i^{n,l} = \frac{k}{c_1 h^2}\left(\varphi_{i+1}^{n,l-1} - 2\varphi_i^{n,l-1} + \varphi_{i-1}^{n,l-1}\right) + \frac{k}{c_1 2h}\left(\psi_{i+1}^{n,l-1} - \psi_{i-1}^{n,l-1}\right) + \frac{\rho_1}{c_1 \Delta t}\hat{\varphi}_i^{n-1}, \\ \frac{b}{h^2}\psi_{i+1}^{n,l} - \left(2\frac{b}{h^2} + k\right)\psi_i^{n,l} + \frac{b}{h^2}\psi_{i-1}^{n,l} = \frac{-\rho_2}{2h\Delta t}\left(\hat{\varphi}_{i+1}^{n,l} - \hat{\varphi}_{i-1}^{n,l}\right) + \frac{\rho_2}{2h\Delta t}\left(\hat{\varphi}_{i+1}^{n-1,l} - \hat{\varphi}_{i-1}^{n-1,l}\right) \\ + \frac{k}{2h}\left(\varphi_{i+1}^{n,l} - \varphi_{i-1}^{n,l}\right) + \frac{\gamma}{2h}\left(\omega_{i+1}^{n,l-1} - \omega_{i-1}^{n,l-1}\right), \\ \frac{k_1}{h^2}\omega_{i+1}^{n,l} + c_3\omega_i^{n,l} + \frac{k_1}{h^2}\omega_{i-1}^{n,l} = \frac{\gamma}{2h}\left(\hat{\psi}_{i+1}^{n,l} - \hat{\psi}_{i-1}^{n,l}\right) - \frac{\rho_3}{\Delta t}\omega_i^{n-1,l-1}, \end{cases} \tag{4.1.4}$$

where $c_1 = \frac{\rho_1}{\Delta t} + \mu_1$, $c_2 = -\left(2\frac{b}{h^2} + k\right)$, $c_3 = -\left(\frac{2k_1}{h^2} + \lambda + \frac{\rho_3}{\Delta t}\right)$, with

$$\begin{cases} \varphi_i^{n,0} = \varphi_i^{n-1},\ \psi_i^{n,0} = \psi_i^{n-1},\ \omega_i^{n,0} = \omega_i^{n-1},\ \varphi_i^{n,l} = \varphi_i^{n-1} + \Delta t\hat{\varphi}_i^{n,l}, \\ \psi_i^{n,l} = \psi_i^{n-1} + \Delta t\hat{\psi}_i^{n,l}, \end{cases}$$

for all $i = 1, ..., N$ and $n = 1, ..., M$, and $l = 1, 2...$

4.2 Matrix form of the discretized problem

The system (4.1.4) can be rewritten as follows

$$\begin{cases} \hat{\Phi}^{n,l} = \dfrac{k}{c_1 h^2} A_1 \Phi^{n,l-1} + \dfrac{k}{c_1 2h} C_1 \Psi^{n,l-1} + \dfrac{\rho_1}{c_1 \Delta t} B_1 \hat{\Phi}^{n-1}, \\ \\ A_2 \Psi^{n,l} = \dfrac{-\rho_2}{2h\Delta t} B_2 \hat{\Phi}^{n,l} + \dfrac{\rho_2}{2h\Delta t} B_2 \hat{\Phi}^{n-1,l} + \dfrac{k}{2h} C_1 \Phi^{n,l} + \dfrac{\gamma}{2h} C_1 \Upsilon^{n,l-1}, \\ \\ A_3 \Upsilon^{n,l} = -\dfrac{\rho_3}{\Delta t} B_1 \Upsilon^{n-1,l-1} - \dfrac{\gamma}{2h} B_2 \hat{\Psi}^{n,l}, \end{cases}$$

with

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & . & . & . & 0 \\ 1 & -2 & . & & & & . \\ 0 & . & . & . & & & . \\ . & & & . & . & . & 0 \\ . & & & & . & -2 & 1 \\ 0 & . & . & . & 0 & 1 & -1 \end{pmatrix}, \; B_1 = Id, \; C_1 = \begin{pmatrix} -1 & 1 & 0 & . & . & . & 0 \\ -1 & 0 & . & & & & . \\ 0 & . & . & . & & & . \\ . & & & . & . & . & 0 \\ . & & & & . & 0 & 1 \\ 0 & . & . & . & 0 & -1 & 0 \end{pmatrix}$$

$$A_2 = Tridiag\left(\frac{b}{h^2}, -c_2, \frac{b}{h^2}\right), \; B_2 = Tridiag\,(-1, 0, 1), \; A_3 = Tridiag\left(\frac{k_1}{h^2}, c_3, \frac{k_1}{h^2}\right).$$

At each time step, we solve the scheme (4.1.4) by an iterative procedure that was stopped when the difference between two successive iterations becomes smaller than a given tolerance ε.

4.3 Approximation of the discrete energy

To approximate the continuous energy (3.1.1), we use the trapezoidal quadrature formula to compute the integral $I = \int_0^1 f(x)dx$

$$I_N = \sum_{i=1}^{N} a_i f(x_i) \approx I,$$

where the weights $\{a_i\}_{i=1}^{N}$ are given by $a_1 = a_N = \dfrac{h}{2}$ and for $i = 2, 3, ..., N-1$, $a_i = h$. Therefore, the discrete energy formula is given by

$$\begin{aligned} E(t_n) \;\approx\; J^n &= \frac{1}{2}\sum_{i=1}^{N} a_i[\rho_1 \left(\hat{\varphi}_i^n\right)^2 \; + k\left(\left(\varphi_x\right)_i^n + \psi_i^n\right)^2 + \frac{\rho_2\rho_1}{k}\left(\left(\hat{\varphi}_t\right)_i^n\right)^2 \\ &+\rho_2\left(\left(\varphi_{tx}\right)_i^n\right)^2 + b\left(\left(\psi_x\right)_i^n\right)^2 + \rho_3\left(\omega_i^n\right)^2], \end{aligned} \tag{4.3.5}$$

with

$$\hat{\varphi}_i^n = \varphi_t\left(x_i, t_n\right), \;\; \left(\hat{\varphi}_t\right)_i^n = \frac{1}{\Delta t}\left(\hat{\varphi}_i^{n+1} - \hat{\varphi}_i^n\right).$$

$$\left(\varphi_x\right)_i^n = \frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2h}, \;\; \left(\psi_x\right)_i^n = \frac{\psi_{i+1}^n - \psi_{i-1}^n}{2h} \text{ and } \left(\varphi_{tx}\right)_i^n = \frac{\hat{\varphi}_{i+1}^n - \hat{\varphi}_{i-1}^n}{2h}.$$

4.4 Numerical illustration

In the next, we describe some numerical examples.

Example 4.4.1. *For this numerical test, we choose the following different values for the coefficients of the system*

$$\begin{aligned} \rho_1 &= 1.1,\; \rho_2 = 10,\; k = 0.01,\; \mu_1 = 10,\; b = 1.5, \\ \gamma &= 0.5,\; \rho_3 = 0.2,\; k_1 = 20,\; \lambda = 2. \end{aligned}$$

We run our code for the following discretization parameters: $N = 200$, $M = 300$, $T = 1$ *and take* $\varepsilon = 10^{-5}$. *With the following initial conditions*

$$\begin{aligned} \varphi_0\left(x\right) &= \frac{17}{20}x^2e^{-2x},\; \varphi_1\left(x\right) = \frac{1}{4}\left(x^3 - \frac{2}{3}x^2\right) \\ \psi_0\left(x\right) &= x^3\left(1-x\right)^2,\; \omega_0\left(x\right) = x^3 - \frac{2}{3}x^2. \end{aligned}$$

Here are the evolution in time of the solutions φ, ψ, ω *and of the discrete energy.*

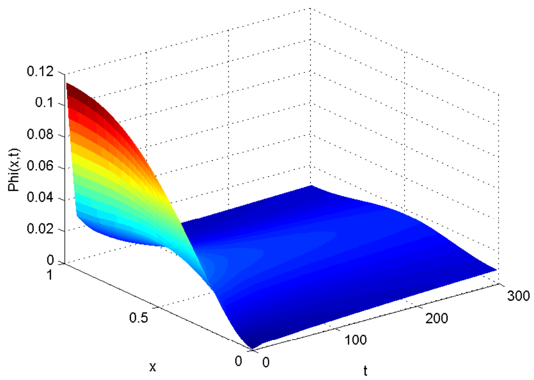

Figure 4.1: Evolution in time of the function φ (Example1)

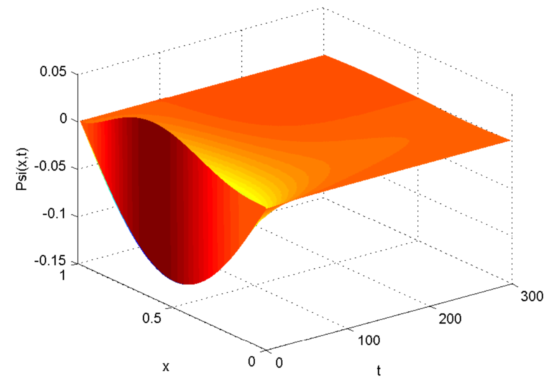

Figure 4.2: Evolution in time of the function ψ (Example1)

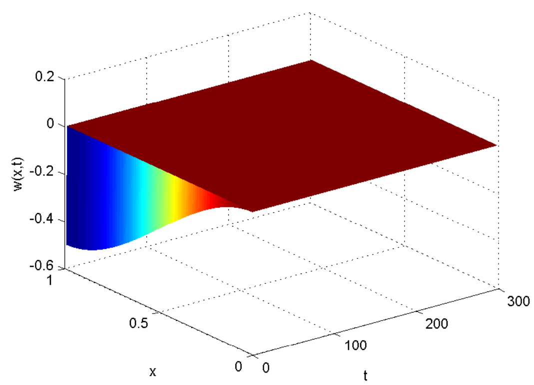

Figure 4.3: Evolution in time of the function ω (Example1)

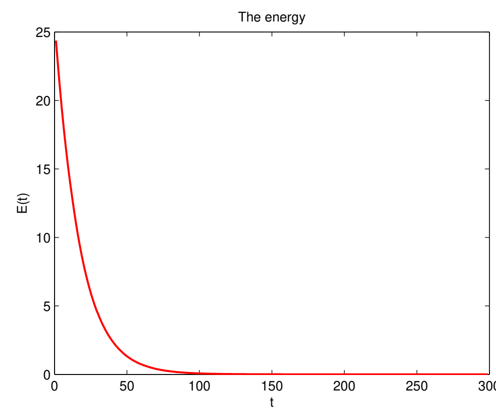

Figure 4.4: Evolution in time of the discrete energy (Example1)

Example 4.4.2. *For this numerical test, we choose the following different values for the coefficients*

$$
\begin{aligned}
\rho_1 &= 0.8,\ \rho_2 = 7,\ k = 0.04,\ \mu_1 = 12,\ b = 2.5, \\
\gamma &= 3.5,\ \rho_3 = 2.8,\ k_1 = 15,\ \lambda = 4.5
\end{aligned}
$$

We run our code for the following discretization parameters: $N = 300$, $M = 500$, $T = 1$ *and take* $\varepsilon = 10^{-5}$. *With the following initial conditions*

$$
\begin{aligned}
\varphi_0(x) &= \frac{1}{5} x^3 e^{-\frac{2}{3}x},\ \varphi_1(x) = 3x^5 (1-x)^3 \\
\psi_0(x) &= \frac{1}{10} \sin(\pi (x+5)),\ \omega_0(x) = 0.
\end{aligned}
$$

Here are the evolution in time of the solutions φ, ψ, ω and of the discrete energy in this case.

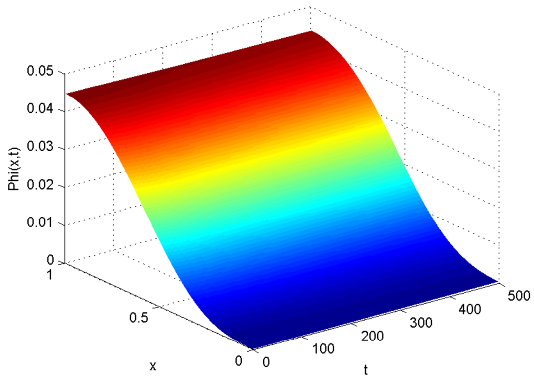

Figure 4.5: Evolution in time of the function φ (Example2)

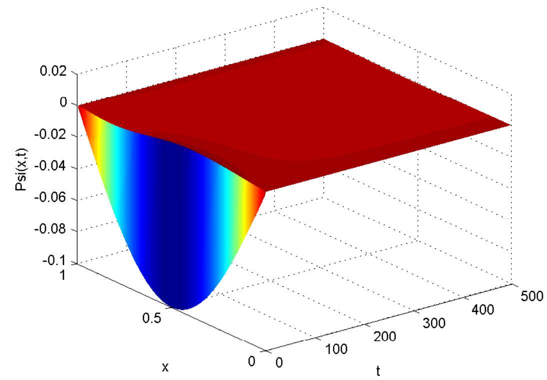

Figure 4.6: Evolution in time of the function ψ (Example2)

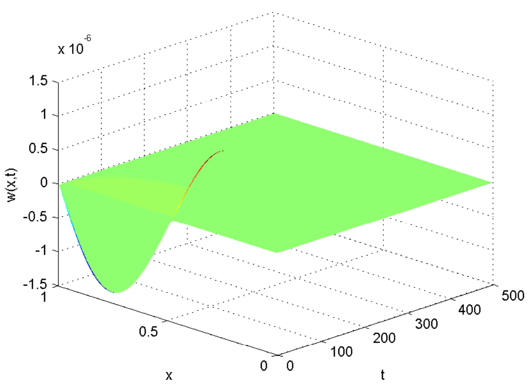

Figure 4.7: Evolution in time of the function ω (Example2)

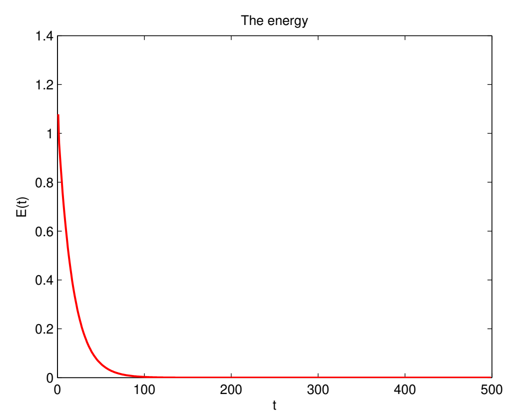

Figure 4.8: Evolution in time of the discrete energy (Example2)

In each above numerical example, the graphics presented in the Figures 4.1–4.3, 4.5–4.7 show the evolution in time of the approximations solutions φ, ψ and ω on the interval $[0, T]$, for different choices of the system parameters and of the initial data. Furthermore, the Figures 4.4 and 4.8 show that the approximate energy (4.3.5) decays in an exponential manner which confirms the main theoretical result obtained.

CONCLUSION

The Bresse–Timoshenko system, especially in the presence of thermal effects, has emerged as a compelling subject of research due to its rich mathematical structure and practical relevance in structural dynamics. In recent years, considerable attention has been devoted to analyzing its stability properties, particularly in the context of thermoelastic and viscoelastic effects. This work has focused on the theoretical analysis and numerical simulation of a heat Bresse–Timoshenko system, highlighting both the existence and uniqueness of solutions as well as the long-term behavior of the system through energy decay analysis. By constructing suitable Lyapunov functionals and employing rigorous mathematical tools, we established conditions for exponential decay of the system's energy.

Beyond the scope of this study, other variants of the Bresse–Timoshenko model—such as those with memory effects, time delays, or diffusion phenomena—offer further opportunities for exploration. These systems may exhibit different types of stability behavior (exponential, polynomial, or general) depending on their structural parameters and the nature of the dissipative mechanisms involved.

Future work may extend these results by incorporating more complex physical effects, refining numerical schemes, or developing control strategies to optimize stability and response in practical applications.

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