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Global Stability of Second Order Rational Difference Equations

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Dedecation

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Finally, I have a great expectation that my study will be beneficial and useful for anyone who is interested in reading this final project.

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Abstract

This brief is devoted to the study of behavior of the solution of certain diffrence equations, more precisely the equation with the rational diffrence of order two.

We study the local and global stability of equilibruim points, and the invariant intervals. We also present the necessary and sufficient conditions for the boundness of the solution of these equations. And we justify these results by numerical tests using Matlab.

Introduction

Difference equation or discrete dynamical system is a diverse field wich impacts almost every branch of pure and applied mathematics. Also,they are the basis of applicable analysis according to L. Euler, P. L. Tchebyheff and A, A. Markov. Every dynamical system $x_{n+1} = f(x_n)$ determines a difference equation and vice versa. Recently, there has been a great interest in studying difference equation systems. One of the reasons for this is the necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, and so forth.

Theory of difference equations plays an inportant role in mathematics. Nonlinear difference equation of order greater than one are of paramount importance in applications . Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay difference equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering.and economics. Many researchers have investigated the behavior of solutions of a system of nonlinear difference equations for example in [4] Beso et al. investigated the boundedness and global asymptotic stability of solution of the folowing difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2},$$

where γ and δ are positive real numbers and the initial conditions x_{-1}, and x_0 are positive real numbers.

Also, in [19] Tasdemir studied the periodicity, boundedness, semi-cycles, global asymptotic stability and rate of convergence of solutions of the following higher order difference equation

$$x_{n+1} = A + B\frac{x_n}{x_{n-m}^2},$$

In this study, we will focus on the dynamics of certain difference equations especially of difference equations of a second order. The propose of this study is to present the necessary and sufficient conditions of local and global stability, boundedness of solution of this equation .

This work is devided into three chapters :

- In the first chapter, we present some definitions and theorems that we will use throughout study.
- In the second chapter, we study the local and the global stability for the following second order rational difference equation

  $$x_{n+1} = ax_n + \frac{bx_n^2 + cx_n x_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1^2}}, \quad n = 0, 1, ...,$$

  where $a,\ b,\ c,\ d,\ \alpha,\ \beta,\ \gamma$ are positive numbers and the initiale values $x_{-1},\ x_0$ are positive numbers.

  Furthermore, we study the boundedness od solutions.
- In the last chapter, we investigate invariant interval and global stability of all positive solutions of the equation

  $$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2....$$

  Where parameters $\alpha,\ \beta,\ A,\ B$ and C are positive, and the initial conditions $x_{-1},\ x_0$ are positive real numbers.
  We give a detailed description of the semi cycles of solution which the equilibrium points are globaly asymptotically stable.

Contents

Chapter 1

Preliminaries

In this preliminary chapter, we recall some general notions about difference equations and the stability with the linearization method. As well as some theorems that proved to be useful to our memoir. For more details, we refer readers to ([5],[8],[14], . . .)

Definition 1.1 (Difference equations)**.** A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n \geq 0, \tag{1.1}$$

where $f : I^{k+1} \to I$ be a continuously differentiable function.The set I is usually an interval of real numbers, or union of intervals. The solution of equation (1.1) obtained from initial point $(x_0, x_{-1}, ..., x_{-k})$ ia a sequence $\{x_n\} \in I$ such that x_n satisfies (1.1) for all $n > 0$. An initial point $(x_0, x_{-1}, ..., x_{-k})$ generates a (forward) solution $\{x_n\}$ by iteration of the function

$$(x_n, x_{n-1}, ..., x_{n-k}) \to f(x_n, x_{n-1}, ..., x_{n-k}) : I^{k+1} \to I.$$

So long as each iterate x_n stays in I. Solutions of (1.1) are called orbits or trajectories.

Remark: A difference equation of order two is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \geq 0, \tag{1.2}$$

Where $f \in C^1[I \times I, I]$ and x_{-1}, x_0 are initial values.
Now we give example for second order equation

Example 1.1. Solve the equation representing the Fibonacci numbers

$$y_{n+1} = y_n + y_{n-1}$$

where

$$y_0 = 0 \quad \text{and} \quad y_{-1} = 1.$$

1. Solve the characteristic equation
   The characteristic equation is:
   $$\lambda^2 - \lambda - 1 = 0$$
   has solutions
   $$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$
2. The general solution is therefore
   $$y_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$
3. Set the values of the constants C_1 and C_2 from the initial values
   $$y_o = 0 \text{ and } y_{-1} = 1$$
   $$\begin{cases} C_1 + C_2 = 0 \\ C_1 \frac{1+\sqrt{5}}{2} + C_2 \frac{1-\sqrt{5}}{2} = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 = \frac{1}{\sqrt{5}} \\ C_2 = -\frac{1}{\sqrt{5}} \end{cases}$$
4. The solution that satisfies the two initial conditions is therefore
   $$y_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$$

Definition 1.2 (Equilibrium Point). A point $\overline{x} \in I$ such that $\overline{x} = f(\overline{x}, \overline{x})$ for all $n \geq 0$, is called an **equilibrium point** of equation (1.2).

Example 1.2. There are two equilibrium points for the equation

$$x_{n+1} = x_n^2 + 2x_n$$

where $f(x) = x^2 + 2x$.To find these equilibrium points, we let $x^2 + 2x = x$, and solve for x. Hence there are two equilibrium points, -1, 0.

Definition 1.3 (Stability). An equilibrium point $\overline{x}$ of equation (1.2) is said to be :
1.**Locally stable** if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x_0, $x_{-1} \in I$, with

$$|x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \epsilon, \quad \text{for all} \quad n \geq -1.$$

Otherwise, the equilibrium $\overline{x}$ is called unstable.
2.**Attractive** if there exists $\mu > 0$ such that for all $x_0, x_{-1} \in I$, with

$$|x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \mu,$$

then

$$\lim_{n\to\infty} x_n = \overline{x}.$$

If $\mu = \infty$, $\overline{x}$ is called **globally attractive** .
3.**Locally asymptotically stable** if it is stable and attractive.
4.**Globally asymptotically stable** if it is stable and globally attractive.

Definition 1.4 (Periodicity). A solution $\{x_n\}_{n>-1}$ of equation (1.2) is called periodic with period p if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n, \quad \text{for all} \quad n \geq -1. \tag{1.3}$$

A solution is called periodic with prime period p if p is the smallest positive integer for which equation (1.3) holds.

The two following definitions give the notion of **semi-cycle** analysis :

Definition 1.5 (A positive semi-cycle)**.** Let $\{x_n\}_{n=-1}^{\infty}$ be a solution to equation (1.2). A positive semi-cycle of the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1.2) consists of a "chain" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to the equilibrium point $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ such that

$$\text{either } l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \overline{x},$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \overline{x}.$$

Definition 1.6 (A negative semi-cycle)**.** Let $\{x_n\}_{n=-1}^{\infty}$ be a solution to equation (1.2). A negative semi-cycle of the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1.2) consists of a "chain" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than $\overline{x}$, with $l > -1$ and $m \leq \infty$ such that

$$\text{either } l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \overline{x},$$

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \overline{x}.$$

Definition 1.7 (Linearized stability analysis)**.** The linearized equation of equation (1.2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = py_n + qy_{n-1} \quad n \in \mathbb{N}, \tag{1.4}$$

where

$$p = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}), \quad q = \frac{\partial f}{\partial v}(\overline{x}, \overline{x})$$

the characteristic equation of equation (1.4) is

$$\lambda^2 - p\lambda - q = 0, \tag{1.5}$$

Next, we set the theorem about linearized stability.

Theorem 1.1. *[14] Let $\overline{x}$ be an equilibrium point of equation* (1.2) *then the following statements are true*

1. *If all roots of equation* (1.5) *lie inside the open unit disk* $|\lambda| < 1$*. Then* $\overline{x}$ *is locally asympotically stable.*

2. *If at least one root of equation* (1.5) *has absolute value greater than one, then* $\overline{x}$ *is unstable.*

3. *A necessary and sufficient condition for the two roots of equation* (1.5) *to be located inside the open unit disk* $|\lambda| < 1$ *is*

$$|p| < 1 - q < 2.$$

In this case, the equilibrium point is locally asymptotically stable equilibrium.

4. *A necessary and sufficient condition for the roots of polynomial (1,10) to have a modulus greater than one is*

$$|q| > 1 \textit{ and } p < |1 - q|.$$

Definition 1.8. The equilibrium point $\overline{x}$ of equation is called

1. **sink** if, no root of equation (1.5) has absolute value equal to one. If there exists a root of equation (1.5) with absolute value equal to one, then the equilibrium point $\overline{x}$ is called non-sink.

2. **A saddle point** if, it is sink and if there exists a root of equation (1.5) with absolute value less than one and another root of equation (1.5) with absolute value greater than one.

3. **source** if all root of equation (1.5) have absolute value greater than one.

The following theorem provides a sufficient condition for the global asymptotic stability of equation (1.5).

Theorem 1.2. *[10] Consider the following difference equation*

$$x_n = f_o(x_n, x_{n-1})x_n + f_1(x_n, x_{n-1})x_{n-1}, \quad n = 0, 1, ..., \tag{1.6}$$

with non-negative initial conditions and

$$f_0, f_1 \in C\left[[0,\infty) \times [0,\infty[, [0,1)\right].$$

Suppose the following assumptions are satisfied:

i) f_o *and* f_1 *are decreasing for each* $x, y \in (0,\infty)$*;*

ii) $f_o(x,x) > 0$ *for all* $x \geq 0$*;*

iii) $f_o(x,y) + f_1(x,y) < 1$ *for all* $x, y \in (0,\infty)$.

Then the zero equilibrium of equation (1.6) is globally asymptotically stable.

Now we give some convergence theorems [5], [11] for second-order difference equations, useful for proving results.

Theorem 1.3. *[13] Consider the difference equation defined by [5], [11]*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, \tag{1.7}$$

with

$$g : [a,b] \times [a,b] \longrightarrow [a,b] \quad a, b \in \mathbb{R}.$$

Suppose that g *is a continuous function such that*

1. $g(x,y)$ *is increasing with respect to* $x \in [a,b]$ *for each* $y \in [a,b]$ *and* $g(x,y)$ *is decreasing with respect to* $y \in [a,b]$ *for each* $x \in [a,b]$.

2. *If* (m, M) *is a solution of the system*

$$m = g(m, M), \quad M = g(M, m).$$

Then

$$m = M.$$

Then the equation (1.7) *has a unique equilibrium point* $\overline{x}$ *and every solution of the equation* (1.7) *converges to* $\overline{x}$.

Theorem 1.4. *[13] Let's consider the difference equation defined by*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1... \tag{1.8}$$

with

$$g : [a, b] \times [a, b] \longrightarrow [a, b] \quad a, b \in R.$$

Suppose that g is a continuous function such that

1. $g(x, y)$ *is decreasing with respect to* $x \in [a, b]$ *for each* $y \in [a, b]$ *and* $g(x, y)$ *is increasing with respect to* $y \in [a, b]$ *for each* $x \in [a, b]$.
2. *If* (m, M) *is a solution of the system*

$$m = g(M, m), \quad M = g(m, M),$$

Then

$$m = M.$$

Then equation (1.8) *has a unique equilibrium point* $\overline{x}$ *and every solution of equation* (1.8) *converges to* $\overline{x}$.

Theorem 1.5. *[13] Let [a, b] be an interval of real numbers and suppose that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \quad a, b \in \mathbb{R}$$

is a continuous function satisfying the following properties :

1. $f(x, y)$ *is decreasing with respect to* $x \in [a, b]$ *for each* $y \in [a, b]$ *and*
2. $f(x, y)$ *is increasing with respect to* $y \in [a, b]$ *for each* $x \in [a, b]$.

The difference equation of equation (1.2) *has no two-periodic solutions in* $[a, b]$.

Then, equation (1.2) *has a unique equilibrium* $\overline{x} \in [a, b]$ *and every solution of equation* (1.2) *converges to* $\overline{x}$.

Theorem 1.6. *[12] Let* $[a, b]$ *be an interval of real numbers and suppose that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \quad a, b \in \mathbb{R},$$

is a continuous function such that :

1. $f(x, y)$ *is decreasing or increasing for each* $x, y \in [a, b]$.
2. *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(M, M), \quad M = f(m, m),$$

then

$$m = M.$$

Therefore, equation (1.2) *has a unique equilibrium point* $\overline{x} \in [a, b]$ *and any solution of equation* (1.2) *converges to* $\overline{x}$.

Theorem 1.7. *[8] Let* $[a, b]$ *an interval of real numbers and suppose that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \quad a, b \in \mathbb{R},$$

is a continuous function satisfying the following properties :

1. $f(x, y)$ *is increasing for each* $x, y \in [a, b]$.
2. *The equation*

$$f(x, x) = x,$$

has a unique positive solution.

Then the equation has a unique equilibrium point in $\overline{x} \in [a, b]$ *and any solution of the equation converges to* $\overline{x}$.

Theorem 1.8. *[10] Let* $I \subseteq [0, +\infty)$ *and suppose that*

$$f \in C\left(I \times I, (0, +\infty)\right)$$

is a function satisfying the following conditions :

1. *f(x, y) is increasing for each* $x, y \in [a, b]$.
2. *The equation* (1.2) *has a unique positive equilibrium point* $\overline{x} \in I$ *and the function* $f(x, x)$ *satisfies the condition :*

$$(x - \overline{x})(f(x, x) - x) < 0 \quad \textit{for all} \quad x \in I - \{\overline{x}\}$$

Then, every positive solution of the equation (1.2) *converges to* $\overline{x}$.

Chapter 2

Qualitative Behavior of a Second-Order Rational Difference Equation

Second order rational difference equation with quadratic terms show a wide variety of dynamic behaviors, It is shown that relying on the parameters and initial values there can be globally attracting equilibrium points.

In our chapter we investigate the boundedness and local stability of solution, the global attractivy of the positive equilibrium point for the rational difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2 + cx_n x_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}^2}, \quad n = 0, 1, ..., \tag{2.1}$$

where the constants a, b, c, d, α, β and γ are positive real numbers and the initial conditions x_{-1} and x_0 are arbitray non zero real numbers.
There is no doubt that the Theory of difference equations problems occurred in biology, physics and economies. In fact, the theory of discrete rational diffrence equations has been greatly analyzed in recent decades.

2.1 Linearized Stability of equation (2.1)

This section proves that equation (2.1) has a unique equilibrium point wihch is asymptotically stable under a certain condition.
The fixed point of equation (2.1) is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2 + c\overline{x}\overline{x} + d\overline{x}^2}{\alpha\overline{x}^2 + \beta\overline{x}\overline{x} + \gamma\overline{x}^2},$$

then

$$\overline{x} - a\overline{x} = \frac{\overline{x}^2(b + c + d)}{\overline{x}^2(\alpha + \beta + \gamma)},$$

$$\overline{x}(1 - a) = \frac{(b + c + d)}{(\alpha + \beta + \gamma)}$$

from which we can obtain the following unique equilbium point :

$$\overline{x} = \frac{(b + c + d)}{(\alpha + \beta + \gamma)(1 - a)}, \quad a \neq 1.$$

Next, we define a function $f : (0, \infty)^2 \longrightarrow (0, \infty)$ as follows:

$$f(u, v) = au + \frac{bu^2 + cuv + dv^2}{\alpha u^2 + \beta uv + \gamma v^2}. \tag{2.2}$$

We now turn to find the following partial derivatives:

$$\frac{\partial f(u, v)}{\partial u} = a + \frac{(2bu + cv)(\alpha u^2 + \beta uv + \gamma v^2) - (bu^2 + cuv + dv^2)(2\alpha u + \beta v)}{(\alpha u^2 + \beta uv + \gamma v^2)^2}$$

$$= a + \frac{(b\beta - c\alpha)u^2 v + 2(b\gamma - d\alpha)uv^2 + (c\gamma - d\beta)v^3}{(\alpha u^2 + \beta uv + \gamma v^2)^2},$$

$$\frac{\partial f(u,v)}{\partial v} = \frac{(cu+2dv)(\alpha u^2 u+\beta uv+\gamma v^2)-(bu^2+cuv+dv^2)(\beta u+2\gamma v)}{(\alpha u^2+\beta uv+\gamma v^2)^2}$$

$$= \frac{2(d\alpha-b\gamma)u^2 v+(d\beta-c\gamma)uv^2+(c\alpha-b\beta)u^3}{(\alpha u^2+\beta uv+\gamma v^2)^2}.$$

Next, evaluating these partial derivatives at the fixed point gives

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = a + \frac{(b\beta-c\alpha)\left(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)}\right)^3+2(b\gamma-d\alpha)\left(\frac{b+c+d}{(1-a)(\alpha+\beta+c)}\right)^3+(c\gamma-d\beta)\left(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)}\right)^3}{\left(\alpha\left(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)}\right)^2+\beta\left(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)}\right)^2+\gamma\left(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)}\right)^2\right)^2}$$

$$= a + \frac{(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)})^3(b\beta-c\alpha+2b\gamma-2d\alpha+c\gamma-d\beta)}{((\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)})^2(\alpha+\beta+\gamma))^2}$$

$$= a + \frac{(b\beta-c\alpha+2b\gamma-2d\alpha+c\gamma-d\beta)}{(\alpha+\beta+\gamma)^2(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)})}$$

$$= a + \frac{(b\beta-c\alpha+2b\gamma-2d\alpha+c\gamma-d\beta)(1-a)}{(\alpha+\beta+\gamma)(b+c+d)}$$

$$= a + \frac{[(b-d)\beta-(2d+c)\alpha+(2b+c)\gamma](1-a)}{(\alpha+\beta+\gamma)(b+c+d)} = -p_1$$

$$\frac{\partial f(\overline{x},\overline{x})}{\partial v} = \frac{2(d\alpha-b\gamma)\overline{x}^3+(d\beta-c\gamma)\overline{x}^3+(c\alpha-b\beta)\overline{x}^3}{(\alpha\overline{x}^2+\beta\overline{x}^2+\gamma\overline{x}^2)^2}$$

$$= \frac{\overline{x}^3(d\alpha-b\gamma+d\beta-c\gamma+c\alpha-b\beta)}{\overline{x}^4(\alpha+\beta+\gamma)^2}$$

$$=\frac{d\alpha-b\gamma+d\beta-c\gamma+c\alpha-b\beta}{(\alpha+\beta+\gamma)^2(\frac{b+c+d}{(1-a)(\alpha+\beta+\gamma)})}$$

$$=\frac{(d\alpha-b\gamma+d\beta-c\gamma+c\alpha-b\beta)(1-a)}{(\alpha+\beta+\gamma)(b+c+d)}$$

$$=\frac{[(d-b)\beta+(2d+c)\alpha-(2b+c)\gamma](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}=-p_2$$

The corresponding linearized diffference equation of (2.1) about the equilibrium point is give by:

$$y_{n+1}+p_1y_n+p_2y_{n-1}=0$$

.

Theorem 2.1. *Suppose that*

$$2|A|<(\alpha+\beta+\gamma)(b+c+d)\quad a<1,$$

where

$$A=(b-d)\beta-(2d+c)\alpha+(2b+c)\gamma.$$

Then, the equilibrium point of equation (2.1) *is locally asymptically stable.*

Proof. As started in Theorem (1.1) the fixed point of equation (2.1) is asymptotically stable if

$$|p_1|+|p_2|<1.$$

This can be written as

$$|a+\tfrac{A(1-a)}{(\alpha+\beta+\gamma)(b+c+d)}|+|\tfrac{A(1-a)}{(\alpha+\beta+\gamma)(b+c+d)}|<1,$$

which can be rearranged as follows:

$$|a(\alpha+\beta+\gamma)(b+c+d)+2A(1-a)|<(\alpha+\beta+\gamma)(b+c+d).$$

Thus,

$$2|A(1-a)| < (1-a)(\alpha+\beta+\gamma)(b+c+d).$$

If $a < 1$, we have

$$2|A| < (\alpha+\beta+\gamma)(b+c+d),$$

The proof is complete.

2.2 Global Attractivity Results

In this section, the global stability of the equilibrium point will be pointed out. There are different cases for equation (2.2) to be increasing or decreasing in the first and second variables. Four cases will be explained and proved in details in the following theorems. It should be noted that each case will be introduced in a specific theorem.

Theorem 2.2. *Let equation* (2.2) *be increasing in the first and the second variable. Then, the fixed point of equation* (2.1) *is a global attractor if* $a \neq 1$.

Proof. Assume that equation (2.2) is increasing in the first and the second variable, and let $(m.M)$ be a solution of the following system:

$$\begin{aligned} m = f(m,m) =& am + \frac{bm^2+cm^2+dm^2}{\alpha m^2+\beta m^2+\gamma m^2}, \\ M = f(M,M) =& aM + \frac{bM^2+cM^2+dM^2}{\alpha M^2+\beta M^2+\gamma M^2}. \end{aligned}$$

Simplifying this gives

$$\begin{aligned} m =& am + \frac{(b+c+d)m^2}{(\alpha+\beta+\gamma)m^2} = \frac{b+c+d+am(\alpha+\beta+\gamma)}{(\alpha+\beta+\gamma)m^2} \\ M =& aM + \frac{(b+c+d)M^2}{(\alpha+\beta+\gamma)M^2} = \frac{b+c+d+aM(\alpha+\beta+\gamma)}{\alpha+\beta+\gamma} \end{aligned}$$

yileds

$$m(\alpha+\beta+\gamma) = am(\alpha+\beta+\gamma) + b + c + d \tag{2.3}$$

$$M(\alpha+\beta+\gamma) = aM(\alpha+\beta+\gamma) + b + c + d \tag{2.4}$$

Substracting equation (2.4) from equation (2.3) yields.

$$(M-m)(\alpha+\beta+\gamma) = a(M-m)(\alpha+\beta+\gamma) + (b+c+d) + (b-c-d), \tag{2.5}$$

implies that

$$(M-m) = a(M-m)$$

$$(1-a)(M-m) = 0.$$

If $a \neq 1$, we have

$$M = m$$

As claimed by Theorem (1.6), the equilibrium point of equation (2.1) is a global attractor.

Theorem 2.3. *Let equation* (2.2) *be decreasing in the first and the second variable.Then, the equilibruim point of equation* (2.1) *is a global attractor.*

Proof. The proof is similar to the previous one and it will be omitted.

Theorem 2.4. *Let equation* (2.2) *be increasing in the first variable and decreasing in the second variable.Then the equilibruim point of equation* (2.1) *is global attractor if* $a < 1$, $\gamma < \alpha + \beta$ *and* $b < d$.

Proof. Let equation (2.2) be increasing in u and decreasing in v, and assume that $(m.M)$ is a solution of the system.

$$\begin{cases} m = & f(m,M) = am + \dfrac{bm^2 + cmM + dM^2}{\alpha m^2 + \beta Mm + \gamma M^2}, \\ \\ M = & f(M,m) = aM + \dfrac{bM^2 + cMm + dm^2}{\alpha M^2 + \beta Mm + \gamma m^2}, \end{cases}$$

Yields

$$\begin{cases} m = & \dfrac{am(\alpha m^2 + \beta mM + \gamma M^2) + bm^2 + cMm + dM^2}{\alpha m^2 + \beta mM + \gamma M^2} \\ \\ M = & \dfrac{aM(\alpha M^2 + bMm + \gamma m^2) + bM^2 + cMm + dm^2}{\alpha M^2 + \beta Mm + \gamma m^2} \end{cases}$$

From which we obtain:

$$\alpha m^3 + \beta m^2 M + \gamma m M^2 = a\alpha m^3 + a\beta m^2 M + a\gamma m M^2 + bm^2 + cmM + dM^2, \quad (2.6)$$

$$\alpha M^3 + \beta M^2 m + \gamma M m^2 = a\alpha M^3 + a\beta M^2 m + a\gamma M m^2 + bM^2 + cMm + dm^2. \quad (2.7)$$

Subtracting equation (2.7) from equation (2.6) and simplifying the result give

$$(m - M)\{(1 - a)[\alpha(m^2 + M^2) + (\alpha + \beta\gamma)mM] + (d - b)(m + M)\} = 0.$$

Hence, if $a < 1$, $\gamma < \alpha + \beta$ and $b < d$, then

$$m = M.$$

As stated by Theorem (1.5), the equilbrium point of equation (2.1) is a global attractor.

Theorem 2.5. *Let equation* (2.2) *be decreasing in the first variables u and inceasing in the second variables v. Then, the equilibrium point of equation* (2.1) *is a global* *attractor if* $a < 1$, $\alpha < \gamma$ *and* $d < b$.

Proof. Suppose that equation (2.2) is decreasing in u and inceasing in v, and let (m, M) be a solution of the following system:

$$m = f(M, m) = aM + \frac{bM^2 + cMm + dm^2}{\alpha M^2 + \beta Mm + \gamma m^2},$$
$$M = f(m, M) = am + \frac{bm^2 + cmM + dM^2}{\alpha m^2 + \beta mM + \gamma M^2},$$

Yields

$$m = \frac{aM(\alpha M^2 + \beta mM + \gamma m^2) + bM^2 + cMm + dm^2}{\alpha M^2 + \beta Mm + \gamma m^2},$$

$$M = \frac{am(\alpha m^2 + \beta mM + \gamma M^2) + bm^2 + cmM + dM^2}{\alpha m^2 + \beta mM + \gamma M^2}.$$

Which can be written as:

$$\alpha mM^2 + \beta Mm^2 + \gamma m^3 = a\alpha M^3 + a\beta M^2 m + a\gamma Mm^2 + bM^2 + cMm + dm^2, \quad (2.8)$$

$$\alpha m^2 M + \beta mM^2 + \gamma M^3 = a\alpha m^3 + \alpha\beta m^2 M + \alpha\gamma mM^2 + bm^2 + cmM + dM^2, \quad (2.9)$$

Substract equation (2.9) from equation (2.8) and simplify the result to have

$$(m - M)\{[\beta - \alpha + a(\beta - \gamma) + \gamma + a\alpha]mM + (\gamma + a\alpha)(m^2 + M^2) + (b - d)(m + M)\} = 0.$$

Or,

$$(m, M)\{[\beta(1 + a) + (1 - a)(\gamma - \alpha)]mM + (\gamma + a\alpha)(m^2 + M^2) + (b - d)(m + M)\} = 0.$$

Hence, if $a < 1$, $\alpha < \gamma$ and $d < b$, then

$$m = M.$$

Therefore,Theorem (1.4) assures that the fixed point of equation (2.1) is a global attractor.

2.3 Existence of Bounded Solution

Here, we will study the existence of bounded solution of equation (2.1).

Theorem 2.6. *Every solution of equation* (2.1) *is bounded if* $a < 1$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (2.1).Then, it follows from equation (2.1) that

$$\begin{aligned}
x_{n+1} &= ax_n + \frac{bx_n^2 + cx_nx_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} \\
&= ax_n + \frac{bx_n^2}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} + \frac{cx_nx_{n-1}}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} + \frac{dx_{n-1}^2}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} \\
&= ax_n + \frac{bx_n^2}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} + \frac{cx_nx_{n-1}}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} + \frac{dx_{n-1}^2}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2} \\
&\leq ax_n + \frac{bx_n^2}{\alpha x_n^2} + \frac{cx_nx_{n-1}}{\beta x_nx_{n-1}} + \frac{dx_{n-1}^2}{\gamma x_{n-1}^2} \\
&\leq ax_n + \frac{b}{\alpha} + \frac{c}{\beta} + \frac{d}{\gamma}.
\end{aligned}$$

by using compraison, we have

$$y_{n+1} = ay_n + \frac{b}{\alpha} + \frac{c}{\beta} + \frac{d}{\gamma}.$$

This difference equation has the following solution:

$$y_n = a^n y_0 + constant,$$

which is asymptotically stable if $a < 1$, and converges to the equilibrium point

$$\overline{y} = \frac{\gamma b\beta + c\alpha\gamma + d\alpha\beta}{\alpha\beta\gamma(1-a)}.$$

Therefore,

$$\lim_{n\longrightarrow\infty} \sup x_n \leq \frac{\gamma b\beta + c\gamma\alpha + d\alpha\beta}{\alpha\beta\gamma(1-a)}.$$

Theorem 2.7. *Every solution of equation* (2.1) *is unbounded if* $a \geq 1$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (2.1).Then it follows from equation (2.1) that

$$x_{n+1} = ax_n + \frac{bx_n^2 + cx_n x_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}^2} > ax_n, \quad \text{for all } n \geq 1.$$

Hence, the right hand side can be written as follows $y_{n+1} = ay_n$, which has the following solution

$$y_n = a^n y_0 + \text{constant}.$$

Since $a > 1$, $\lim_{n\to\infty} y_n = \infty$. Then, by using ratio test $\{x_n\}_{n=-1}^{\infty}$ is unbounded from above.

2.4 Numerical Examples

For confirming the theoretical results ,we consider some numerical examples which represent different types of solutions to equation (2.1) as follows:

Example 2.1. Consider the equation

$$x_{n+1} = 0.4x_n + \frac{2x_n^2 + 3x_n x_{n-1} + x_{n-1}^2}{3x_n^2 + x_n x_{n-1} + 4x_{n-1}^2} \tag{2.6}$$

with initial conditions $x_{-1} = 0.1, \quad x_0 = 0.2$.

In this case, the conditions $a < 1$, $\alpha < \gamma$, $d < b$ hold.

Then, form theorem (2.7) the equilibrium point $\overline{x}$ of equation is globally asymptotically stable. See figure (2.1).

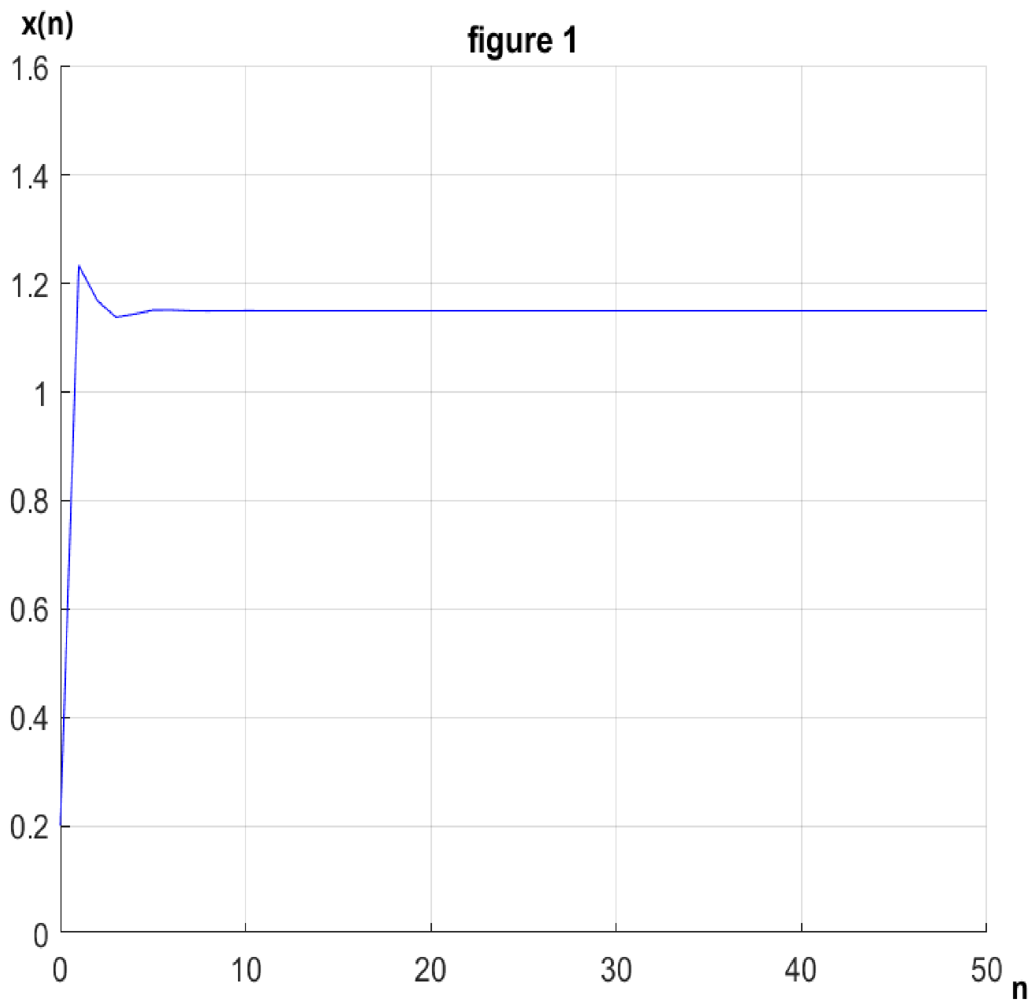

Figure 2.1: Plot of the solution $\{x_n\}$, $n \geq 0$ of equation (2.4) with the initial values $x_{-1} = 0.1, \quad x_0 = 0.2$.

Example 2.2. Consider the equation

$$x_{n+1} = 0.01x_n + \frac{2x_n^2 + x_n x_{n-1} + 3x_{n-1}^2}{5x_n^2 + 7x_n x_{n-1} + 4x_{n-1}^2}, \tag{2.7}$$

with the initial condition $x_{-1} = 0.5$, $x_0 = 0.85$.
In this case, the conditions $a < 1$, $\gamma < \alpha + \beta$, $b < d$ hold. Then from Theorem (2.4) the equilibrium point $\overline{x}$ of equation (2.7) is global attractor. See figure (2.2)

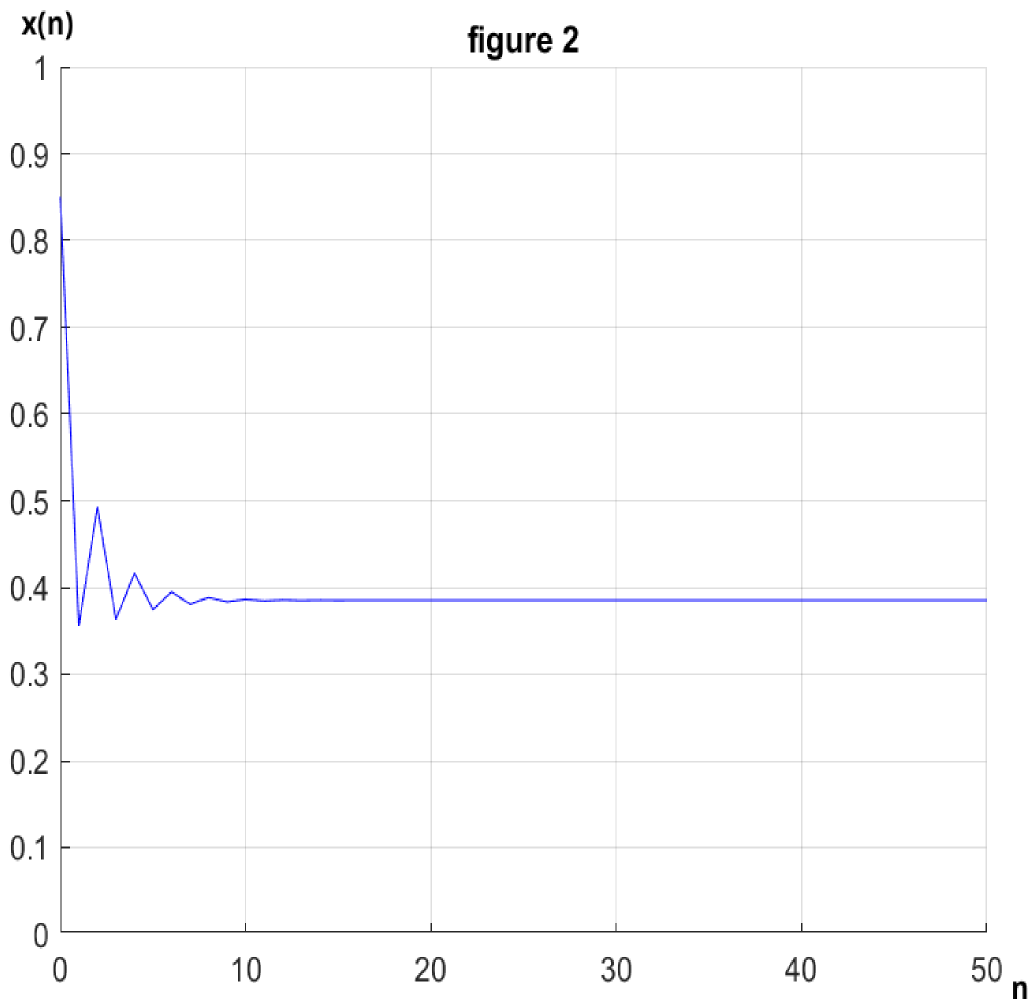

Figure 2.2: Plot of the solution $\{x_n\},\ n \geq 0$ of equation (2.7) with the initial values $x_{-1} = 0.5, \quad x_0 = 0.85$.

Example 2.3. Consider the equation

$$x_{n+1} = 0.006x_n + \frac{0.01x_n^2 + 8x_n x_{n-1} + 7x_{n-1}^2}{0.9x_n^2 + 6.35x_n x_{n-1} + 2x_{n-1}^2} \tag{2.8}$$

with the initial values $x_{-1} = -4$, $x_0 = 5$.
In this case, the conditions of theorems (2.3) are verifie. Then every solution of equation (2.8) converges to the equilibrium point $\overline{x}$.

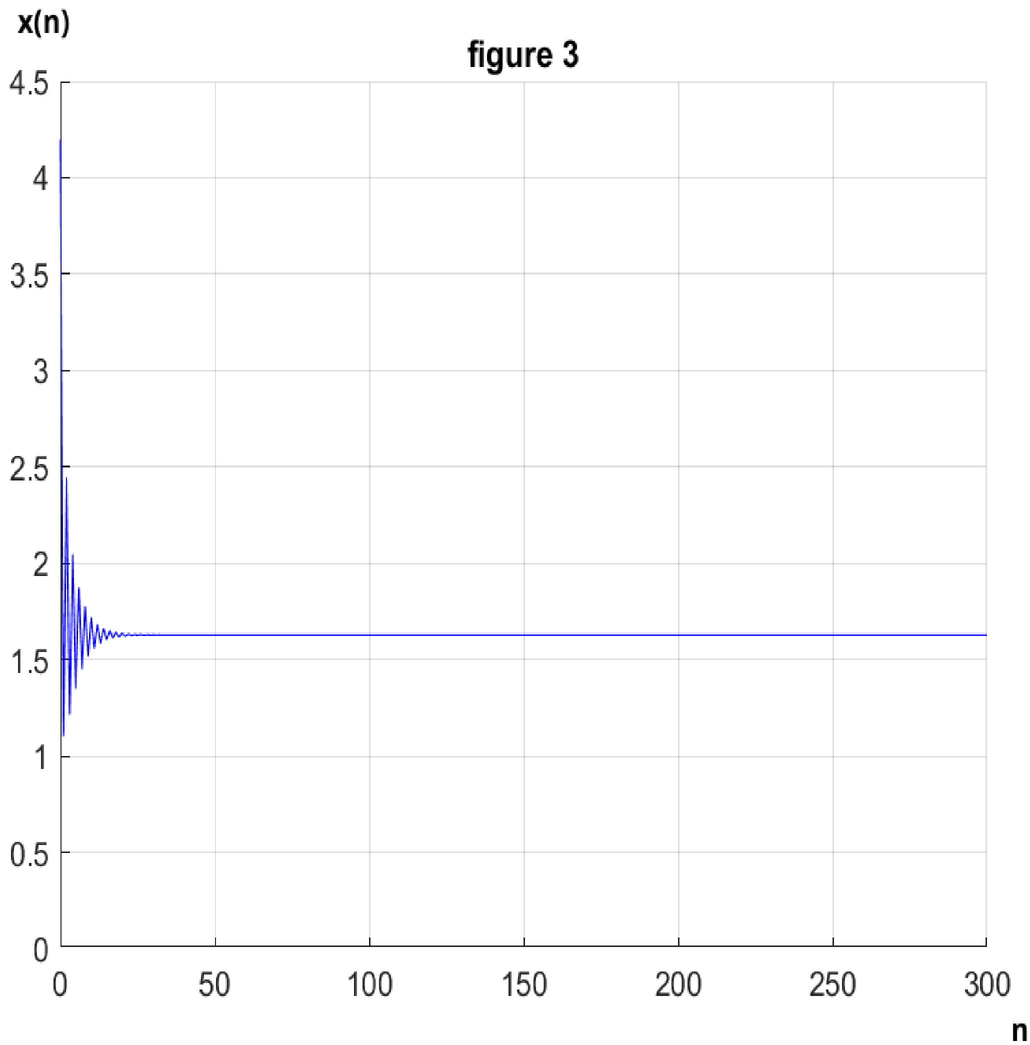

Figure 2.3: Plot the solution $\{x_n\}_{n\geq 0}$, of equation with the initial values $x_{-1} = -4, \quad x_0 = 5$.

Chapter 3

Dynamics of Second Order Rational Difference Equation

Our goal in this chapter is to study the local and global stability of equilibrium points, as well as the boundedness of solutions of the following rational difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, ..., \tag{3.1}$$

where the parameters α, β, A, B, C are positive, and the initial conditions x_{-1} and x_0 are arbitrary non-negative real numbers. We give detailed description of the semi cycles of solutions and determine conditions under which the equilibrium points are glolabally stable.

3.1 Equilibrium Points

First, note that the equilibrium points of equation (3.1) are the non-negative solutions of the equation

$$\overline{x} = \frac{\alpha + \beta \overline{x}}{A + (B + C)\overline{x}} \tag{3.2}$$

Equivalent to

$$(B + C)\overline{x}^2 - (\beta - A)\overline{x} - \alpha = 0. \tag{3.3}$$

$\overline{x} = 0$ is an equilibrium point of equation (3.1) if and only if

$$\alpha = 0 \quad \text{and} \quad A > 0, \tag{3.4}$$

when (3.4) is satisfied, in addition to the zero equilibrium, The equation (3.1) has a positive equilibrium if and only if

$$\beta > A,$$

In fact, in this case, the positive equilibrium of equation (3.1) is unique and given by

$$\overline{x} = \frac{\beta - A}{B + C}.$$

In the case

$$\alpha = 0 \text{ and } A = 0,$$

the only equilibrium point of equation (3.1) is positive and given by

$$\overline{x} = \frac{\beta}{B + C},$$

Finally, in the case where $\alpha > 0$, the only positive equilibrium point of equation (3.1) is given by

$$\overline{x} = \frac{\beta - A + \sqrt{(\beta - A)^2 + 4\alpha(B + C)}}{2(B + C)}, \tag{3.5}$$

of the quadratic equation (3.3).

In summary, it is interesting to note that when equation (3.1) has a positive equilibrium point $\overline{x}$, then $\overline{x}$ is unique, it satisfies Equations (3.2) and (3.3), and it is given by (3.1). This observation simplifies the study of the stability of the positive equilibrium point of equation (3.1).

3.2 Stability of equilibrium points

3.2.1 Stability of Zero Equilibrium

Consider the function $f : [0, \infty[\times[0, \infty[\longrightarrow [0, \infty[$ the function defined by

$$f(u, v) = \frac{\alpha + \beta u}{A + Bu + Cv},$$

the derivatives

$$f_u(u, v) = \frac{\beta(A + Bu + Cv) - B\beta u}{(A + Bu + Cv)^2},$$

and

$$f_v(u, v) = \frac{-C\beta u}{(A + Bu + Cv)^2}.$$

If $\overline{x}$ denotes an equilibrium point of equation (3.1), then the associated linearized equation to equation (3.1) around $\overline{x}$ is:

$$y_{n+1} - py_n - qy_{n-1} = 0,$$

with

$$p = f_u(u, v) \quad \text{and} \quad q = f_v(u, v).$$

The local stability of $\overline{x}$ is described by the Linearization Stability Theorem (1.1).
For the zero equilibrium of equation (3.1), we have

$$p = \frac{\beta}{A} \text{ and } q = 0$$

and thus the linearized equation associated with equation (3.1) around the zero equilibrium point is :

$$y_{n+1} - \frac{\beta}{A}y_n = 0 \quad , \; n = 0, 1, ...,$$

For the stability of the zero equilibrium of equation (3.1), in addition to the Local Stability Theorem (1.3), we have the Global Stability Theorem (1.1). Therefore, from these two theorems, we obtain the following global result:

Theorem 3.1. *[5] Suppose $A > \beta$. Then, the zero equilibrium point of the equation*

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n + Cx_{n-1}}, n = 0, 1, ...$$

is globally asymptotically stable if

$$\beta \leq A$$

and unstable if

$$\beta > A.$$

Furthermore, the zero equilibrium point is :

- *a sink if $A > \beta$,*
- *a saddle point if $A < \beta < A + 2\beta$,*
- *a source if $\beta > A + 2\beta$.*

3.2.2 Stability at the positive Equilibrium

As mentioned in the previous section, equation (3.1) has a positive equilibrium when

$$\alpha = 0 \text{ and } \beta > A, \text{ or } \alpha > 0.$$

In these cases, by using the change variable

$$x_n = \frac{A}{B} y_n,$$

then

$$x_{n+1} = \frac{A}{B} y_{n+1}$$

and

$$x_{n-1} = \frac{A}{B} y_{n-1}$$

By subtituing into the equation (3.1) we get :

$$\frac{A}{B}y_{n+1} = \frac{\alpha + \beta(\frac{A}{B}y_n)}{A + B(\frac{A}{B}y_n) + C(\frac{A}{B}y_{n-1})}, \quad n = 0, 1, ...$$

$$\frac{A}{B}y_{n+1} = \frac{\alpha + \frac{\beta A}{B}y_n}{A + Ay_n + \frac{CA}{B}y_{n-1}} = \frac{\alpha + \frac{\beta A}{B}y_n}{A(1 + y_n + \frac{C}{B}y_{n-1})}, \quad n = 0, 1, ...$$

$$y_{n+1} = \frac{B}{A}\frac{\alpha + \frac{\beta A}{B}y_n}{A(1 + y_n + \frac{C}{B}y_{n-1})} = \frac{\frac{B\alpha}{A^2} + \frac{\beta}{A}y_n}{1 + y_n + \frac{C}{B}y_{n-1}}.$$

Finally, we obtain

$$y_{n+1} = \frac{p + qy_n}{1 + y_n + ry_{n-1}}, \quad n = 0, 1, ... \tag{3.6}$$

such as

$$p = \frac{\alpha B}{A^2}, \quad q = \frac{\beta}{A}, \quad r = \frac{C}{B}$$

Which $p, q, r \in (0, \infty)$ and $y_{-1}, y_0 \in (0, \infty)$.
The equation (3.6) has a unique equilibrium point $\overline{y}$ given by

$$\overline{y} = \frac{p + q\overline{y}}{1 + \overline{y} + r\overline{y}}$$

this equilibrium point is the non-negative solution of this equation

$$\overline{y}^2(r + 1) - (q - 1)\overline{y} - p = 0,$$

and given by:

$$\overline{y} = \frac{q - 1 + \sqrt{(q - 1)^2 + 4p(r + 1)}}{2(r + 1)},$$

and according to stability Theorem (1.1), we have the following result :

Theorem 3.2. *The equilibrium point $\overline{y}$ of equation* (3.6) *is locally asymptotically* *stable for all values of* p, q *and* r.

3.3 The boundedness of solutions

The main result of this section is the following theorem, which establishes the boundedness of the solutions of equation (3.1).

Theorem 3.3. *Every solution of equation* (3.6) *is bounded.*

Proof. For $n \geq 0$, we have

$$y_{n+1} = \frac{p + qy_n}{1 + y_n + ry_{n-1}} \leq \frac{p + qy_n}{1 + y_n} \leq \frac{max\{p, q\}(1 + y_n)}{(1 + y_n)} = k_2,$$

If the solution $\{y_n\}$ is bounded then $y_n, y_{n-1} \leq M$

$$y_{n+1} = \frac{p + qy_n}{1 + y_n + ry_{n-1}} \geq \frac{p}{1 + y_n + ry_{n-1}} \geq \frac{p}{1 + (1 + r)M} = k_1,$$

Therefore, each solution is bounded.

3.4 Invariant intervals

Definition 3.1. I is an invariant interval for equation (3.6) if it satisfies

$$y_N,\ y_{N+1} \in I,\ N \geq 0 \Longrightarrow y_n \in I \quad \text{for} \quad n \geq N.$$

The following results give the invariant intervals for equation (3.6).

Lemma 3.1. Equation (3.6) has the following invariant intervals

(a) $\left[0, \frac{q - 1 + \sqrt{(q-1)^2 + 4p}}{2}\right]$, if $p \leq q$.

(b) $\left[\frac{q - p}{qr}, q\right]$ if $q < p < q(rq + 1)$.

(c) $\left[q, \frac{q - p}{p}\right]$, if $p > q(rq + 1)$.

Proof. Let

$$g(x) = \frac{p+qx}{1+x} \quad \text{and} \quad b = \frac{q-1+\sqrt{(q-1)^2+4p}}{2},$$

and

$$g'(x) = \frac{q-p}{(1+x)^2},$$

as $p \leq q$ we have $g'(x) > 0$, so g is an increasing and also continuous function on the closed bounded interval $\left[0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}\right]$ thus it reaches its bounds, $i.e., g(b) \leq b$.

Next, we use (3.6) and see that if $y_{k-1}, y_k \in [0, b]$ then

$$y_{k+1} = \frac{p+qy_k}{1+y_k+ry_{k-1}} \leq \frac{p+qy_k}{1+y_k} = g(y_k) \leq g(b) \leq b.$$

Hence the result (a).

(b) Let the function

$$f(x,y) = \frac{p+qx}{1+x+ry}$$

and its derivative

$$f_x(x,y) = \frac{q-p+qry}{(1+x+ry)^2} = \frac{qr(y-\frac{p-q}{qr})}{(1+x+ry)^2}$$

as $y > \dfrac{p-q}{qr}$ we have $f_x(x,y) > 0$, therefore f it is an increasing function with respect to $\overline{x}$ using equation (3.6) and we see that if $y_{k-1}, y_k \in [\frac{p-q}{qr}, p]$ so

$$\begin{aligned} y_{k+1} =& \frac{p+qy_k}{1+y_k+ry_{k-1}} \\ &= f(y_k, y_{k-1}) \\ &\leq f(q, \frac{p-q}{qr}) = q, \end{aligned}$$

and also by using the condition $p < q(rq+1)$, we obtain

$$\begin{aligned} y_{k+1} =& \frac{p + qy_k}{1 + y_k + ry_{k-1}} \\ =& f(y_k, y_{k-1}) \\ & \geq f(\frac{p-q}{qr}, q) \\ & = \frac{q(pr + p - q)}{(rq)^2 + rq + p - q} > \frac{p-q}{qr} \end{aligned}$$

hence the result (b)

(c) It is clear that the function

$$f(x, y) = \frac{p + qx}{1 + x + ry}$$

is decreasing with respect to x for $y < \dfrac{p-q}{qr}$, using the equation and we see that if $y_{k-1}, y_k \in [q, \dfrac{p-q}{qr}]$ then

$$\begin{aligned} y_{k+1} =& \frac{p + qy_k}{1 + y_k + ry_{k-1}} \\ =& f(y_k, y_{k-1}) \geq f(\frac{p-q}{qr}, \frac{p-q}{qr}) = q, \end{aligned}$$

and also by using the condition $p > q(rq+1)$, we obtain

$$\begin{aligned} y_{k+1} =& \frac{p + qy_k}{1 + y_k + ry_{k-1}} \\ =& f(y_k, y_{k-1}) \leq f(q, q) = \frac{p + q^2}{1 + (r+1)q} < \frac{p-q}{qr}. \end{aligned}$$

hence the result (c).

3.5 Analysis of the semi-cycles

Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (3.6). Then the following identities are true for $n \geq 0$.

$$\begin{aligned} y_{n+1} - q = & \frac{p + qy_n}{1 + y_n + ry_{n-1}} - q \qquad (3.7) \\ = & \frac{p + qy_n - q(1 + y_n + ry_{n-1})}{1 + y_n + ry_{n-1}} \\ = & \frac{p - q - qry_{n-1}}{1 + y_n + ry_{n-1}} \\ = & \frac{qr(\frac{p-q}{qr} - y_{n-1})}{1 + y_n + ry_{n-1}} \end{aligned}$$

$$\begin{aligned} y_{n+1} - \frac{p-q}{qr} = & \frac{(p + qy_n)qr - (p - q)(1 + y_n + ry_{n-1})}{(1 + y_n + ry_{n-1})qr} \\ = & \frac{qr(q - \frac{p-q}{qr})y_n + qr(y_{n-1} - \frac{p-q}{qr}) + pr(q - y_{n-1})}{qr(1 + y_n + ry_{n-1})} \qquad (3.8) \end{aligned}$$

$$y_n - y_{n+4} = \frac{qr(y_n - \frac{p-q}{qr})y_{n+1} + (y_n - q)(y_{n+1}y_{n+3} + y_{n+3} + y_{n+1} + ry_ny_{n+3}) + y_n + ry_n^2 - p}{(1 + y_{n+3})(1 + y_{n+1} + ry_n) + r(p + qy_{n+1})}, \qquad (3.9)$$

$$\begin{aligned} y_{n+1} - \overline{y} = & \frac{p + qy_n}{1 + y_n + ry_{n-1}} - \overline{y} \\ = & \frac{p + qy_n - \overline{y}(1 + y_n + ry_{n-1})}{1 + y_n + ry_{n-1}} \qquad (3.10) \\ = & \frac{p + qy_n - \overline{y} - \overline{y}y_n - r\overline{y}y_{n-1}}{1 + y_n + ry_{n-1}} \qquad (3.11) \end{aligned}$$

And $\overline{y}$ is an equilibrium point of equation (3.1), so we have

$$\overline{y} = \frac{p + q\overline{y}}{1 + (r + 1)\overline{y}}$$

equivalent to

$$\overline{y}^2(r + 1) + (1 - q)\overline{y} = p.$$

Replacing p in equation (3.1), we obtain:

$$\begin{aligned} y_{n+1} - \overline{y} &= \frac{r\overline{y}^2+\overline{y}^2-q\overline{y}+qy_n-\overline{y}y_n-r\overline{y}y_{n-1}}{1+y_n+ry_{n-1}} \\ &= \frac{(\overline{y}-q)(\overline{y}-y_n)+\overline{y}r(\overline{y}-y_{n-1})}{1+y_n+ry_{n-1}}, \end{aligned} \tag{3.12}$$

the proofs of the following two lemmas are consequences of identities (3.7) and (3.12)

Lemma 3.2. Suppose

$$p > q(qr+1)$$

and let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (3.6). Then the following results are true :

(i) If $\forall N \geq 0$, $y_n > \dfrac{p-q}{qr}$, then $y_{N+2} < q$;

(ii) If $\forall N \geq 0$, $y_n = \dfrac{p-q}{qr}$, then $y_{N+2} = q$;

(iii) If $\forall N \geq 0$, $y_n < \dfrac{p-q}{qr}$, then $y_{N+2} > q$;

(iv) If $\forall N \geq 0$, $q < y_n < \dfrac{p-q}{qr}$, then $q < y_{N+2} < \dfrac{p-q}{qr}$;

(v) If $\forall N \geq 0$, $\overline{y} \geq y_{n-1}$ and $\overline{y} \geq y_N$, then $y_{N+2} \geq \overline{y}$;

(vi) If $\forall N \geq 0$, $\overline{y} < y_{n-1}$ and $\overline{y} \leq y_N$, then $y_{N+2} < \overline{y}$;

(vii) $q < \overline{y} < \dfrac{p-q}{qr}$.

Proof. We will prove (i) by induction, according to (3.6) we have :

$$y_{n+1} - q = \frac{qr(\frac{p-q}{qr} - y_{n-1})}{1 + y_n + ry_{n-1}},$$

For $N = 0$ we have : $y_o > \frac{p-q}{qr} \Rightarrow y_2 < q$,

$$y_2 - q = \frac{qr(\frac{p-q}{qr} - y_0)}{1 + y_1 + ry_0}, \tag{3.13}$$

According to the assumption ($y_o > \frac{p-q}{qr}$) and identity (3.13), it is clear that $y_2 < q$.
for $N = n$, we have : $y_n > \frac{p-q}{qr} \Rightarrow y_{n+2} < q$,

$$y_{n+2} - q = \frac{qr(\frac{p-q}{qr} - y_n)}{1 + y_{n+1} + ry_n},$$

According to the assumption ($y_n > \frac{p-q}{qr}$) it is clear that $y_{n+2} < q$.
The same reasoning for (ii) and (iii).
For (iv) remains to be proven if $y_N > q \Rightarrow y_{N+2} < \frac{p-q}{qr}$, using identity (3.8) and by induction we have :
For $N = 0$ (the same reasoning as (i)),
For $N = n$: we have $y_n > q \Rightarrow y_{n+2} < \frac{p-q}{qr}$

$$y_{n+2} - \frac{p-q}{qr} = \frac{qr(q - \frac{p-q}{qr})y_{n+1} + qr(y_n - \frac{p-q}{qr}) + pr(q - y_n)}{qr(1 + y_{n+1} + ry_n)} \tag{3.14}$$

According to the assumption ($q < y_0 < \frac{p-q}{qr}$) and identity(1.4), it is clear that $y_{n+2} < \frac{p-q}{qr}$.

Lemma 3.3. We suppose that

$$q < p < q(qr+1)$$

and let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (3.6). Then the following results are true :

(i) If $\forall\, N \geq 0,\, y_n < \dfrac{p-q}{qr}$, then $y_{N+2} > q$;

(ii) If $\forall\, N \geq 0,\, y_n = \dfrac{p-q}{qr}$, then $y_{N+2} = q$;

(iii) If $\forall\, N \geq 0,\, y_n > \dfrac{p-q}{qr}$, then $y_{N+2} < q$;

(iv) If $\forall\, N \geq 0,\, y_n > \dfrac{p-q}{qr}$ and $y_N < q$, then $q > y_{N+2} > \dfrac{p-q}{qr}$;

(v) $\dfrac{p-q}{qr} < \overline{y} < q$.

Suppose that

$$p = q(qr+1)$$

and let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (3.6). Then

$$y_{n+1} - q = \frac{qr(q - y_{n-1})}{1 + y_n + ry_{n-1}} \tag{3.15}$$

and when $qr < 1$, then

$$\lim_{n\to\infty} y_n = \overline{y}, \tag{3.16}$$

Identity (3.15) is obtained by a simple calculation, the limit (3.16) is a consequence obtained in the case $qr \in (0, \infty)$ and equation (3.6) has no two-periodic solution. Here we present an analysis of the semi-cycle of solutions of equation (3.6) when $p = q(1 + rq)$, in this case the equation (3.6) will be :

$$y_{n+1} = \frac{q + rq^2 + qy_n}{1 + y_n + ry_{n_1}}, \quad n = 0, 1, ... \tag{3.17}$$

And the unique equilibrium point is $\overline{y} = q$.
Assuming that $p = q + rq^2$ and let $\{y_n\}_{n=-1}^{\infty}$ be a non-trivial solution of the equation (3.6). Then this solution is oscillatory and the sum of the lengths of two consecutive semi-cycles, excluding the first one, is equal to four. More precisely, the following results are true for any $K \geq 0$.

(i) $y_{-1} > q$ and $y_0 \leq q \Rightarrow y_{4k-1} > q, \quad y_{4k} \leq 1, \quad y_{4k+1} < q$ and $y_{4k+2} \geq q$;

(ii) $y_{-1} < q$ and $y_0 \geq q \Rightarrow y_{4k-1} < q, \quad y_{4k} \geq 1, \quad y_{4k+1} > q$ and $y_{4k+2} \leq q$;

(iii) $y_{-1} > q$ and $y_0 \geq q \Rightarrow y_{4k-1} > q, \quad y_{4k} \leq 1, \quad y_{4k+1} < q$ and $y_{4k+2} \leq q$;

(iv) $y_{-1} < q$ and $y_0 \leq q \Rightarrow y_{4k-1} < q, \quad y_{4k} \leq 1, \quad y_{4k+1} > q$ and $y_{4k+2} \geq q$;

3.6 Global Asymptotic Stability

The following lemma establishes when $p \neq q(qr+1)$ each solution of equation (3.6) is eventually trapped in one of the three invariant intervals of equation (3.6) described in lemma 3.1 More precisely, we have :

Lemma 3.4. Let I be the interval defined as follows:

$$\begin{aligned}
I =& [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}], \quad \text{if } p \leq q, \\
I =& [\frac{p-q}{qr}, q], \quad \text{if } q < p < q(qr+1); \\
I =& [q, \frac{p-q}{qr}], \quad \text{if } p > q(qr+1);
\end{aligned}$$

then every solution of equation (3.6) is in I.

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (3.6)
First, suppose that $p \leq q$, then it is clear that $\overline{y} \in I$ let

$$b = \frac{q-1+\sqrt{(q-1)^2+4p}}{2}$$

And observe that

$$y_{n+1} - b = \frac{(q-p)(y_n - b) - r(p+q)y_{n-1}}{(1 + y_n + ry_{n-1})(1+b)}, \quad n \geq 0$$

Therefore, if for a certain $N, y_n \leq b$.
Then $y_{N+1} < b$. Now we assume the opposite that

$$y_n > b \quad \text{for} \quad n > 0$$

Then, $y_n > \overline{y}$ for $n \geq 0$ and therefore

$$\lim_{n \longrightarrow \infty} y_n = \overline{y} \in I$$

Which is a contradiction.
Next, suppose that

$$q < p < q(qr+1),$$

and, we prove by contradiction if the solution $\{y_n\}_{n=-1}^{\infty}$ is not in the interval I. Then by lemma (3.2), there exists $N > 0$ such that, one of the following three cases is true:

(i) $y_N > q, \quad y_{N+1} > q, \quad$ and $y_{N+2} < \dfrac{p-q}{qr}$;

(ii) $y_N > q, \quad y_{N+1} < \dfrac{p-q}{qr}, \quad$ and $y_{N+2} < \dfrac{p-q}{qr}$;

(iii) $y_N > q, \quad \dfrac{p-q}{qr} \leq y_{N+1}, \quad$ and $y_{N+2} < \dfrac{p-q}{qr}$;

It is also observed that:
if $y_N \geq q$, then $y_N + ry_N^2 - p > 0$, and if $y_N \leq \dfrac{p-q}{qr}$ then $y_N + ry_n^2 - p < 0$.
The desired contradiction is now obtained using identity (3.9) from which it follows that for $j \in \{0, 1, 2, 3\}$, each subsequence $\{y_{n+4kj}\}_{k=0}^{\infty}$ with all its terms outside of the interval I converges monotonically and enters the interval I. Similarly for

$$p > q(qr + 1).$$

Using the monotonicity of the function

$$f(x, y) = \frac{p + qx}{1 + x + ry}.$$

In each of the intervals in lemma (3.1), with the properties of the convergence Theorems (1.8) and (1.5), we can obtain results of global asymptotic stability for the solutions of equation (3.6). For example, the following results are true for equation (3.6):

Theorem 3.4.

(a) If

$$p \geq q + q^2 r.$$

Or

$$p < \frac{q}{1+r}.$$

Then the equilibrium $\overline{y}$ *of equation* (3.6) *is globally asymptotically stable.*

(b) If

$$p \leq q$$

Or

$$q < p < q + q^2 r$$

And one of the following conditions is also satisfied

(i) $q \leq 1$

(ii) $r \leq 1$

(iii) $r > 1 \quad$ *and* $\quad (q-1)^2(r-1) \leq 4p.$

Then the equilibrium $\overline{y}$ *of equation* (3.6) *is globally asymptotically stable.*

Proof.

(a) The proof follows from lemma (3.1) and Theorem (1.5) and Theorem (1.8)

(b) From lemma (3.1), we see that when $y_{-1}, y_0 \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$,

then $y_n \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$ for all $n \geq 0$.
It is easy to verify that $\overline{y} \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$ and that $[0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$, the function f is increasing in x and decreasing in y in this interval.
We use Theorem (1.3), so it remains to be shown that if

$$m = f(m, M) \quad \text{and} \quad M = f(M, m).$$

Then

$$M = m$$

This system is written as:

$$m = \frac{p + qm}{1 + m + rM}$$

and

$$M = \frac{p + qM}{1 + m + rM}$$

Yileds

$$m + m^2 + rMm = p + qm \tag{3.18}$$

and

$$M + M^2 + rMm = p + qM \tag{3.19}$$

Substracting equation (3.19) from equation (3.18) we get

$$(M - m) + M^2 - m^2 = q(M - m),$$

if $M \neq m$ we have :

$$(M - m)(1 + M + m) = q(M - m),$$

implies that

$$M + m = q - 1.$$

Then, by multiplying equation (3.18) by M, and equation (3.19) by m, we get :

$$mM(M - m) + rmM(m - M) = p(m - M)$$
$$(m - M)[-mM + rmM] = p(m - M)$$

Since $m \neq M$ implies that

$$mM(r-1) = p$$

Yileds

$$mM = \frac{p}{r-1},$$

Therefore m and M are the solutions of the equation

$$m^2 - (q-1)m + \frac{p}{r-1} = 0$$

where well

$$(r-1)m^2 + (r-1)(1-q)m + p = 0$$

Clearly now, if condition (ii) or (iii) is satisfied, $m = M$, hence the result.
For the proof when $q < p < q + qr^2$, using lemma (3.1) and Theorem (1.3).

3.7 Numerical applications

To confirm the results of this part, we consider the following two numerical examples :

First, we have the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A - Bx_n + Cx_{n-1}}, n = 0, 1... \tag{3.20}$$

Example 3.1. If we take $(\alpha, \beta, A, B, C) = (1, 1, 2, 1.1)$, the equation (3.20) takes the form

$$x_{n+1} = \frac{1 + x_n}{2 + x_n + x_{n-1}}, \tag{3.21}$$

by the change of variables

$$x_n = 2y_n,$$

we obtain

$$y_{n+1} = \frac{\frac{1}{4} + \frac{1}{2}y_n}{1 + y_n + y_{n-1}}, \quad n = 0, 1... \tag{3.22}$$

The equilibrium points of equation (3.22) are the non-negative solutions of the equation

$$\overline{y} = \frac{\frac{1}{4} + \frac{1}{2}\overline{y}}{1 + 2\overline{y}}$$

or equivalent

$$8\overline{y}^2 + \overline{y} - 1 = 0$$

The equation (3.22) has a unique positive equilibrium point :

$$\overline{y} = \frac{1}{4}.$$

(3.14), with the initial values $x_{-1} = 0,1, x_0 = 0,9.$

The positive equilibrium of this equation is locally and globally asymptotically stable.(see figure (3.1))

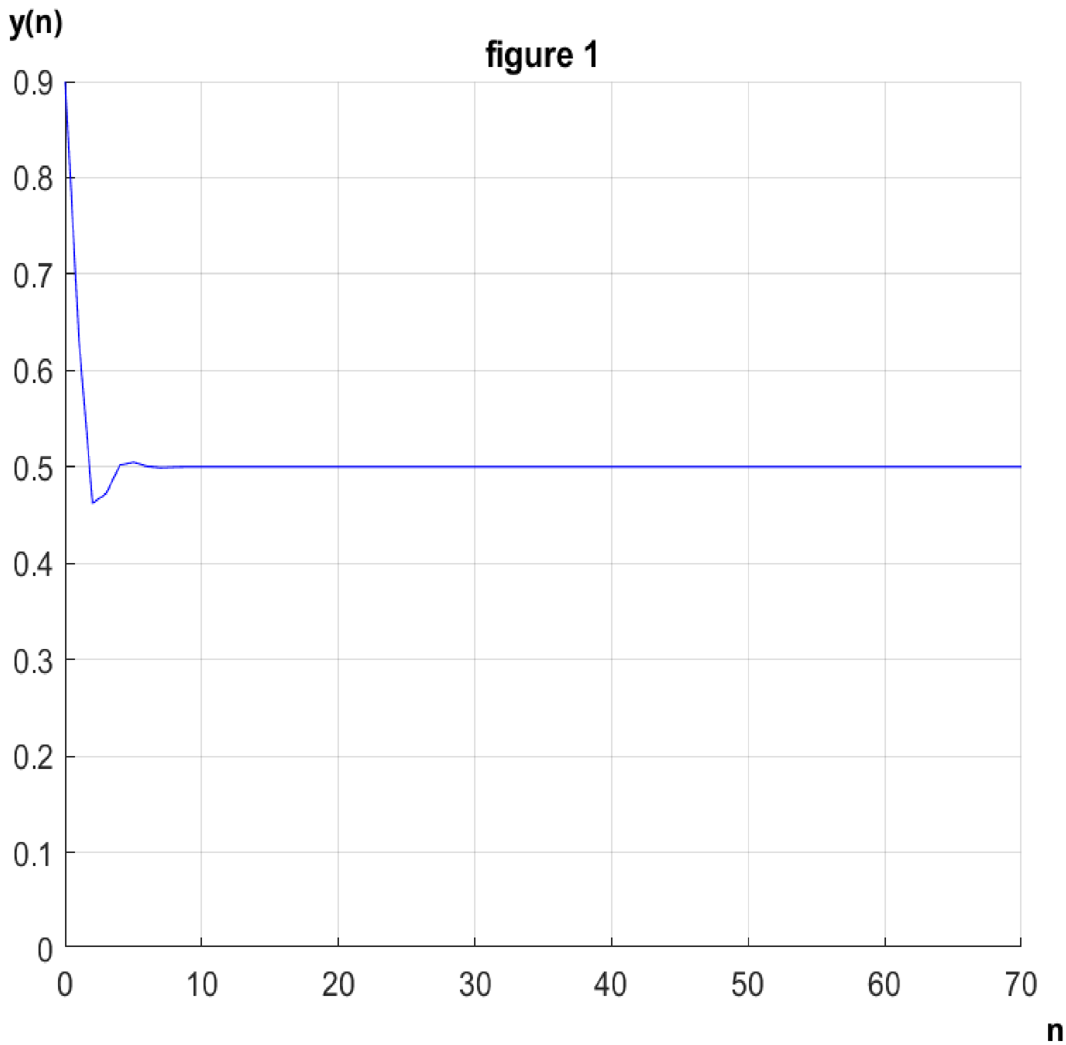

Figure 3.1

This graph represents the behavior of the solution of equation (3.22), with the initial values $y_{-1} = 0, \quad y_0 = 0.9$.

Example 3.2. If we take $(\alpha, \beta, A, B, C) = (2, 3, 1, 1, 2)$, the equation (3.20) takes the form

$$x_{n+1} = \frac{2 + 3x_n}{1 + x_n + 2x_{n-1}} \tag{3.23}$$

by change of variable

$$x_n = y_n,$$

we obtain

$$y_{n+1} = \frac{2 + 3y_n}{1 + y_n + 2y_{n-1}}, \quad n = 0, 1... \tag{3.24}$$

The equilibrium points of equation (3.24) are the non-negative solutions of the equation

$$\overline{y} = \frac{2 + 3\overline{y}}{1 + 3\overline{y}}$$

where equivalent

$$3\overline{y}^2 - 2\overline{y} - 2 = 0.$$

The equation (3.24) has a unique positive equilibrium :

$$\overline{y} = 1.21.$$

But the positive equilibrium of this equation is locally and globally asymptotically Stable.(see figure (3.2))

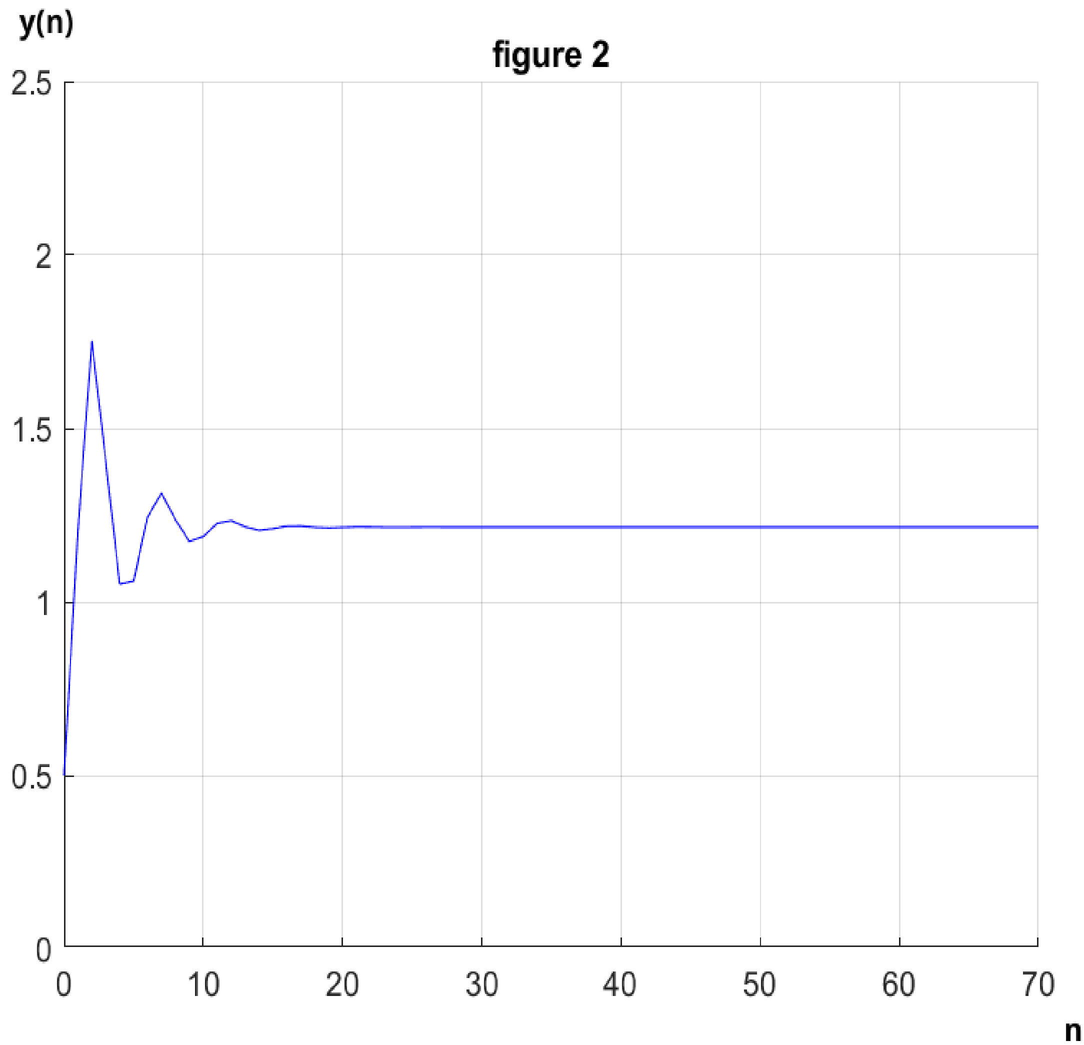

Figure 3.2

This graph represents the behavior of the solution of equation (3.23), with the initial values $y_{-1} = 0.7, \quad y_0 = 0.5$.

Conclusion

This study is related to convergence result of second order rational diffrence equation. Firstly, we investigate the unique positive equilibrium point of this equation (2.1), then we analyse the boundedness of solution of this equation. Secondly, we analyzed the dynamics of equation (3.1), we also study the invariant intervales and the semi cycles. As a result of this, we obtained that the equilibrium points are globaly asymptotically stable.In addition we presented some numerical examples to verify our theoritical results.

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