**People's Democratic Republic of Algeria**
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# Final Year Dissertation

For the diploma of
**Middle School Teaching Professor**

**Theme:**

# Differential Equations: Theoretical and Numerical Study

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All praise is due to Allah, the Lord of the worlds.

# Dedication

﴿ ومَا توْفيقِي الّا بالله عَليهِ توكَلْتُ واِليْه أنيبْ﴾

And my success is only from Allah.

Upon Him I have relied and to Him I return.

(Qur’an, 11:88)

To myself…

To the little dreamer inside me, who stayed strong despite the exhaustion,

Thank you for not giving up, for believing in the dream and walking towards it step by step.

To my beloved mother

To the one who gave me everything, her love, care, and patience,

Thank you, for rising me well and for being the guiding light in my life.

To my dear father

To the dear one whose name I carried with pride,

To the one whom God endowed with dignity and reverence.

To my beautiful sisters, **Amani**, **Houyem** and **Aya**

You are the sweetest part of my life, my memories, and my heart.

You were my laughter when it faded, and my helping hand when I stumbled.

To my dear friends

To each of you who shared a moment with me, who supported me with a word, especially **Assil** and **Kawter**.

Thank you for being part of this journey.

*Lyna*

# Dedication

﴿ ومَا توْفيقِي الّا باللّه عَليهِ تَوكَلْتُ واِليْه أنِيبْ﴾

And my success is only from Allah.

Upon Him I have relied and to Him I return.

(Qur’an, 11:88)

I dedicate this work to :

My beloved parents, my constant support and source of strength.

Your encouragement and presence have guided me through every step of this journey. I am deeply grateful for all the sacrifices you made for me.

To my dear siblings.

And to my sincere friends : **Karim**, **Islam** and **Ayman**.

Thank you for standing by me and offering your help whenever I needed it.

I send you my sincere thanks and heartfelt prayers.

I hope this achievement brings you pride, as you truly deserve all my love and respect.

Abd Rahmen

# Contents

# General Introduction

Differential equations represent one of the cornerstones of applied mathematics, providing a powerful framework for expressing the relationship between a function and its derivatives. This allows for the modeling and analysis of a wide range of dynamic phenomena across physics, biology, and engineering. Since the era of Newton and Euler, differential equations have played a pivotal role in understanding motion, heat conduction, wave propagation, and other time-dependent processes. Broadly speaking, differential equations are classified into two main types: Ordinary Differential Equations (ODEs), which involve derivatives with respect to a single independent variable, and Partial Differential Equations (PDEs), which involve derivatives with respect to multiple independent variables.

This study explores both the theoretical and numerical aspects of differential equations and is structured into four main chapters. The first chapter addresses ordinary differential equations, introducing fundamental definitions, classifications, and the most prominent analytical techniques used to solve them. The second chapter focuses on first-order linear partial differential equations, presenting their general form, the method of characteristics, and practical applications such as modeling transport phenomena and wave behavior. The third chapter delves into second-order partial differential equations, discussing key equations like the heat equation, wave equation, and Laplaces equation. It emphasizes the classification of these equations into elliptic, parabolic, and hyperbolic types, and presents solution methods such as separation of variables and mathematical transforms. Finally, the fourth chapter is devoted to the practical applications of partial differential equations, where theoretical knowledge is combined with numerical methods to tackle real-world problems. MATLAB is used to perform numerical simulations and obtain approximate solutions to boundary value and eigenvalue problems using tools such as Fourier series and integral transforms.

Through this integrated approach, the study aims to equip the reader with a solid and practical understanding of differential equations, bridging abstract mathematical theory with computational techniques and real-life applications in science and engineering.

Chapter 1

# Ordinary differential equations

## Introduction

It can be said that the differential equation into all branches of science as most of the governing relation in physics and engineering that appear in differential equation.

## 1.1 Generalities

**Definition 1.1.1.** *A differential equation is a relationship between an independent variable $x$ and dependent variable $y$,the general form of a differential equation can be expressed as follows*

$$F\left(y\,,\,\frac{dy}{dx}\,,\,\frac{d^2y}{dx^2}\,,\,\ldots\,,\,\frac{d^ny}{dx^n}\right) \tag{1.1}$$

### 1.1.1 The order of a differential equation

**Definition 1.1.2.** *The order of a differential equation is the order of highest derivative (also known as differential coefficient) present in the equation.*

**Example.** in the equation

$$y''' + 2xy'' + 2xy = 0.$$

the order of this equation is 3

### 1.1.2 The degree of a differential equation

**Definition 1.1.3.** *The degree of a differential equation is defined as the power of highest derivative that appears in the equation ,provided the equation is a polynomial in the derivatives*

**Example.**

$$y''' + 2x(y'')^4 + 2xy = 0.$$

the degree of this equation is 4.

### 1.1.3 The Cauchy problem

In the first order or higher accompanied by initial condition for example if the differential equation:

$$\begin{cases} \frac{dy}{dx} = f(x, y), \\ y(x_0) = y_0. \end{cases} \tag{1.2}$$

The first order differential equation are a type of equation that involves first. While there are multiple methods to solve certain types of these equation, there is no single comprehensive methods than can be applied to solve all first order differential equation

### 1.1.4 General, Particular, and Singular Solutions

To solve or integrate a differential equation on an interval $I \subseteq \mathbb{R}$ or over all of $\mathbb{R}$ means to determine the set of all its solutions.

The set of solutions for an $n$ the order differential equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{1.3}$$

generally depends on $n$ arbitrary constants $\lambda_1, \lambda_2, \ldots, \lambda_n$.

#### Definition (General and Particular Solutions)

1. The family of solutions $(y_\lambda)$, indexed by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, is called the *general solution* (or integral).

2. A *particular solution* is obtained by imposing an initial condition on $y_\lambda$.

In some cases, besides the general solution, there exist particular solutions $y = \varphi_0(x)$ , $y = \varphi_1(x), \ldots$ that cannot be derived from any choice of $\lambda$. These are called *singular solutions.*

**Example.** Consider the equation $y^2 + (yy')^2 = 1$. It can be verified that the general solutions are:

$$y_\lambda = \pm\sqrt{1 - (x - \lambda)^2}, \quad \lambda \in \mathbb{R},$$

along with two additional functions $\bar{y}_1 = -1$ and $\bar{y}_2 = 1$. The solutions $\bar{y}_1$ and $\bar{y}_2$, which do not belong to the family $y_\lambda$, are singular solutions.

### 1.1.5 Maximal Solutions

**Definition 1.1.4.** *Let $y : I \to \mathbb{R}$ and $\tilde{y} : \tilde{I} \to \mathbb{R}$ be two functions where $I \subseteq \tilde{I}$. Then $\tilde{y}$ is said to be a prolongation of $y$ if $\tilde{y}|_I = y$*

**Definition 1.1.5** (Maximal Solution)**.** *A solution $y : I \to \mathbb{R}$ of the equation (1.3) is said to be a maximal solution if dose not able any prolongation.*

**Example.** The function $y : x \to \frac{1}{x}$ defined on $]0\,,+\infty[$ is a maximal solution of the following equation

$$y' + y^2 = 0.$$

**Theorem 1.1.1.** *Every solution $y$ of (1.3) can be prolonged into a maximal solution $\tilde{y}$ not necessary unique.*

### 1.1.6 Global solutions

**Definition 1.1.6.** *Let $(I\,,y)$ be a solution of the equation (1.3). Then $(I\,,y)$ is said to be a global solution in $I$ if it is defined on whole interval $I$.*

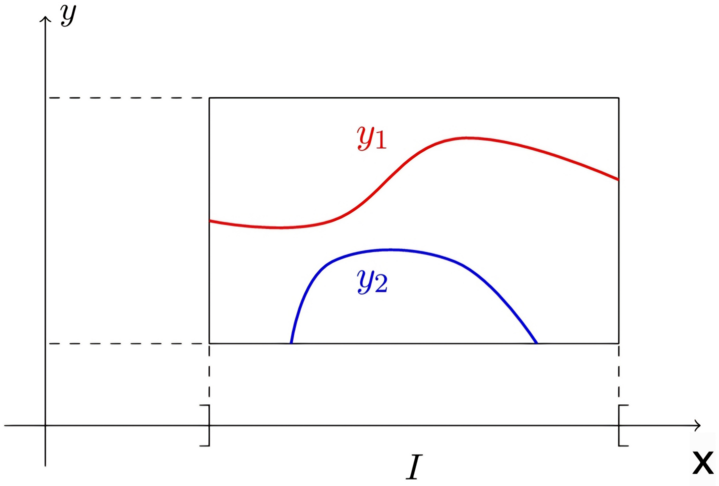


Figure 1.1: Graphs of global solutions

## 1.2 First order and first degree differential equations

The first order and first degree differential equation take the general form

$$\frac{dy}{dx} = F(x,y). \tag{1.4}$$

### 1.2.1 Types of first order differential equations

#### Separable differential equations

Separable differential equation are types of first order differential equation that can be expressed :

$$\frac{dy}{dx} = f(x)g(y).$$

The condition for this method : continuity $f(x)$ and $g(y)$ must be continuous in their respective domaine and initial condition for use the integration of their solving the equation, the solution to this method :

$$\frac{dy}{dx} = f(x)g(y),$$

$$\int \frac{1}{g(y)}dy = \int f(x)dx.$$

**Example.** Suppose the following equation

$$\frac{dy}{dx} = xy, \tag{1.5}$$

Equivalent

$$\frac{dy}{y} = xdx,$$

Now, by the integration the two members, we obtain

$$\int \frac{dy}{y} = \int xdx,$$

Thus

$$y = K \exp\left\{\frac{x^2}{2}\right\}.$$

### 1.2.2 Exact ordinary differential equations

An exact ordinary differential equation is of the form

$$M(x\,,\,y)dx + N(x\,,\,y)dy = 0. \tag{1.6}$$

We say about (1.6) it is exact ordinary differential equation $(x, y) \in D$ if the condition holds

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}.$$

**Example.** Let the following equation

$$2xydx + x^2dy = 0. \tag{1.7}$$

we have

$$M(x\,,\,y) = 2xy \quad , \quad N(x\,,\,y) = x^2$$

Then we obtain
$$\frac{\partial M}{\partial x}(x\,,\,y)=2x \quad , \quad \frac{\partial N}{\partial y}(x\,,\,y)=2x$$
Since $\frac{\partial M}{\partial x}(x\,,\,y)=\frac{\partial N}{\partial y}(x\,,\,y)=2x$, thus, the differential equation (1.7) is exact $D$

**Example.**

$$2xydx+x^2dy=0. \tag{1.8}$$

$$M(x,y)=2xy \qquad N(x,y)=x^2$$

$$M_y=2x \qquad N_x=2x$$

The differential equation (1.8) is exact in $D$

### 1.2.3 The solution of exact ordinary differential equations

Now that we have a test with which to determine exactness let us proceed to solve exact differential equation if (1.3) exact in a domain $D$, them exists a function $U$ such that

$$M(x,y)=\frac{\partial U(x,y)}{\partial x} \quad \text{and} \quad N(x\,,\,y)=\frac{\partial U(x,y)}{\partial y}, \quad \forall (x\,,\,y)\in D$$

Then the equation may be written

$$\frac{\partial U(x,y)}{\partial x}dx+\frac{\partial U(x,y)}{\partial y}dy=0. \tag{1.9}$$

this last equation equivalent

$$M(x\,,\,y)dx+N(x\,,\,y)dy=0 \quad \Leftrightarrow \quad dU(x\,,\,y)=0. \tag{1.10}$$

The relation $U(x,y)=c$ is obviously a solution of (1.9), where $c$ is a arbitrary constant we summarize this observation in the following

#### Standard method

In this method we must find $U(x,y)$ such that

$$M(x,y)=\frac{\partial U(x,y)}{\partial x} \quad \text{and} \quad N(x,y)=\frac{\partial U(x,y)}{\partial y},$$

**Example.** Solving the following equation Solve the following equation

$$xy^2dx+(x^2y-\cos(y))dy=0. \tag{1.11}$$

We have

$$\int\frac{\partial U(x,y)}{\partial x}dx=\int M(x,y)dx=xy^2 \quad \text{and} \quad \int\frac{\partial U(x,y)}{\partial y}=\int N(x,y)dy=x^2y-\cos(y),$$

Then

$$U(x,y)=\int M(x,y)dx=\int xy^2dx=\frac{1}{2}x^2y^2+\phi(y),$$

Hence we obtain

$$\frac{\partial U(x,y)}{\partial y}=x^2y+\varphi'(y),$$

But also

$$\frac{\partial U(x,y)}{\partial y}=N(x,y)=x^2y-\cos(y),$$

And so

$$\varphi'(y)=-\cos(y)$$

$$\varphi(y)=-\sin(y)+c_0,$$

Thus

$$U(x,y)=\frac{-1}{2}x^2y^2-\sin(y)+c_0.$$

### Method of grouping

We shall now solve the differential equation of last example by grouping the terms in such way that it's left member appears as the sum of certain exact differential, we write (1.11) as

$$dU(x,y)=xy^2dx+(x^2y-\cos(y))dy=0, \tag{1.12}$$

In the form

$$dU(x,y)=(xy^2dx+x^2y)dy-\cos(y)dy,$$

we now recognize this as

$$dU(x,y)=d(\frac{1}{2}x^2y^2)-d(\sin(y))=dA,$$

form this we have $$U(x,y)=\frac{1}{2}x^2y^2-\sin(y)=A \qquad A\in\mathbb{R}.$$

### Method of comparison

We must find the function $U$ once of integral $M(x,y)$ with respect to $x$ and another one integral $N(x,y)$ with respect to $y$.

**Example.** We have

$$dU(x,y)=xy^2dx+(x^2y-\cos(y))dy=0.$$

Then

$$M(x,y)=xy^2 \quad \text{and} \quad N(x,y)=x^2y-\cos(y),$$

Integral $M(x,y)$ with respect $x$, we obtain

$$\begin{aligned} U(x,y) = & \int M(x,y)dx = \int xy^2 dx, \\ = & \frac{1}{2}x^2y^2 + \varphi(y) = A. \end{aligned} \tag{1.13}$$

Now we Integral $N(x,y)$ with respect to $y$ we get

$$\begin{aligned} U(x,y) = & \int N(x,y)dy = \int (xy^2 - \cos(y))dy, \\ = & \frac{1}{2}x^2y^2 - \sin(y) + \phi(x) = A. \end{aligned} \tag{1.14}$$

Let us compare the two statements (1.13) and (1.14) we find

$$\varphi(y) = -\sin(y) \qquad and \qquad \phi(x) = 0,$$

Thus

$$U(x,y) = \frac{1}{2}x^2y^2 - \sin(y) = A \qquad\qquad A. \in \mathbb{R}$$

### 1.2.4 Integrating Factors

Given the differential equation

$$M(x,y)dx + N(x,y) = 0. \tag{1.15}$$

it's not exact that is

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}.$$

We can transform it (1.15) into an essentially equivalent if so we can proceed to solve the resulting exact equation by one of above produces $I(x,y)$. Then $I(x,y)$ is called an **integrating factor** of the differential equation (1.15)

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0. \tag{1.16}$$

is exact that is

$$\frac{\partial I(x,y)M(x,y)}{\partial y} = \frac{\partial I(x,y)N(x,y)}{\partial x}.$$

**How find the integrating factor $I(x,y)$?**

If the function in $x$ $\Longleftrightarrow$ $(I_y = 0)$ than mine $I(x) = I(x,y)$ and $[\frac{dI(x,y)}{dy} = 0;\ I'(x) = I_x]$

$$I(x,y)M_y = I'(x) + I(x,y)N_x,$$

there so

$$I'(x) = \frac{I(x,y)(M_y - N_x)}{N(x,y)},$$

and

$$\frac{I'(x)}{I(x,y)} = \frac{I(x,y)(M_y - N_x)}{N(x,y)}.$$

$$I(x) = \exp\left\{\int \frac{M_y - N_x}{N(x,y)}\right\}. \tag{1.17}$$

but if the function in $y \iff I_x = 0$ and $I(y) = I(x,y)$.

The integrating factor $I(y)$ be

$$I(y) = \exp\left\{\int \frac{N_x - M_y}{M(x,y}\right\}.$$

**Example.** consider the differential equation

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0.$$

This equation take a form

$$M(x,y) = 3y + 4xy^2 \qquad ; \qquad N(x,y) = 2x + 3x^2y$$

$$\frac{\partial M(x,y)}{dy} = 3 + 8xy \qquad ; \qquad \frac{\partial N(x,y)}{dx} = 2 + 6xy$$

Since

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$

Let $I(x,y) = x^2y$ them the corresponding differential equation of the form (1.16)

$$(3x^2y^2 + 4x^3y^3)dx + (2x^2y + 3x^4y^2)dy = 0.$$

The equation is exact because

$$\frac{\partial I(x,y)M(x,y)}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial I(x,y)N(x,y)}{\partial x}.$$

We can find the integrating factor , we use (1.17)

## 1.3 Linear Equations

**Definition 1.3.1.** *A first-order linear differential equation is an equation that is linear with respect to the unknown function and its derivative. It has the form*

$$a(x)y' + b(x)y = c(x). \tag{1.18}$$

*where $a, b$ and $c$ are given functions of $x$ continuous in the domain where the equation (1.18) is to be solved*

**Comment.** The function $c$ is called the source term of the differential equation, while $a$ and $b$ are called the coefficients.

If $c(x) = 0$, the equation (1.18) is called a *linear homogeneous equation* or an equation *without a source term.* that is

$$a(x)y' + b(x)y = 0. \tag{1.19}$$

The zero function is always a solution. Other solutions can be found by writing

$$\frac{y'}{y} = -\frac{b(x)}{a(x)} \quad \text{or} \quad \frac{dy}{y} = -\frac{b(x)}{a(x)}\,dt,$$

and integrating both sides. This yields:

$$\ln|y(x)| = -\int g(x)\,dx + K, \quad \text{where} \quad g(x) = \frac{b(x)}{a(x)}, \quad K \in \mathbb{R}$$

For each value of $K$, this gives two solutions. One always positive

$$y = e^K \exp\left\{-\int g(x)\,dx\right\},$$

and the other always negative

$$y = -e^K \exp\left\{-\int g(x)\,dx\right\}.$$

All these solutions, including the zero solution, can be combined by stating that the general

solution of (1.19) is

$$y_h = C\exp\left\{-\int g(x)\,dx\right\} \quad \text{where} \quad g(t) = \frac{b(x)}{a(x)}, \quad C \in \mathbb{R}.$$

If the value of the solution at $x = 0$ is given, it is often written as

$$y(x) = y(0)\exp\left\{-\int_0^x g(s)\,ds\right\}$$

.

**Remark.** The function $(x, y) \mapsto \mu(x, y) = \exp\left\{\int g(t)\,dt\right\}$, where $g(x) = \frac{b(x)}{a(x)}$ is an integrating factor for the differential equation $y' + g(x)y = 0$.

Suppose $x \mapsto y_p(x)$ is a particular solution of (1.18). By substituting $y = y_p + u$ into (1.18), we obtain

$$a(x)y_p' + b(x)y_p + a(x)u' + b(x)u = c(x).$$

Since $a(x)y_p' + b(x)y_p = c(x)$, it follows that $a(x)u' + b(x)u = 0$, meaning $u$ is a solution of the homogeneous equation (1.19).

Therefore, the general solution of the non-homogeneous equation (1.18) is the sum of a particular solution of (1.18) and the general solution of the associated homogeneous equation (1.18):

$$y(x) = \underbrace{y_p(x)}_{\substack{\text{Particular solution}\\ \text{of (5 1)}}} + \underbrace{y_h(x)}_{\substack{\text{Solution of the}\\ \text{homogeneous equation (5.2)}}} = y_p(x) + Ce^{-\int g(x)\,dx}, \quad g(x) = \frac{b(x)}{a(x)}.$$

### 1.3.1 Integration factor

The linear differential equation :

$$dy + [P(x)y - Q(x)]dx = 0 \tag{1.20}$$

we have

$$M(x,y) = P(x)y - Q(x). \quad \text{and} \qquad N(x,y) = 1,$$

Then

$$M_y(x\,,\,y) = P(x) \quad \text{and} \quad N_x(x\,,\,y) = 0.$$

the integrating factor is

$$I(x) = \exp\left\{\int \frac{M_y - N_x}{N(x,y)}\right\}. \tag{1.21}$$

Multiplying the equation (1.20) though by this integrating factor

$$I(x)dy + I(x)(P(x)y - Q(x))dx = 0.$$

it is exact.

**Example.** Solve the differential equation

$$y' - y = e^{2x}. \tag{1.22}$$

here we have $P(x) = -1$

and hence an integrating factor is:

$$I(x) = \exp\left\{\int P(x)dx\right\},$$

hence

$$I(x) = \exp\left\{\int -1dx\right\},$$

Hence

$$I(x) = e^{-x}.$$

Multiplying the equation (1.22) though by this integrating factor $I(x)$

$$e^{-x}dy-(ye^{-x}+e^{-x})dx=0. \tag{1.23}$$

here we have $M(x,y)=-ye^{-x}-e^{-x}$ and $N(x,y)=e^{-x}$ Thus

$$\frac{\partial M(x,y)}{\partial Y}=-e^{-x}=\frac{\partial N(x,y)}{\partial x}.$$

Now (1.23) exact. the solution is

$$d(ye^{-x})+d(e^{x})=dA,$$

So

$$ye^{-x}+e^{x}=A,$$

Thus

$$y=Ae^{x}-e^{2x}. \quad A\in\mathbb{R}.$$

## 1.4 Nonlinear Differential Equations

### 1.4.1 Bernoulli Equation

**Definition 1.4.1.** *An equation of the form*

$$\frac{dy}{y}+P(x)y=Q(x)y^{n}. \tag{1.24}$$

*is called a Bernoulli differential equation*

And it leads to a linear differential equation transformation $v=y^{n-1}$ reduces the Bernoulli equation (1.24) to a linear equation in $v$

*Proof.* we first multiply equation (1.24) by $y^{-n}$ there by expressing it in the equation from

$$y^{-n}\frac{dy}{dx}+P(x)y^{-n}=Q(x). \tag{1.25}$$

if we let $v=y^{1-n}$ the equation (1.25) transforms into

$$\frac{1}{1-n}\frac{dv}{dx}+P(x)v=Q(x)$$

the equation it's a linear ■

### 1.4.2 Equation of the form

$$f'(y)y' + f(y)P(x) = Q(x). \tag{1.26}$$

Where $P(x), Q(x)$ and $f(y)$ are a continuous function
if let

$$\begin{cases} v = f(y), \\ \frac{dv}{dx} = y'f'(y). \end{cases}$$

the equation (1.26) transform into

$$\frac{dv}{dx} + P(x)v = Q(x). \tag{1.27}$$

the equation (1.27) is a linear

### 1.4.3 Biccard's method

Let the Cauchy problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y), \\ y(x_0) = y_0. \end{cases} \tag{1.28}$$

It has a solution in domain $D$
The basic idea of Riccard is that by integrating equation (1.28) we obtain the following integral equation.

$$y(x) = y_0(x) + \int_{x_0}^{x} f(s, y(s))ds. \tag{1.29}$$

to obtain approximate solution $y(x)$ for the integral equation we follow the following steps

**1** we put $y_0(x) = y_0$ and substitute $y(s)$ of the first condition $y_0$ it is the integral equation (1.29)

**2** let

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(s, y_0(s)ds \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(s, y_1(s)ds \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))ds \end{aligned}$$

### 1.4.4 Existence and uniqueness theorem in differential equations

The existence and uniqueness theorem in differential equation addresses the condition under which a differential equation has a unique this is particularly relevant in the context of initial value problems ,often referred to as the Cauchy problem , the Cauchy problem is

$$\begin{cases} \dfrac{dy}{dx} = f(x,y), \\ y(x_0) = y_0. \end{cases}$$

**Theorem 1.4.1.** *if* $f(x,y)$ *and* $\dfrac{\partial f(x,y)}{\partial y}$ *continuous and bounded in the rectangular* $D$*:*

$$D = \left\{(x,y) \in \mathbb{R}^2, |x-x_0| < a; |y-y_0| \leqslant b\right\}$$

*If we have*

$$\alpha = \min\left(a, \frac{b}{M}\right) \quad \wedge \quad M = \max f(x,y)$$

*Then, the Cauchy problem has a unique solution on a given interval* $I_\alpha$ *defined by*

$$I_\alpha = [x - \alpha_0; x + \alpha_0]$$

**Example.** In the rectangular $D$:

$$D = \left\{(x,y) \in \mathbb{R}^2;\ 0 \leqslant x < \frac{1}{2}\ ;\ |y| < 1\right\}$$

prove this problem

$$\begin{cases} y' = x^2 + y^2, \\ y(0) = 1. \end{cases} \tag{P}$$

**1** It has a unique solution in domaine $D$

**2** Using the Riccard method , find the three first approximation for the following problem.

$$\begin{cases} y' = y, \\ y(0) = 1. \end{cases} \tag{S}$$

**Solution.** **1** prove (P) it have a unique solution in domain $D$

we have

$$f(x,y) = x^2 + y^2 \quad \wedge \quad \frac{\partial f(x,y)}{\partial y} = 2x$$

the function $f(x,y)$ and $\dfrac{\partial f(x,y)}{dy}$ are continuous and bounded that's mine (P) has a unique solution

**2** find three approximation with using Riccard method

we have

$$\begin{cases} y' = y & x_0 = 0 \\ y(0) = 1 & y_0 = 1 \end{cases}$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s)ds, \quad n \in \mathbb{N}$$

for the first approximation $n = 1$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds \iff y_1(x) = 1 + \int_0^x ds \iff y_1(x) = 1 + [s]_0^x \iff y_1(x) = 1 + x$$

the second approximation $n = 2$

$$\begin{aligned} y_2(x) = y_0 + \textstyle\int_{x_0}^{x} f(s, y_1(s))ds &\iff y_2(x) = 1 + \textstyle\int_0^x (1+s)ds \\ &\iff y_2(x) = 1 + \left[\frac{s^2}{2} + s\right]_0^x \iff y_2(x) = 1 + \frac{x^2}{2} + x \end{aligned}$$

The third approximation $n = 3$

$$\begin{aligned} y_3(x) = y_0 + \textstyle\int_{x_0}^{x} f(s, y_2(s))ds &\iff y_3(x) = 1 + \textstyle\int_0^x (1 + \frac{s^2}{2} + s)ds \\ &\iff y_3(x) = 1 + \left[ s + \frac{s^3}{6} + \frac{s^2}{2}\right]_0^x \\ &\iff y_3(x) = \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \end{aligned}$$

hence the first three approximation are

$$\begin{cases} y_1(x) = 1 + x \\ y_2(x) = 1 + x + \frac{x^2}{2} \\ y_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \\ \vdots \\ y_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} \end{cases}$$

So

$$y_n(x) = \sum_{n=0}^{n} \frac{x^n}{n!} = e^x$$

## 1.5 Second order differential equations

These equations can be reduced to first-order equations.

### 1.5.1 Linear second-order differential equations

Consider the linear second-order differential equations

$$y'' + a(x)y' + b(x)y = c(x). \tag{1.30}$$

**Definition 1.5.1.** *Two solutions $y_1$ and $y_2$ of equation (1.30) are called independent on an interval $I$ if there does not exist a real constant $k$ such that $y_2(x) = ky_1(x)$ for all $x \in I$.*

**Remark.** The functions $y_1$ and $y_2$ being independent here means they are linearly independent in the context of vector spaces.

#### Homogeneous linear second-order equations

If $c(t) \equiv 0$, the equation (1.31) is called a homogeneous linear equation or an equation without a right-hand side:

$$y'' + a(x)y' + b(x)y = 0. \tag{1.31}$$

**Case where two independent particular solutions are known**

If $y_1$ and $y_2$ are two independent solutions of the differential equation $y'' + a(x)y' + b(x)y = 0$, then the general solution of this differential equation is:

$$y = \lambda y_1 + \mu y_2,$$

where $\lambda$ and $\mu$ are arbitrary constants.

**Case where a particular solution is known**

Consider the homogeneous differential equation:

$$y'' + a(x)y' + b(x)y = 0$$

for which a particular solution $y_1$ is known. The method to find the general solution involves performing a change of function by setting $y(x) = y_1(x)v(x)$, where $v$ is the new unknown function. Substituting this into the equation gives:

$$y_1''v + 2y_1'v' + v''y_1 + a(x)(y_1'v + y_1v') + b(x)y_1v = 0.$$

Since $y_1$ is a solution of the differential equation, we simplify to:

$$y_1v'' + \left(2y_1' + a(t)y_1\right)v' = 0.$$

Let $w = v'$. Then $w$ satisfies a first-order linear differential equation:

$$\frac{w'}{w} = -2\frac{y_1'(x)}{y_1(x)} - a(x).$$

The general solution for $w$ is:

$$w(x) = \lambda\left(y_1(x)\right)^{-2} e^{-\int a(x)\,dx}, \quad \lambda \in \mathbb{R}.$$

Integrating $w$, we find $v$:

$$v(x) = \int w(x)\,dx + \mu, \quad \mu \in \mathbb{R}.$$

Thus, the general solution of $y'' + a(x)y' + b(x)y = 0$ is:

$$y(x) = y_1(x)\left(\lambda \int w(x)dx + \mu y_1(x)\right). \qquad \lambda, \mu \in \mathbb{R}$$

that mine

$$y(x) = y_1(x)\left(\lambda \int [(y_1(x))^{-2}\, e^{-\int a(x)\,dx}]\,dt + \mu y_1(x)\right).$$

**Nonhomogeneous linear second-Order equations**

If $c(x) \neq 0$, the equation $y'' + a(x)y' + b(x)y = c(x)$ is called a nonhomogeneous linear second-order differential equation or an equation with a forcing term. The equation $y'' + a(x)y' + b(x)y = 0$ is its associated homogeneous equation.
As with first-order linear differential equations, the general solution of the nonhomogeneous equation $y'' + a(x)y' + b(x)y = c(x)$ is equal to the sum of the general solution of the associated homogeneous equation and a particular solution of the nonhomogeneous equation.

### 1.5.2 Variation of Constants

Suppose we know the general solution $y = \lambda y_1 + \mu y_2$, $\lambda$, $\mu \in \mathbb{R}$ of the homogeneous equation $y'' + a(x)y' + b(x)y = 0$. We can then seek the general solution using the method of variation of constants.

The principle of this method is to treat $\lambda$ and $\mu$ as functions of the variable $x$. We look for solutions in the form $y(x) = \lambda(x)y_1(x) + \mu(x)y_2(x)$. Substituting this function into the nonhomogeneous equation and simplifying, we obtain:

$$2\lambda' y_1' + 2\mu' y_2' + \lambda'' y_1 + \mu'' y_2 + a(x)(\lambda' y_1 + \mu' y_2) = c(x).$$

By imposing the additional condition $\lambda' y_1 + \mu' y_2 = 0$,
we observe that $\lambda'' y_1 + \mu'' y_2 = -(\lambda' y_1' + \mu' y_2')$. Consequently, the derivatives $\lambda'$ and $\mu'$ must satisfy the system:

$$\begin{cases} \lambda' y_1 + \mu' y_2 = 0, \\ \lambda' y_1' + \mu' y_2' = c(x). \end{cases}$$

Solving this system yields:

$$\lambda' = \frac{-c(x)y_2}{y_1y_2' - y_1'y_2}, \quad \mu' = \frac{c(x)y_1}{y_1y_2' - y_1'y_2}.$$

Hence, the general solution of the nonhomogeneous equation $y'' + a(c)y' + b(x)y = c(x)$ is:

$$y = y_1 \int \frac{-c(x)y_2}{y_1y_2' - y_1'y_2}\,dx + y_2 \int \frac{c(x)y_1}{y_1y_2' - y_1'y_2}\,dx + \alpha y_1 + \beta y_2, \quad \alpha,\, \beta \in \mathbb{R}.$$

### 1.5.3 Linear second-order equations with constant coefficients

A linear second-order differential equation with constant coefficients is an equation of the form

$$y'' + ay' + by = c(x),$$

where $a$ and $b$ are real constants, and $c(x)$ is a given continuous function on an interval $I \subset \mathbb{R}$.

As in the case of non-constant coefficients, we begin by solving the associated homogeneous equation (or the equation without a forcing term):

$$y'' + ay' + by = 0.$$

We seek solutions of the form $y = e^{\lambda x}$, where $\lambda \in \mathbb{R}$. Substituting this into the homogeneous equation, we obtain:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Since the exponential function is never zero, a non-trivial solution requires:

$$\lambda^2 + a\lambda + b = 0.$$

This is called the **characteristic equation** (or auxiliary equation) of the homogeneous equation. The roots $r$ are found using the quadratic formula:

$$\lambda = \frac{-a \pm \sqrt{b^2 - 4ac}}{2}.$$

Three cases may arise based on the discriminant $\Delta = b^2 - 4ac$:

**Case 1:** If $\Delta > 0$, there are two distinct real roots $\lambda_1$ and $\lambda_2$. The functions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ form a fundamental set of linearly independent solutions. The general solution of the homogeneous equation $y'' + ay' + by = 0$ is:

$$y_h = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}, \quad \alpha, \beta \in \mathbb{R}.$$

**Case 2:** If $\Delta = 0$, there is one repeated real root $\lambda_0$. In this case, $y_1 = e^{\lambda_0 x}$ is a solution. A second linearly independent solution is $y_2 = xe^{\lambda_0 x}$. Thus, the general solution of the homogeneous equation becomes:

$$y_h = (\alpha x + \beta)e^{\lambda_0 x}, \quad \alpha, \beta \in \mathbb{R}.$$

**Case 3:** If $\Delta < 0$, there are two distinct complex conjugate roots of the general form $z_1 = \alpha - i\beta$ and $z_2 = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. In this case, the functions $y_1 = e^{\alpha t}\cos(\beta t)$ and $y_2 = e^{\alpha t}\sin(\beta t)$ form a fundamental set of linearly independent solutions to the homogeneous equation. The general solution of the homogeneous equation is:

$$y_h = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right), \quad c_1, c_2 \in \mathbb{R}.$$

This can also be expressed in the alternative forms:

$$y_h = c_1 e^{\alpha x}\cos(\beta x + c_2) \quad \text{or} \quad y_h = c_1 e^{\alpha x}\sin(\beta x + c_2), \quad c_1, c_2 \in \mathbb{R}.$$

**Example.** **(a)** Solve $y'' + 4y' + 3y = 0$.

The characteristic equation is $\lambda^2 + 4\lambda + 3 = 0$. The roots are $\lambda_1 = -3$ and $\lambda_2 = -1$. Thus, the general solution is:

$$y = \alpha e^{-3t} + \beta e^{-t}, \quad \alpha, \beta \in \mathbb{R}.$$

**(b)** Solve $y'' + 4y' + 9y = 0$.

The characteristic equation is $z^2 + 4z + 9 = 0$. The roots are $z_1 = -2 - i\sqrt{5}$ and $z = -2 + i\sqrt{5}$. Therefore, the general solution is:

$$y = e^{-2x}\left(c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x)\right), \quad c_1, c_2 \in \mathbb{R}.$$

**(c)** Solve $y'' + 6y' + 9y = 0$.

The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, which has a repeated real root $\lambda_1 = -3$. The general solution is:

$$y = (\alpha x + \beta)e^{-3x}, \quad \alpha, \beta \in \mathbb{R}.$$

## Finding a particular solution

To solve the nonhomogeneous equation $y'' + ay' + by = c(x)$, we first find a particular solution $y_p$. The general solution is then:

$$y = y_p + y_h,$$

where $y_h$ is the general solution of the associated homogeneous equation.

### 1.5.4 Particular solution by the method of variation of constants

As seen in the general case of linear second-order equations, a particular solution to the equation $y'' + ay' + by = c(x)$ can be obtained using the method of variation of constants (see page 10):

$$y = y_1 \int \frac{-c(x)\, y_2}{y_1 y_2' - y_1' y_2}\, dx + y_2 \int \frac{c(x)\, y_1}{y_1 y_2' - y_1' y_2}\, dx + \alpha y_1 + \beta y_2, \quad \alpha, \beta \in \mathbb{R},$$

where $y_1$ and $y_2$ are two independent particular solutions of the homogeneous equation $y''+ay'+by = 0$. If $\lambda_1$ and $\lambda_2$ are distinct roots of the characteristic polynomial $\lambda^2+a\lambda+b$, then $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. In this case, using integration by parts, the general solution of the nonhomogeneous equation can be written in a single formula:

$$y = e^{\lambda_1 x} \int \left( e^{(\lambda_2 - \lambda_1)x} \left( \int e^{-\lambda_2 x} c(x)\, dx \right) \right) dx.$$

**Example.** Let us find a solution to the equation $y'' - y' - 2y = e^{-x}$. The characteristic equation $\lambda^2 - \lambda - 2 = 0$ has roots $\lambda_1 = -1$ and $\lambda_2 = 2$. The general solution is therefore:

$$\begin{aligned} y &= e^{-x} \int \left( e^{3x} \left( \int e^{-2x} e^{-x}\, dx \right) \right) dx \\ &= e^{-x} \int \left( e^{3x} \left( -\frac{1}{3} e^{-3x} + c_1 \right) \right) dx \\ &= e^{-x} \left( \frac{1}{3} \lambda e^{3x} - \frac{1}{3} x + c_2 \right) \\ &= \frac{1}{3} c_1 e^{2x} + \left( c_2 - \frac{1}{3} x \right) e^{-x}, \quad c_1, c_2 \in \mathbb{R}. \end{aligned}$$

## 1.6 Higher order differential equations

The order differential equation take a form

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

then

$$p = \frac{dy}{dx} \quad , \quad f(x, y, p) = 0 \tag{1.32}$$

If the degree $p$ is greater than 1 the equation (1.32) called higher order differential equation

**Example.**

$$(p)^3 + 2p + 1 = 0$$

**Definition 1.6.1.** *An ordinary equation of order $n$ in the dependent variable $y$ and the independent variable $x$ is an equation ,on can be expressed in the form :*

$$p^{(n)} + f_1(x,y)p^{(n-1)} + f_2(x,y)p^{(n-2)} + ... + f_n(x,y) = 0 \tag{1.33}$$

**Definition 1.6.2.** ***(Wronskian:)*** *Let $y_1(x)$ and $y_2(x)$ be two differentiable functions on an interval $I$.*
*The* Wronskian *of these two functions is defined using a determinant:*

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

*The two differentiable functions $y_1(x)$ and $y_2(x)$ are linearly independent if and only if their Wronskian $W(y_1, y_2)$ is not identically zero.*

**Example.** The functions $x \mapsto \sin x$ and $x \mapsto e^x$ are independent.

The functions $f : x \mapsto 3e^{-x}\sin x$ and $g : x \mapsto 5e^{-x}\sin tx$ are not linearly independent, since $g = \frac{5}{3}f$.

### 1.6.1 The equation solve in $p$

To analyze the expression (1.33) which is considered $p$ polynomial of degree $n$, in terms of linear factors, it take a form :

$$[p - F_1(x,y)][p - F_2(x,y)]...[p - F_n(x,y)] = 0$$

$$\begin{cases} p - F_1(x,y) = 0 \\ p - F_2(x,y) = 0 \\ \vdots \\ p - F_n(x,y) = 0 \end{cases} \Rightarrow \begin{cases} \frac{dy}{dx} = F_1(x,y) \\ \dfrac{dy}{dx} = F_2(x,y) \\ \vdots \\ \dfrac{dy}{dx} = F_n(x,y) \end{cases}$$

we get a $n$ the first order and first degree differential equation ,you can use various method and the solution be

$$\begin{cases} y_1(x,y,c) = 0 \\ y_2(x,y,c) = 0 \\ \vdots \\ y_n(x,y,c) = 0 \end{cases}$$

we write the solution

$$[y_1(x,y,c)][y_2(x,y,c)]...[y_n(x,y,c] = 0$$

**Example.** solve this equation

$$p^3 - 2p^2 - p = 0 \tag{S}$$

We can write (S) whit this form

$$p(p-2)(p-1)=0$$

$$\begin{cases} p=0 \\ p-2=0 \\ p-1=0 \end{cases} \Rightarrow \begin{cases} \frac{dy}{dx}=0 \\ \frac{dy}{dx}=2 \\ \frac{dy}{dx}=1 \end{cases} \Rightarrow \begin{cases} y=c \\ y=2x+c \\ y=x+c \end{cases}$$

the solution is

$$(y-c)(y-2x-c)(y-x-c)=0$$

**Secondly:** If differential equation can be written in the form:

$$y=f(x,p) \tag{1.34}$$

By differentiating both sides of the equation with respect to $x$, we obtain:

$$p=\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{dp}{dx}$$

The last relation can be expressed as:

$$\phi\left(\frac{dp}{dx},p,x\right)=0 \tag{1.35}$$

By solving this, we can obtain $p$ as a function of $x$. Subsequently, by eliminating $p$ from (1.34) and solving (1.35) (if possible), we derive $y$ as a function of $x$.
This covers the solution process for first-order differential equations using substitution and elimination methods.

$$p=g(x,c) \tag{1.36}$$

from (1.34) and (1.36) we have

$$y=f(x,g(x,c))$$

**Example.** we have

$$y=x+\ln p \tag{T}$$

we derivative in $x$

$$\begin{aligned}
\frac{\partial y}{\partial x} =&p=1+\frac{1}{p}\frac{dp}{dx}\\
dx &=\frac{dp}{(p-1)p}\\
x+c &=\ln\frac{p-1}{p}\\
ke^{x} &=\frac{p-1}{p} && ,k=e^{c}\\
&=1-\frac{1}{p}\\
1-ke^{x} &=\frac{1}{p}\\
p &=\frac{1}{1-ke^{x}} && (1)
\end{aligned}$$

from (T) and (1)

$$y = x + \ln \frac{1}{1 - ke^x}$$

**Thirdly:** If differential equation can be written in the form:

$$x = f(y, p)$$

By differentiating with respect to $y$, we obtain:

$$\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}$$

The last relation can be written as:

$$\phi\left(\frac{dp}{dy}, p, y\right) = 0$$

Solving the differential equation yields:

$$p = g(y, c)$$

This relation, together with the original differential equation, forms the parametric equations of the general solution. By eliminating $p$ from one of them (if possible), we can derive:

$$x = y(x, c)$$

which represents the general solution of the differential equation.

**Example.** Find the general solution and the singular solution of the equation:

$$p^3 - 2xyp + 4y^2 = 0$$

**Solution.** Rewrite the equation as:

$$2x = \frac{p^2}{y} + \frac{4y}{p}$$

Differentiate both sides with respect to $y$:

$$\frac{2}{p} = \frac{2p}{y}\frac{dp}{dy} - \frac{p^2}{y^2} + 4\left(\frac{1}{p} - \frac{4}{p^2}\frac{dp}{dy}\right)$$

Simplifying, we obtain:

$$\left(p - 2y\frac{dp}{dy}\right)\left(2y^2 - p^3\right) = 0$$

This gives two cases:

$$2y^2 - p^3 = 0 \qquad (1)$$

$$p - 2y\frac{dp}{dy} = 0 \qquad (2)$$

### Integrating case (2)

Integrate equation (2):

$$\frac{dp}{p} = \frac{dy}{2y}$$

This yields:

$$\ln p = \frac{1}{2}\ln y + \ln c \quad \Rightarrow \quad p^2 = cy$$

Substitute $p^2 = cy$ into the original equation:

$$(cy)^{3/2} - 2xy(cy)^{1/2} + 4y^2 = 0$$

Simplify to obtain the general solution:

$$c(c-2x)^2 = 16y$$

### Singular solution from case (1)

From equation (1):

$$p^3 = 2y^2 \quad \Rightarrow \quad p = \sqrt[3]{2y^2}$$

Substitute $p = \sqrt[3]{2y^2}$ into the original equation:

$$2y^2 - 2xy\sqrt[3]{2y^2} + 4y^2 = 0$$

Simplify to find:

$$3\sqrt[3]{\frac{y}{2}} = x \quad \Rightarrow \quad y = 2\left(\frac{x}{3}\right)^3$$

# Chapter 2

# Linear First-Order PDEs

## Introduction

The general first-order partial differential equation (PDE) for a two-variables function, denoted as $u = u(x, y)$, can be expressed in the form:

$$F(x, y, u, u_x, u_y) = 0 \tag{2.1}$$

Here, $u_x$ and $u_y$ represent the partial derivatives of $u$ with respect to $x$ and $y$, respectively. The function $F$ establishes a functional relationship between $u$ and its partial derivatives, and the independent variables $x$ and $y$. The general first-order partial differential equation for a function $u = u(x_1, \ldots, x_n)$ of $n$ independent variables, denoted as $x_1, \ldots, x_n$, can be represented as:

$$F(x, u, \nabla u) = 0 \tag{2.2}$$

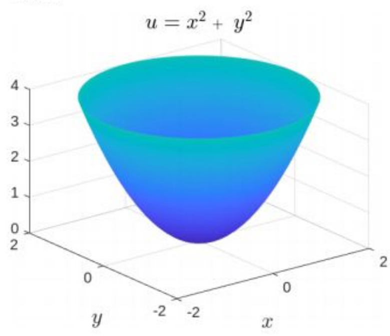
Here, $\nabla u$ is a vector denoted as $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, which comprises the partial derivatives of $u$ with respect to each independent variable $x_1, \ldots, x_n$.

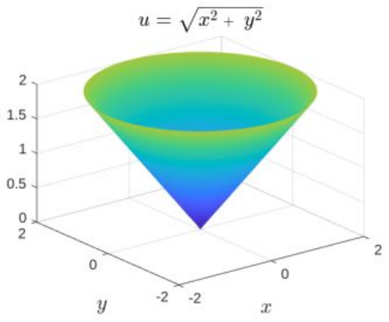
**Definition 2.0.1.** *A classical solution of the equation $F(x, u, \nabla u) = 0$, for $x \in \mathbb{R}^n$, $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, is a smooth function $u = u(x)$ defined on an open set $\Omega \subset \mathbb{R}^n$ such that $F(x, u(x), \nabla u(x)) = 0$ is an identity for all $x \in \Omega$.*

For example, it is possible to verify that functions of the form $u = h(x^2 + y^2)$ for arbitrary smooth functions $h$ is the classical solution of the equation

$$yu_x - xu_y = 0.$$

For example, the function $u = x^2 + y^2$ is a classical solution to the equation for all $(x, y) \in \mathbb{R}^2$, while $u = \sqrt{x^2 + y^2}$ is a solution only on $\mathbb{R}^2 - \{(0, 0)\}$. The graph of these two solutions shown below. Observe that the graph of a classical solution of a first-order PDE in two variables $x$, $y$ is a smooth surface:





## 2.1 Classification of first-order PDEs

In this chapter, we will exclusively focus on the study of first-order PDEs falling within the categories of linear, semi-linear, and quasi-linear equations.

### 2.1.1 Linear equations.

The general form of a linear first-order PDE for a function $u(x) = u(x_1, \ldots, x_n)$ is given by:

$$\sum_{j=1}^{n} v_j(x)\, \partial_j u(x) + v_0(x) u(x) = r(x) \tag{2.3}$$

for some (usually) continuous functions $v_j(x)$ and $r(x)$.

### 2.1.2 Semi-linear equations.

A semi-linear equation is characterized by the general form:

$$\sum_{j=1}^{n} v_j(x)\, \partial_j u(x) = r(x, u) \tag{2.4}$$

The difference between a linear and semi-linear equation is that a semi-linear equation can be nonlinear with respect to $u$ (and not with the partial derivatives $u_x$ and $u_y$).

### 2.1.3 Quasi-linear equations.

A quasi-linear equation assumes the general form:

$$\sum_{j=1}^{n} v_j(x, u)\, \partial_j u(x) = r(x, u) \tag{2.5}$$

The difference between a quasi-linear and semi-linear equation is that in the former case, the coefficients of partial derivatives are functions of $u$ as well.

### 2.1.4 Fully-nonlinear equations.

A fully nonlinear equation is an equation where one or all of the partial derivatives are nonlinear. For example, the equation:

$$|u_x|^2 + |u_y|^2 = 1,$$

is a fully nonlinear first-order equation for $u = u(x, y)$.

**Example.** We can classify the following first-order equations according to their linearity:

**1** $u_x + u_y = e^x u$
It is classified as semi-linear, since all derivatives and the unknown function appear linearly, and the coefficients depend only on the independent variables.

**2** $xu_x + yu_y = e^u$
It is classified as semi-linear, as the highest-order derivatives appear linearly, but the right-hand side is a nonlinear function of the unknown

**3** $u_x(\ln(u_x)) = 1 + u$
It is classified as fully nonlinear, because the derivative appears inside a nonlinear function (the logarithm).

**4** $u_x + u_y + uu_x = 1$
It is classified as quasi-linear, since the derivatives appear linearly, but the coefficients depend on the unknown function itself.

**Example.** Consider the following first-order partial differential equation:

$$u_x + u_y = -u.$$

We can classify this equation as **linear**, since the unknown function and its derivatives appear linearly, and the coefficients depend only on the independent variables.

We write the characteristic system:

$$\frac{dx}{1} = \frac{dy}{1} = -\frac{du}{u}.$$

From $\frac{dx}{1} = \frac{dy}{1}$, we get:

$$x - y = c_1.$$

From $\frac{dx}{1} = \frac{du}{-u}$, we solve:

$$\frac{du}{u} = -dx \quad \Rightarrow \quad \ln|u| = -x + c_2 \quad \Rightarrow \quad u = Ce^{-x}.$$

Since $C$ is constant along the characteristic $x-y=c_1$, we write:

$$u(x,y)=f(x-y)e^{-x},$$

where $f$ is an arbitrary function.

We can verify that this function satisfies the PDE

$$u_x=-f(x-y)e^{-x}+f'(x-y)e^{-x}, \quad u_y=-f'(x-y)e^{-x}.$$

Adding

$$u_x+u_y=-f(x-y)e^{-x}=-u.$$

So

$$u(x,y)=f(x-y)e^{-x}.$$

## 2.2 Characteristic method and ODEs along curves

The characteristic method is a powerful technique for solving first-order partial differential equations, and it is especially useful for semi-linear and quasi-linear equations. By using this method, one can derive the general solution to such equations. Furthermore, the characteristic method has a geometric interpretation that can be illustrated through the Cauchy problem.

### 2.2.1 Introductory remark: ODE along a curve

In our study of ordinary differential equations (ODEs), we explored equations of the form:

$$\frac{du}{dx}=f(x,u).$$

Here, $u=u(x)$ represents a single-variable function. Geometrically, we interpret the $x$-variable as the $x$-axis in the standard direction. The solution to this equation consists of a one-parameter family of functions $u=u(x,c)$, where $c\in\mathbb{R}$, such that for any $x$ within the domain of $u$, the following relation holds:

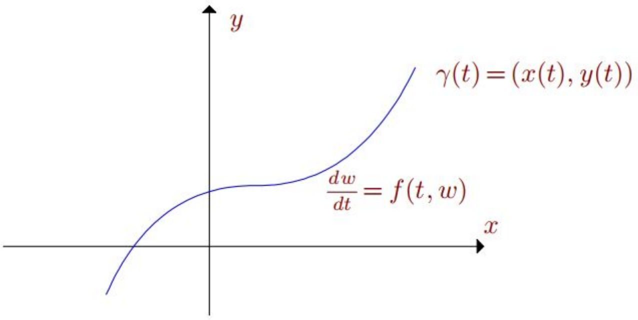
$$\frac{d}{dx}u(x,c)=f(x,u(x,c)).$$

Now, lets shift our focus to a parametric curve $\gamma(t)$ in the $xy$-plane. An ordinary differential equation (first-order) along $\gamma(t)$ takes the form:

$$\frac{d}{dt}u\circ\gamma=f(t,u\circ\gamma).$$

Here, $u\circ\gamma$ is defined at any $t$ as $(u\circ\gamma)(t)=u(\gamma(t))$. If we denote $w=u\circ\gamma$, we arrive at the equation:

$$\frac{dw}{dt}=f(t,w).$$

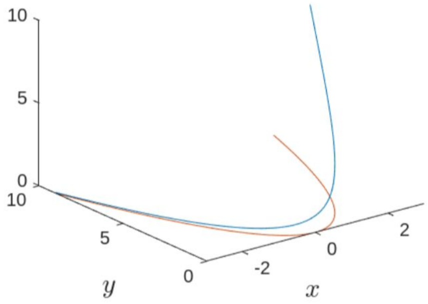


For instance, consider $\gamma(t)$ given by $\gamma(t)=[t,t^2]$, a parabola in the $xy$-plane along which the differential equation

$$\frac{dw}{dt}=w,$$

is defined. Suppose $u$ at the point $(0,0)$ is 1, corresponding to $\gamma(0)$. Then, we obtain $w$ as

$$w(t)=u(\gamma(t))=e^t.$$



**Remark.** Solving differential equations along curves can sometimes result in non-valid solutions. For instance, lets consider the circle $\mathcal{C}$ represented by the parametric curve

$$\gamma(t)=(\cos(t),\sin(t)),$$

for $t$ in the interval $[0,2\pi]$, along with the initial value problem

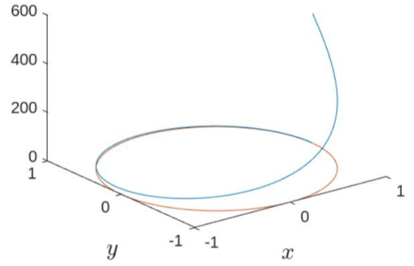
$$\frac{dw}{dt}=w,\quad w(0)=1,$$

defined for $w(t)=u(\gamma(t))$. The solution to this equation is $w(t)=e^t$ for $t\in[0,2\pi]$. However, the function $u$ is not continuous on the circle because

$$u(\gamma(0))=u(1,0)=1,$$

and

$$u\left(\lim_{t\to 2\pi}\gamma(t)\right)=u(\gamma(0))=1\neq\lim_{t\to 2\pi}u(\gamma(t)).$$



### 2.2.2 A simple type of equations

Lets begin with the following simple equation

$$u_x + v(x\,,\,y)u_y = 0, \tag{2.6}$$

where $u$ is a smooth two-variable function, $u = u(x, y)$. Well relate the independent variable $y$ to $x$ through the equation

$$\frac{dy}{dx} = v(x\,,\,y)$$

and assume that the solution to this equation is expressed as $y = Y(x\,,\,c)$, where $c$ is a parameter of the solution to this ordinary differential equation (ODE). This family of curves is known as the characteristic curves of the given partial differential equation (PDE). The reason is that along each curve $y = Y(x\,,\,c)$, the PDE reads as an ODE

$$\frac{d}{dx}u(x\,,\,Y(x\,,\,c)) = 0. \tag{2.7}$$

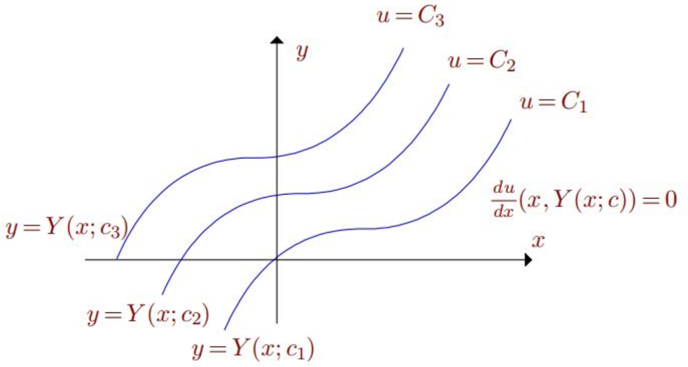
Note that, by the chain rule, we have

$$\frac{d}{dx}u(x\,,\,Y(x\,,\,c)) = u_x(x\,,\,y) + \frac{dy}{dx}u_y(x\,,\,y) = u_x(x\,,\,y) + v(x\,,\,y)u_y(x\,,\,y).$$

Equation (2.7) can be simply solved for the constant function

$$u(x, Y(x\,,\,c)) = C,$$

where $C$ is constant along the characteristic curve $y = Y(x\,,\,c)$ for a fixed $c$. Therefore, $C$ is a function of $c$, written as $C = h(c)$, where $h$ is an arbitrary function.



Let us assume that the equation $y = y(x\,,,c)$ can be solved for $c$ as $c = g(x\,,\,y)$. Then, we can express $u(x\,,\,y)$ as

$$u(x,y) = h(g(x,y)). \tag{2.8}$$

for an arbitrary smooth function $h$. Now, lets verify that the solution (2.8) satisfies equation (2.6)

$$\begin{cases} u_x = h'(g(x,y))\, g_x \\ u_y = h'(g(x,y))\, g_y \end{cases}$$

This implies

$$u_x + v(x,y)u_y = h'(g(x,y))\,(g_x + v(x,y)g_y)$$

Utilizing the relation $c = g(x,y)$, we have

$$0 = g_x dx + g_y dy, \quad \Rightarrow \quad g_x + \frac{dy}{dx} g_y = 0.$$

Substituting this into our previous equation

$$(h(g(x,y)))_x + v(x,y)(h(g(x,y)))_y = 0.$$

This confirms that the solution (2.8) satisfies equation (2.6).

**Example.** Consider the partial differential equation

$$u_x + u_y = 0.$$

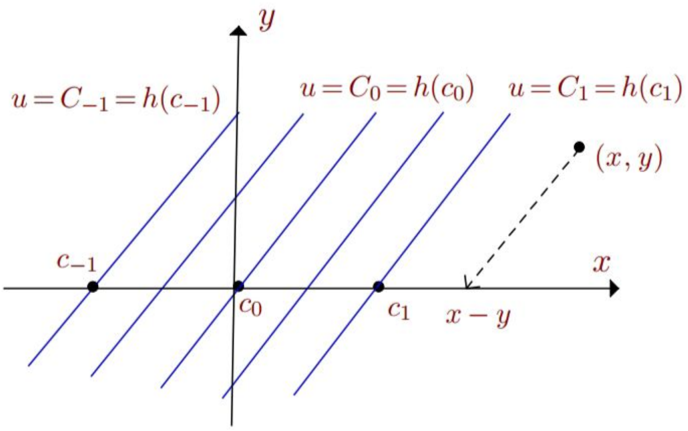
To apply the characteristic method, we begin by finding the characteristic equation

$$\frac{dy}{dx} = 1.$$

This equation has the solution $y = x+c$, where $c$ is a parameter. Now, lets consider the characteristic family, denoted as $\{c = y-x, \quad c \in \mathbb{R}\}$. Along each characteristic curve, $u$ remains constant due to the equation $\frac{du}{dx} = 0$. Thus, we can express $u$ as $u = C$ along each line $c = y - x$, where $C$ depends on $c$. Consequently, we have

$$u(x,y) = h(y-x).$$

Here, $h$ is an arbitrary smooth function. The characteristic curves in this case are straight lines with a slope of 1.



**Remark.** In the example we solved earlier, we obtained the solution $u$ in terms of an arbitrary function $h(y-x)$. Consequently, any function of the form $u=\sin(y-x)$, $u=e^{-(y-x)^2}$, $u=(y-x)^3+x-y$, and so on, satisfies the given partial differential equation $u_x+u_y=0$. This type of solution is known as a general solution.

The concept of a general solution here is akin to the concept of the general solution for a first-order ordinary differential equation that typically contains a constant parameter rather than an arbitrary function. In subsequent discussions, we will explore how to determine the specific form of the arbitrary function $h$ with the help of auxiliary conditions for the problem.

### 2.2.3 Characteristic method for semi-linear PDEs

Lets consider the following equation:

$$v_1(x,y)u_x+v_2(x,y)u_y=v_3(x,y,u), \tag{2.9}$$

where $v_1, v_2$, and $v_3$ are smooth functions. The objective is to transform this partial differential equation into a set of first-order ordinary differential equations along characteristic curves.

Recall the differential of a two-variable function $u=u(x,y)$ as $du=u_x dx+u_y dy$. Comparing the expression of $du$ with equation (2.9) implies the following system:

$$\frac{du}{v_3(x,y,u)}=\frac{dx}{v_1(x,y)}=\frac{dy}{v_2(x,y)}. \tag{2.10}$$

By relating $x$ to $y$ through the characteristic equation:

$$\frac{dy}{dx}=v(x,y), \tag{2.11}$$

where $v=\frac{v_2}{v_1}$, we obtain a family of curves $\gamma_c : c=g(x,y)$, where $c$ is an arbitrary constant. The given PDE reduces to the following equation along each $\gamma_c$:

$$\frac{du}{dx}=\frac{v_3}{v_1}. \tag{2.12}$$

Suppose this equation is solved for $u = U(x, c, C)$, where $C$ depends on $\gamma_c$ and thus can be expressed as $C = h(c)$ for an arbitrary smooth function $h$. Hence, the general solution can be expressed as:

$$u = U\left(x\, ,\, g(x,y)\, ,\, h(g(x\, ,\, y))\right).$$

**Example.** Lets solve the following partial differential equation:

$$xu_x + yu_y = xy.$$

The characteristic equation is given by:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xy}$$

Hence

$$\frac{dy}{dx} = \frac{y}{x},$$

which we can solve to obtain $y = cx$. The ordinary differential equation for $u$comes:

$$\frac{du}{dx} = y.$$

Substituting $y = cx$ into this equation, we have:

$$\frac{du}{dx} = cx.$$

This ordinary differential equation can be solved to find:

$$u = \frac{1}{2}cx^2 + C.$$

Here, $C$ be expressed as an arbitrary smooth function in terms of $c$. Therefore, the general solution can be written as:

$$u(x, y) = \frac{xy}{2} + h\left(\frac{y}{x}\right).$$

**Theorem 2.2.1.** *The general solution of equation (2.9) is*

$$u = U(x\, ,\, g(x,y)\, ,\, h(g(x\, ,\, y))), \tag{2.13}$$

*where $h$ an arbitrary smooth function, $c = g(x\, ,\, y)$ is the equation of characteristic curves solution of the equation (2.11), and $U$ the solution of the equation (2.12).*

*Proof.* We have

$$\begin{cases} u_x = U_x + U_c g_x + U_C h'(g) g_x, \\ u_y = U_c g_y + U_C h'(g) g_y. \end{cases}$$

Multiplying the first equation by $v_1$ and the second one by $v_2$, we obtain

$$v_1u_x + v_2u_y = v_1U_x + U_c(v_1g_x + v_2g_y) + U_Ch'(g)(v_1g_x + v_2g_y).$$

By the equation $c = g(x, y)$ ,we have

$$0 = g_xdx + g_ydy,$$

and by the equality

$$\frac{dx}{v_1} = \frac{dy}{v_2},$$

we obtain

$$v_1g_x + v_2g_y = 0.$$

Hence, we obtain the equality

$$v_1u_x + v_2u_y = v_1U_x.$$

On the other hand, from the equation $u = U(x, c, C)$, we have $du = U_xdx$. Utilizing the equation

$$\frac{du}{dx} = \frac{v_3}{v_1},$$

yields $v_1U_x = v_3$ ,that proves the equation $v_1u_x + v_2u_y = v_3$.

■

**Definition 2.2.1.** *Given a fixed value of $c$ in the real numbers, the curve $\gamma_c$, which is the solution of equation (2.11), is referred to as a characteristic curve of the differential equation (2.9). Since equation (2.9) reduces to an ordinary differential equation when evaluated along $\gamma_c$ for any fixed $c$, we obtain an infinite system of ordinary differential equations for the family of characteristic curves $\{\gamma_c, c \in \mathbb{R}\}$. The system (2.10) is known as the characteristic system of the associated partial differential equation.*

**Example.** The existence of a general solution, even for linear first-order PDEs, is not always a trivial question. Consider the equation

$$xu_x + yu_y = \alpha u,$$

where $\alpha$ is a constant. We will examine three cases: $\alpha = 0, -1$ and 1.

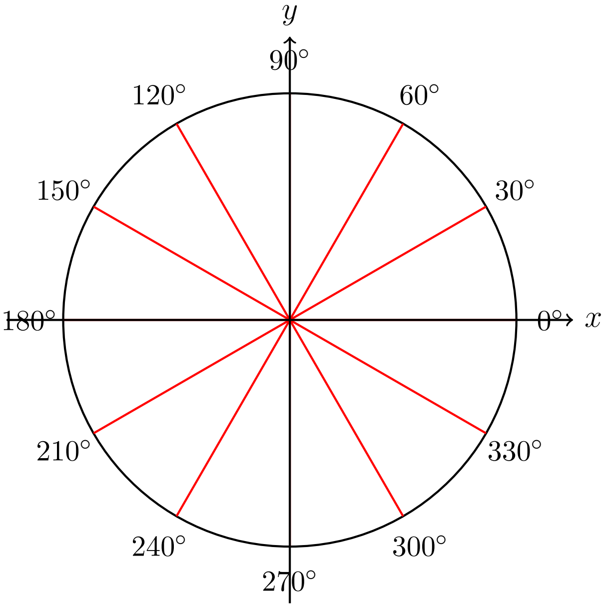
For $\alpha = 0$, the characteristic system is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{0},$$

and the equation for the characteristic curves in the $xy$-plane is

$$y\,dx + x\,dy = 0.$$

The general solution of this simple ODE is $y = cx$, which is shown in the following figure



Along the characteristic line $\gamma_c$, we have $\dfrac{du}{dx} = 0$, and thus $u$ is constant along $\gamma_c$. Furthermore, all characteristic lines intersect at the origin, so$\gamma_c$ carries the information of $u$ at the origin. This implies that $u(x, y) = u(0, 0) = C$, a constant for all $(x, y)$. There is no other solution of the equation in this case.

Now, consider $\alpha = -1$. We will show that the only possible solution is $u \equiv 0$. In this case, the solution $u$ along $\gamma_c$ satisfies the ODE

$$\frac{du}{dx} = -\frac{1}{x}u,$$

and therefore, $u = \dfrac{C}{x}$ along $\gamma_c : y = cx$ with respect to $x$. On the other hand, the PDE implies $u(0, 0) = 0$ and thus $C = 0$, implying that $u(x, y)$is identically zero in this case.

For $\alpha = 1$, the solution $u$ satisfies the ODE

$$\frac{du}{dx} = \frac{1}{x}u,$$

with the solution $u = Cx$ for a constant $C$. The general solution in this case is

$$u(x, y) = f\left(\frac{y}{x}\right)x.$$

The form of the solution imposes a restriction on the form of the function $f$ if $u$ is assumed to be smooth inside the unit disk. For example, for $f(z) = z^2$, the solution is not even continuous at the origin.

## 2.3 Semi-Linear equation in general dimension

Now, we generalize the method for semi-linear PDEs in general dimension. Consider the following equation

$$\sum_{j=1}^{n} v_j(x)\frac{\partial u}{\partial x_j} = r(x,u), \quad x \in \mathbb{R}^n, \tag{2.14}$$

where $v_j(x)$ are smooth functions of $x = (x_1, \dots, x_n)$. To solve this equation, we first consider the characteristic system

$$\frac{dx_1}{v_1(x)} = \cdots = \frac{dx_n}{v_n(x)} = \frac{du}{r(x)}.$$

The characteristic equation in the space $(x_1, \dots, x_n)$ is

$$\frac{dx_1}{v_1(x)} = \cdots = \frac{dx_n}{v_n(x)}.$$

By taking $x_1$ as the independent variable, we can write the system as

$$\begin{cases} \dfrac{dx_2}{dx_1} = \dfrac{v_2(x)}{v_1(x)} \\ \quad\vdots \\ \dfrac{dx_n}{dx_1} = \dfrac{v_n(x)}{v_1(x)} \end{cases} \tag{2.15}$$

We can solve system (2.15) for a given parameter $c_2 = (c_2, \dots, c_n)$ as the implicit system

$$c_2 = g_2(x), \quad \dots, \quad c_n = g_n(x).$$

We assume that the above implicit solutions are solvable for $x$ in terms of $x_1$ and $c$, so we can write

$$x_2 = x_2(x_1; c), \quad \dots, \quad x_n = x_n(x_1; c).$$

The intersection of surfaces

$$c_2 = g_2(x), \quad \dots, \quad c_n = g_n(x),$$

reduces to a smooth curve $\gamma_c, c = (c_2, \dots, c_n)$, which is the same as the parametric curves

$$\gamma_c : \{x_2 = x_2(x_1; c), \dots, x_n = x_n(x_1; c)\}.$$

Next, we consider the derivative of $u$ along the curve $\gamma_c$, which is given by

$$\frac{du}{dx_1} = \frac{r(x,u)}{v_1(x)}. \tag{2.16}$$

By solving this equation, we obtain a function $= U(x_1, c_2, \dots, c_n, C)$. Therefore, taking $C = h(c)$, for an arbitrary smooth function $f$, the general solution to equation (2.14) can be expressed as

$$u = U(x_1, g_2(x), \dots, g_n(x), h(g_2(x), \dots, g_n(x))).$$

**Example.** Lets solve the partial differential equation

$$u_x + u_y + u_z = 0.$$

The characteristic equations in the space $(x, y, z)$ is given by

$$\frac{dy}{dx} = 1, \quad \frac{dz}{dx} = 1,$$

which is solved for $c_2 = y - x$, $c_3 = z - x$. The characteristic curves parameterized by $x$ is

$$\gamma_c = (x, x + c_2, x + c_3)$$

for $c = (c_2, c_3)$. The given PDE along $\gamma_c$ becomes

$$\frac{du}{dx} = 0,$$

which implies $u = C$. Replacing $C$ by $h(c_1, c_2)$ for an arbitrary smooth function $h$, we obtain the general solution

$$u(x, y, z) = h(y - x, z - x).$$

## 2.4 Characteristic method for quasi-linear equations

The method of characteristics for quasi-linear partial differential equations may lead to the emergence of new phenomena, such as shocks, which we will discuss later. Let's consider the following equation

$$v_1(x\,,\,y\,,\,u)\,u_x + v_2(x\,,\,y,\,,\,u)\,u_y = v_3(x,\,,\,y,\,,\,u). \tag{2.17}$$

The coefficients of the quasi-linear equations depend on $u$. The characteristic system of the PDE is

$$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{du}{v_3}. \tag{2.18}$$

Let $c_1 = \phi(x\,,\,y\,,\,u)$ and $c_2 = \psi(x, y, u)$ be implicit solutions of the characteristic system. We have the following theorem

**Theorem 2.4.1.** *Let $v_1, v_2$, and $v_3$ be smooth functions in equation (2.17), and let $c_1 = \phi(x, y, u)$ and $c_2 = \psi(x, y, u)$ be implicit solutions of its associated characteristic system. Then, the general solution of the equation in implicit form is given by $f(\phi, \psi) = 0$ for any smooth function $f$ satisfying $f_u \neq 0$.*

**Example.** Let's consider the following equation called the Burgers equation

$$u_x + u\,u_y = 0.$$

The characteristic system is given by

$$\frac{dx}{1} = \frac{dy}{u} = \frac{du}{0}.$$

By taking $x$ as the independent variable and setting $y = y(x)$, we obtain

$$\frac{dy}{dx} = u, \quad \frac{du}{dx} = 0.$$

This system is solved for $c_1 = u$ and $c_2 = y - xu$. The general implicit solution of the equation is derived by $f(c_1, c_2) = 0$, for arbitrary smooth function $f$, or equivalently $f(u, y - xu) = 0$. Alternatively, we can write the solution as $u = g(y - xu)$ for arbitrary smooth function $g$. Note that $u$ in both cases is in implicit form.

**Example.** Consider the equation

$$u_x + y\,u\,u_y + z\,u\,u_z = 0.$$

To solve this equation, we need to determine the characteristic system, which is given by

$$\frac{dy}{dx} = yu, \quad \frac{dz}{dx} = zu, \quad \frac{du}{dx} = 0.$$

Solving this system of differential equations, we obtain $c_1 u = e^{-x} y$, $c_2 u = e^{-x} z$, and $c = u$, where $c, c_1$ and $c_2$ are arbitrary constants. Substituting these values into the equation, we get the general implicit solution as

$$f\left(u, e^{-x} y \cdot \frac{1}{u}, e^{-x} z \cdot \frac{1}{u}\right) = 0.$$

Alternatively, the solution can be expressed as

$$u = g\left(\frac{ye^{-x}}{u}, \frac{ze^{-x}}{u}\right).$$

*Proof.* Since $f(\phi(x, y, u(x, y)), \psi(x, y, u(x, y)))$ is identically zero, taking derivatives with respect to $x$ and $y$ gives

$$\begin{cases} f_\phi(\phi_x + u_x\phi_u) + f_\psi(\psi_x + u_x\psi_u) = 0, \\ f_\phi(\phi_y + u_y\phi_u) + f_\psi(\psi_y + u_y\psi_u) = 0. \end{cases}$$

Note that $f_u \neq 0$ implies that $f_\phi$ and $f_\psi$ cannot both be identically zero. This in turn implies that the following determinant is zero

$$\begin{vmatrix} \phi_x + u_x\phi_u & \psi_x + u_x\psi_u \\ \phi_y + u_y\phi_u & \psi_y + u_y\psi_u \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we get

$$(\phi_u\psi_y - \psi_u\phi_y)u_x + (\phi_x\psi_u - \phi_u\psi_x)u_y = \phi_y\psi_x - \phi_x\psi_y. \tag{2.19}$$

On the other hand, we have

$$\begin{cases} d\phi = v_1\phi_x + v_2\phi_y + v_3\phi_u = 0, \\ d\psi = v_1\psi_x + v_2\psi_y + v_3\psi_u = 0. \end{cases}$$

that gives

$$\frac{v_1}{\phi_y\psi_u - \phi_u\psi_y} = \frac{v_2}{\phi_u\psi_x - \phi_x\psi_u} = \frac{v_3}{\phi_x\psi_y - \phi_y\psi_x}. \tag{2.20}$$

Matching relations (2.19) and (2.20) gives $v_1u_x + v_2u_y = v_3$, which completes the proof.

■

## 2.5 Theoretical aspects

### 2.5.1 The geometrical interpretation of a first-order PDE

Consider the partial differential equation

$$v_1(x,y,u)u_x + v_2(x,y,u)u_y = v_3(x,y,u). \tag{2.21}$$

Here, $v_1, v_2$, and $v_3$ are functions of $x, y$, and $u$. Let $z = u(x,y)$ be a solution to this equation. This function defines a surface in the space $(x,y,z)$

$$S : (x, y, u(x,y)).$$

Recall that the vector $\vec{n} = (-u_x, -u_y, 1)$ is perpendicular to $S$ at any point $(x,y,z)$ on $S$. Now, assume that the vector field $\vec{V} : (x,y,z) \mapsto (v_1, v_2, v_3)$ is given in the space. The partial differential equation (2.21) can then be stated from the geometrical point of view as follows

### 2.5.2 Geometrical interpretation of a first-order PDE

Given a vector field $\vec{V} : (x,y,z) \mapsto (v_1, v_2, v_3)$, the equation of a tangent surface $z = u(x,y)$ to $\vec{V}$ is given by the partial differential equation

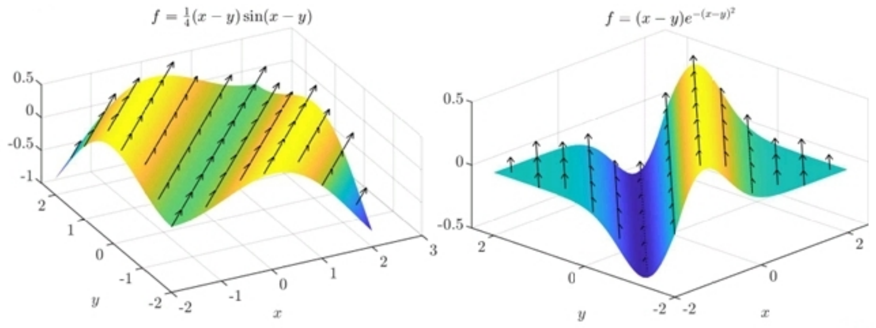
$$v_1(x,y,u)\, u_x + v_2(x,y,u)\, u_y = v_3(x,y,u).$$

Conversely, the solution of the above partial differential equation defines a surface $z = u(x,y)$ that is locally tangent to the vector field $\vec{V}$.

For example, consider the equation: $u_x + u_y = 0$. The vector field is

$$\vec{V} : (x, y, z) \mapsto (1, 1, 0).$$

The figure below shows two different surfaces that are tangent to this vector field at all points of the surfaces



In fact, it can be seen that all surfaces generated by the function $u = f(y - x)$ for arbitrary smooth $f$ have this tangent property since

$$\vec{V} \cdot \vec{n} = (1, 1, 0) \cdot (f'_x, -f'_y, 1) = 0.$$

### 2.5.3 Parametric Solution Surfaces

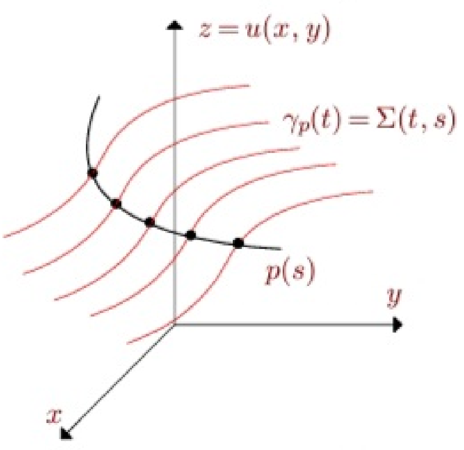
Now, consider a curve $\gamma(t)$ on the solution surface (yet unknown) $S$. This curve is tangent to the vector field $V$ at any point on the curve. Therefore, the equation of $\gamma(t)$ is determined by the following system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = v_1(x(t), y(t), z(t)) \\ \dfrac{dy}{dt} = v_2(x(t), y(t), z(t)) \\ \dfrac{dz}{dt} = v_3(x(t), y(t), z(t)) \end{cases} \tag{2.22}$$

This first-order system of ordinary differential equations has a unique solution if an initial condition is set for the system. Let's assume that we know a point $p_0$ on $S$. We can set the initial condition as

$$\gamma(0) = p_0 : \begin{cases} x(0) = x_0 \\ y(0) = y_0 \\ z(0) = z_0 \end{cases}$$

With this initial condition, a curve $\gamma_{p_0}(t)$ is obtained on $S$. In this way, to determine $S$, we need a family of curves $\{\gamma_p(t)\}$ where $p$ lies on a curve on $S$ as shown below



For example, assume the curve $p(s) = (s, 0, s\sin s)$ lies on $S$ of the solution of the equation $u_x + u_y = 0$. The solution surface is spanned by the curves determined by the following system of ODEs

$$\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 1 \\ \dfrac{dz}{dt} = 0 \end{cases}$$

accompanied with the conditions $x(0) = s$, $y(0) = 0$, $z(0) = s\sin s$. This system is solved as

$$x(t,s) = t + s, \quad y(t,s) = t, \quad z(t,s) = s\sin s.$$

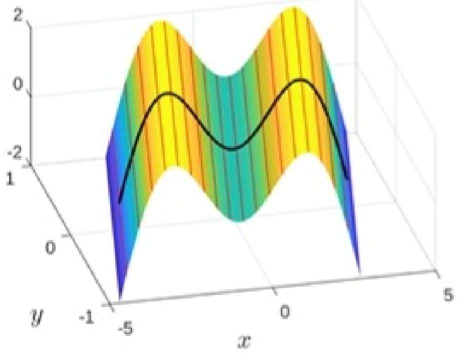
In this way, we obtain a parametric surface

$$\sum(t,s) = (t + s, t, s\sin(s)).$$

It is simply seen that this parametric surface is algebraically represented as

$$u(x,y) = (y - x)\sin(y - x).$$

The figure below depicts the data line $\rho(s)$ in black and the space characteristic curves $\gamma_p(t)$ in red.



**Remark.** The advantage of using parametric form for representing solutions is that it can represent very complicated surfaces, whereas explicit functions $z = u(x, y)$ represent only restricted classes of surfaces. The example below further clarifies this point.

**Example.** Consider the following problem

$$\begin{cases} 0.2xu_x - u_y = y \\ u\Big|_{x=y^2} = s \end{cases}$$

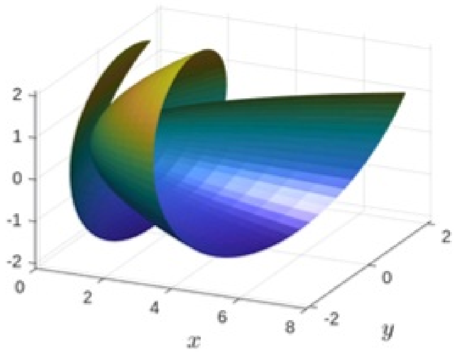
Here $u$ is given along the curve $x = y^2$ in the $xy$-plane. This data can be parameterized in terms of $s$ as: $\rho(s) = (s^2, s, s)$. The system of characteristic equations is

$$\begin{cases} \frac{dx}{dt} = 0.2x \\ \frac{dy}{dt} = -1 \\ \frac{du}{dt} = y \end{cases}$$

with the initial conditions $x_0 = s^2$, $y_0 = s$, and $z_0 = s$. The parametric representation of the integral surface is obtained as

$$\Sigma(t, s) = \left(s^2 e^{0.2t}, s(\cos t - \sin t), s(\sin t + \cos t)\right).$$

As shown in the figure below, the solution of the equation represents a complicated surface which cannot be expressed by an explicit function $z = u(x, y)$.



Even though this surface is not associated with an explicit function, it is possible to define an explicit function that locally coincides with this surface. Thus, classical solutions of partial differential equations can only be defined locally in a neighborhood of the data curve.

### 2.5.4 Cauchy Problem

In this section, we demonstrate how to obtain the particular solution of a given first-order PDE from the general solution, using an auxiliary condition or additional information about the solution. This process is similar to deriving particular solutions from the general solutions of ODEs using initial conditions.

**Definition 2.5.1.** *A problem of the form*

$$\begin{cases} v_1(x,y,u)u_x + v_2(x,y,u)u_y = v_3(x,y,u), \\ u|_C = g. \end{cases} \tag{2.23}$$

*where $C$ is a curve in an open set $\Omega \subset \mathbb{R}^2$ in the $xy$-plane, is called a Cauchy problem.*

Let's consider the following problem

$$\begin{cases} u_x + u_y = 0 \\ u|_{\{y=0\}} = \frac{1}{1+2x^2} \end{cases}.$$

The general solution to the equation is $u = h(y-x)$ for an arbitrary smooth function $h$. Utilizing the initial condition $u(x,0) = \frac{1}{1+2x^2}$, we get

$$\frac{1}{1+2x^2} = h(-x),$$

and thus $h(y-x) = \frac{1}{1+2(y-x)^2}$. Therefore, the particular solution to the given Cauchy problem is

$$u(x,y) = \frac{1}{1+2(y-x)^2}.$$

Examples below highlight some of the issues that can arise when attempting to extract a particular solution from a general solution.

**Example.** Consider the Cauchy problem:

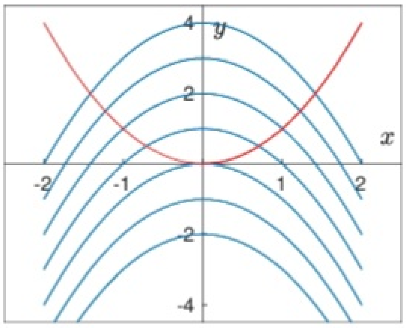
$$\begin{cases} u_x - 2x\,u_y = 0 \\ u|_{\{y=x^2\}} = x \end{cases}.$$

The general solution of the PDE is $u = f(y+x^2)$ for an arbitrary smooth function $f$. We can use the auxiliary condition $u = x$ on the curve $C$: $y = x^2$ to determine $f$, which gives

$$x = f(2x^2).$$

However, this equation cannot be solved uniquely because $f(2)$ can have two possible values, $\pm 1$, leading to a contradiction.

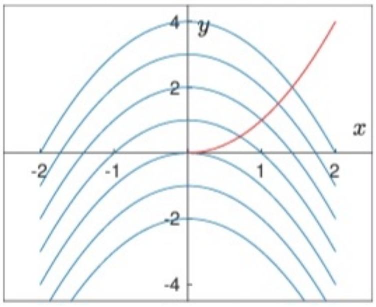
This problem occurs because the characteristic curves of the PDE, $y = -x^2 + c$, intersect the data curve $C$: $y = x^2$ at more than one point, resulting in multiple

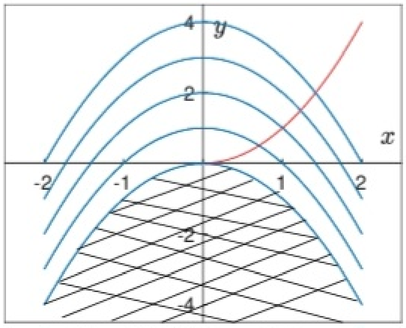


possible values for $u$. To resolve this issue, we can consider only one branch of the data curve, such as $y = x^2$ for $x \geq 0$, to obtain a unique solution

$$u(x,y) = \sqrt{\frac{y+x^2}{2}} \quad \text{for} \quad y \geq -x^2.$$

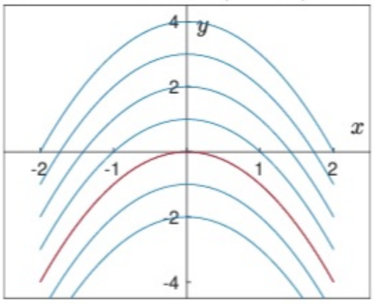
However, this solution is still not unique in the region $y < -x^2$, where the characteristic curves do not intersect the data curve. The value of $u$ along these curves can be chosen arbitrarily. If we change the data curve to $y = -x^2$, which is also a characteristic curve, there is no unique solution in the regions $y > -x^2$ and $y < -x^2$. In this case, the condition $x = f(0)$ cannot be solved from the general solution $u = f(y + x^2)$ and the data curve $y = -x^2$.





**Example.** Consider the following equation

$$\begin{cases} -yu_x + xu_y = u \\ u|_C = x \end{cases},$$



where $C$ is the $x$-axis for $x \geq 0$. To apply the method of characteristics, we first find the equation of the characteristic curves. Using the characteristic equation

$$\frac{dx}{-y} = \frac{dy}{x},$$

we get $x^2 + y^2 = c$, where $c$ is a positive constant. Thus, the characteristic curves are circles centered at the origin.

In a previous exercise, we showed that this equation does not have any smooth solution inside a disk. This can be verified by attempting to solve the equation explicitly. Taking $x$ as the independent variable, we can rewrite the PDE as

$$\frac{du}{dx} = -\frac{1}{y}u.$$

To solve this equation, we need to express $y$ as a function of $x$. However, this is not possible using the implicit function $x^2+y^2 = c$. Parametric Representation An alternative approach is to use the parametric representation of the characteristic curves. Let $t$ be a parameter and consider the characteristic system

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = x \end{cases}.$$

The solution to this system is

$$\gamma_p(t) = (s\cos(t), s\sin(t)),$$

where $s$ is a non-negative parameter. Note that we used the data curve $C$ to write the initial point of the characteristic curve as $\gamma_s(0) = (s, 0)$. Now, we can express $u$ in terms of $t$ as follows

$$\frac{du}{dt} = u,$$

which is a separable ODE with solution

$$u(\gamma_s(t)) = u(\gamma_s(0))e^t = u(s)e^t = se^t.$$

To determine the domain of $t$, note that $\gamma_s(0) = \gamma_s(2\pi)$, which implies

$$u(\gamma_s(2\pi)) = u(\gamma_s(0)) = s.$$

However, we also have

$$u(\gamma_s(2\pi)) = se^{2\pi},$$

So we conclude that the domain of $t$ can not be $[0, 2\pi]$. Note that for $x = s\cos t$, $y = s\sin t$ and $u = se^t$, we obtain the integral surface

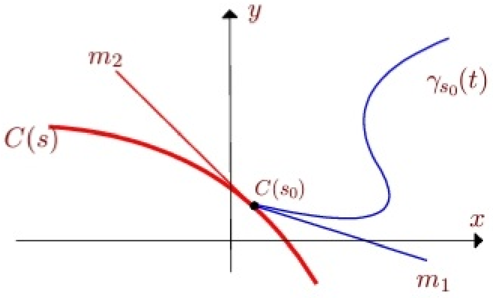
$$\Sigma(t, s) = (s\cos t, s\sin t, se^t),$$

The solution can be put in the algebraic form as:

$$u(x, y) = \sqrt{x^2 + y^2}\, e^{a\tan\left(\dfrac{y}{x}\right)}.$$

### 2.5.5 Well-Posedness and Existence of Integral Surfaces

As we observed in previous examples, if the data curve of a Cauchy problem is not a characteristic curve, a solution can be extended locally. The figure below illustrates this schematically. Here, the data curve $C$ is parametrized by $s$ as $C = C(s)$, and the planar characteristic curves $\gamma$ are parameterized by $t$. Note that $\gamma_s(t)$ is a characteristic curve passing through $C(s)$ at $t = 0$. Let $s_0$ be a point on $C(s)$ in its domain. If $\gamma_{s_0}(0)$ and $C'(s_0)$ are non-parallel, then there exists a $t_0 > 0$ such that $\gamma_{s_0}(t)$ exists for $0 \le t < t_0$.



**Theorem 2.5.1.** *Consider the Cauchy problem*

$$\begin{cases} v_1(x, y)\, u_x + v_2(x, y)\, u_y = v_3(x, y, u) \\ u|_C = f \end{cases}$$

*where $v_1, v_2$ and $v_3$ are smooth functions, and $C$ is a smooth curve in the xy-plane. Assume there exists $(x_0, y_0) \in C$ such that*

$$C'(x_0, y_0) \nparallel (v_1(x_0, y_0), v_2(x_0, y_0)). \tag{2.24}$$

*Then there exists an open neighborhood $\Omega$ of $(x_0, y_0)$ and a smooth function $u = u(x, y)$ on $\Omega$ that solves the given Cauchy problem.*

*Proof.* The proof of the theorem is based on a standard theorem on the existence and uniqueness of the solution to ordinary differential equations. Note that if condition (2.24) holds, then due to the continuity of $v_1, v_2$, and $C'$ at $(x_0, y_0)$, the condition holds for an open neighborhood of $(x_0, y_0)$. Then, the existence of a domain $\Omega$ for the solution $u(x, y)$ is reduced to the existence and uniqueness of the ordinary differential equation

$$\frac{du}{dt} = v_3(x(t), y(t), u).$$

However, it is important to note that the theorem only provides a sufficient condition for the existence of a solution, and there may be cases where the condition is not satisfied but a solution still exists. For instance, consider the problem

$$\begin{cases} u_x + \sqrt{y}\, u_y = 0 \\ \qquad u(x, 0) = f(x) \end{cases}$$

This problem has the solution $u(x, y) = f(x - 2\sqrt{y})$ which is defined for $y \geq 0$, even though it is not generally differentiable on the $x$-axis.

■

Chapter 3

# Second order PDEs

## Introduction

Partial differential equations (PDEs) can be classified in various ways based on their mathematical properties, structure, and behavior. These classifications are important because they determine the appropriate methods for solving the equations and provide insights into the physical phenomena they model.

## 3.1 Linear partial differential equations

**Definition 3.1.1.** *A partial differential equation is called linear if it is linear in the unknown function and its derivatives. In other words, the unknown function and all its derivatives appear only to the first power and are not multiplied together.*

A partial differential equation is called linear if it is linear in the unknown function and its derivatives. In other words, the unknown function and all its derivatives appear only to the first power and are not multiplied together.

The general form of a second-order linear partial differential equation can be written as:

$$\alpha\phi + a \bullet \nabla\phi + B : \nabla(\nabla\phi) = f \tag{3.1}$$

Where $\alpha$ , $a$ , $B$ and $f$ are coefficients that may depend on the independent variables but not on the unknown function $\phi$ or its derivatives.

### 3.1.1 Properties of linear partial differential equations

**The principle of superposition applies**

if $\phi_1$ and $\phi_2$ are solutions, then any linear combination $c_1\phi_1 + c_2\phi_2$ is also a solution - Methods such as separation of variables, Fourier transforms, and Green's functions can be applied - Well- developed theoretical framework exists for existence, uniqueness, and regularity of solutions

**Examples of linear partial differential equations**

Heat equation:

$$\frac{\partial \phi}{\partial t} - \mu \nabla \phi = f. \tag{3.2}$$

Wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \mu \nabla \phi = f. \tag{3.3}$$

Laplace equation: $\delta \phi = 0$ Nonlinear partial differential equation

**Definition 3.1.2.** *A partial differential equation is nonlinear if it contains nonlinear terms involving the unknown function or its derivatives. This could include products of the function with itself, products of different derivatives, or functions of the unknown or its derivatives.*

### 3.1.2 Properties of nonlinear partial differential equations

The principle of superposition does not apply - Often exhibit complex behaviors such as shock formation, solutions, and chaos - Generally more difficult to solve analytically - May require specialized techniques or numerical methods

## 3.2 Classification by order

The order of a partial differential equation is determined by the highest derivative that appears in the equation.

### 3.2.1 First-Order partial differential equations

A first-order partial differential equation involves only first derivatives of the unknown function. The general form with $n$ independent variables is

$$F\left(x_1\,, x_2\,, \ldots\,, x_n\,, w\,, \frac{\partial w}{\partial x_1}\,, \frac{\partial w}{\partial x_2}\,, \ldots\,, \frac{\partial w}{\partial x_n}\right) = 0. \tag{3.4}$$

First-order partial differential equations are often solved using the method of characteristics, which reduces the partial differential equation to a system of ordinary differential equations.

### 3.2.2 Examples

**Transport equation:**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{3.5}$$

**Hamilton-Jacobi equation:**

$$\frac{\partial u}{\partial t} + H(x\,,\,\nabla u) = 0. \tag{3.6}$$

### 3.2.3 Examples

All the classical equations of mathematical physics (heat, wave, Laplace) - Einstein's field equations in general relativity - Schrödinger equation in quantum mechanics

## 3.3 Higher-Order partial differential equations

partial differential equations of order three or higher appear in various specialized applications.

### 3.3.1 Examples

Biharmonic equation (fourth-order):

$$\Delta^2 u = 0. \tag{3.7}$$

Used in elasticity theory - Korteweg-de Vries equation (third-order):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{3.8}$$

used in fluid dynamics - Equation of transverse vibration of elastic rod (fourth-order):

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0. \tag{3.9}$$

## 3.4 Classification of Second-Order PDEs

Second-order partial differential equations are particularly important in applications and have a special classification system based on the nature of their characteristics.

For a second-order partial differential equation in two variables of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial x} + \frac{\partial^2 u}{\partial y^2} + \text{lower order terms} = 0. \tag{3.10}$$

The classification depends on the discriminant

$$B^2 - 4AC$$

1. **Elliptic partial differential equations** ($B^2-4AC<0$): Elliptic partial differential equations typically model equilibrium or steady-state phenomena. They have no real characteristic curves.

   **Properties:** - Solutions tend to be smooth - Boundary value problems are well-posed - Information propagates in all directions - Often model steady-state or equilibrium situations

   **Examples.** Laplace equation:

$$\Delta u(x\,,\,y\,,\,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{3.11}$$

   Poisson equation:

$$\Delta u(x\,,\,y\,,\,z) = f. \tag{3.12}$$

   Helmholtz equation:

$$\Delta u + k^2 u = 0. \tag{3.13}$$

   Physical Applications: - Electrostatic potentials - Steady-state temperature distributions - Gravitational fields - Steady fluid flow around obstacles

2. **Parabolic partial differential equations** ($B^2-4AC=0$) Parabolic partial differential equations typically model diffusion processes or heat conduction. They have one repeated real characteristic direction.

   **Properties:** Initial value and boundary value problems - Information propagates with infinite speed - Smoothing effect on initial data - Often model time-dependent diffusion processes

   **Examples.** Heat equation:

$$\frac{\partial u}{\partial t} - \mu \Delta u = f. \tag{3.14}$$

   Diffusion equation:

$$\frac{\partial u}{\partial t} = D\Delta u. \tag{3.15}$$

   Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}(\mu p) + \left(\frac{1}{2}\right)\frac{\partial^2 u}{\partial x^2}(\sigma^2 p). \tag{3.16}$$

   **Physical Applications:** Heat conduction - Diffusion of substances - Brownian motion - Option pricing in finance (Black-Scholes equation)

3 **Hyperbolic partial differential equations** $(B^2-4AC>0)$: Hyperbolic partial differential equations typically model wave propagation. They have two distinct real characteristic directions.
**Properties:** - Initial value and boundary value problems - Information propagates along characteristics with finite speed - Can develop discontinuities (shocks) even from smooth initial data - Often model wave phenomena or transport with finite propagation speed.

**Examples.** - Wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \mu \Delta \phi = f.$$

- Transport equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

- Telegraph equation:

$$\frac{\partial^2 u}{\partial t^2} + 2a\frac{\partial u}{\partial t} - c^2 \delta u = 0.$$

Physical Applications: - Sound waves - Electromagnetic waves - Seismic waves - Traffic flow

### 3.4.1 Other Classification Schemes

Homogeneous and Non-homogeneous

- A partial differential equation is homogeneous if all terms contain the dependent variable or its derivatives.
- A partial differential equations is non-homogeneous if it contains terms that do not involve the dependent variable or its derivatives.

Quasi-linear partial differential equations

- A partial differential equation is quasi-linear if all the terms with the highest order derivatives of dependent variables occur linearly, but terms with lower-order derivatives can occur in any manner.

Systems of partial differential equations

- Many physical phenomena are modeled by systems of coupled partial differential equations rather than a single equation.
  Examples include:
  - Maxwell's equations in electromagnetism

- Navier-Stokes equations in fluid dynamics

- Einstein's field equations in general relativity Solution Methods Based on Classification
  The classification of a partial differential equations often determines the appropriate solution methods:

- Elliptic partial differential equations: Finite Elements compatible with LBB conditions

- Parabolic partial differential equations: Finite difference on the parabolic variable and a time loop on each elliptic subsystem; better stability with implicit time schemes

- Hyperbolic partial differential equations: Upwinding, Petrov-Galerkin, Characteristics-Galerkin, Discontinuous-Galerkin, or Finite Volumes methods When a system changes type (e.g., from subsonic to supersonic flow in aerodynamics), special care must be taken as difficulties like shock discontinuities may arise.

## 3.5 Sturm Liouville problems and eigenfunction expansion

Consider the Sturm Liouville eigenvalue problem:

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+[Q(x)+\lambda r(x)]\,y=0, \quad a\leq x\leq b.$$

$$\alpha y(a)+\beta y'(a)=0.$$

$$\gamma y(b)+\delta y'(b)=0.$$

The first equation is a linear second-order ordinary differential equation. We assume that the coefficients of this ordinary differential equation are real functions satisfying

$$P, P', Q, r \in C([a,b]),$$

$$P(x), r(x) > 0, \forall x \in [a,b]$$

We also assume that

$$\alpha,\beta,\gamma,\delta\in\mathbb{R}, \quad |\alpha|+|\beta|>0, \quad |\gamma|+|\delta|>0.$$

Under these assumptions the eigenvalue problem is called **a regular Sturm Liouville problem** . if either of the functions P or r vanishes at least at one end point. or is discontinuous there, or if the problem is defined on an infinite interval, then the Sturm Liouville problem is said to be **singular**.

**Remark.** It is always possible to transform a general linear second-order ordinary differential equation into the Sturm-Liouville form:

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=f.$$

indeed, suppose that

$$A(x)y''+B(x)y'+C(x)y=F(x).$$

Is an arbitrary linear second-order ordinary differential equations such that $A$ is a positive continuous function. We denote by $P$ the integration factor

$$P(x)=\exp\left(\int\frac{B(x)}{A(x)}dx\right).$$

Multiplying by $P(x)/A(x)$, We obtain

$$P(x)y''+P'(x)y'+\frac{P(x)}{A(x)}C(x)y$$

$$=(P(x)y')'+Q(x)y=f,$$

Where

$$Q(x)=[P(x)/A(x)]C(x)$$

, and

$$\mathrm{f(x)}=[P(x)/A(x)]F(x)$$

.

### 3.5.1 Regular, periodic and singular Sturm-Liouville problems

In the three problems listed below, the following assumptions are made. Assume that the functions $P(x)$,$P'(x)$,$Q(x)$ and $r(x)$are continuous on the interval
$a\leq x\leq b$, and that $P(x)>0$ and $r(x)>0$ on the bounded interval $a\leq x\leq b$.

**A regular Sturm-Liouville problem on** $[a,b]$

We further assume that $P(x)>0$ and $r(x)>0$ at the ends of the interval. find number for which there is a non-trivial solution to the ordinary differential equations.

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+[Q(x)+\lambda r(x)]\,y=0,\quad a\leq x\leq b$$

Regular boundary conditions

$$\begin{aligned} \alpha y(a)+\beta y'(a)=0, &\quad \alpha^2+\beta^2\neq 0 \\ \gamma y(b)+\delta y'(b)=0. &\quad \gamma^2+\delta^2\neq 0 \end{aligned}$$

**A periodic Sturm-Liouville problem on** $[a,b]$

We further assume that $P(a)=P(b)$. find number for which there is a non-trivial solution to the ordinary differential equations.

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+\left[Q(x)+\lambda r(x)\right]y=0, \quad a\leq x\leq b$$

Regular boundary conditions

$$\begin{aligned} y(a)&=y(b). \\ y'(a)&=y'(b). \end{aligned}$$

**A singular Sturm-Liouville problem on** $[a,b]$

An Sturm-Liouville problem is said to be singular if on a finite interval at least one of the regularity properties fails or on an infinite interval $(-\infty,+\infty)$, $(a,+\infty)$ and $(-\infty,b)$. Also a singular Sturm-Liouville problem is to find numbers for which there exists a non-trivial solution of the ordinary differential equation.

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+\left[Q(x)+\lambda r(x)\right]y=0, \quad a\leq x\leq b$$

With one of the following three types of boundary conditions:

- **Type 1**: $P(a)=0$ and $\gamma y(b)+\delta y'(b)=0$.
- **Type 2**: $P(b)=0$ and $\alpha y(a)+\beta y'(a)=0$.
- **Type 3**: $P(a)=P(b)=0$. Then there are no boundary conditions specified, but the solutions must be bounded functions on the interval $[a,b]$.

## 3.6 Analytical methods for solving second order partial differential equations

### 3.6.1 The method of separation of variables

The history of this method is hidden in the fog that inevitably surrounds the past; it was used by a number of mathematicians, but usually applied only to very specific problems, from about the middle of the 18th century. Certainly L'Hospital, d'Alembert, Daniel Bernoulli and Euler employed the technique, and a case can be made that L'Hospital was the first. However, it was not until J.B.J. Fourier (1768-1830) that a complete and systematic development was presented (in about 1807, for the problem of heat conduction); thereafter it became a standard procedure in the armoury of applied mathematicians, at least for certain types of partial differential equation.

### Introducing the method

we will describe the fundamental principles that underpin the method of separation of variables by considering the classical wave equation.

$$\frac{\partial^2 U}{\partial t^2} - c^2 \cdot \frac{\partial^2 U}{\partial x^2} = 0.$$

( $c > 0$,constant ) which is of Hyperbolic type.

The separable solution is written as $U(x,t) = X(x) \cdot T(t)$, for suitable functions $X$ and $T$

The wave equation then becomes

$$XT'' - c^2 X'' T = 0.$$

where the primes denote derivatives with respect to the corresponding arguments of the functions. It is convenient to divide throughout by $XT$:

$$\frac{T''}{T} - c^2 \cdot \frac{X''}{X} = 0,$$

or

$$\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T}.$$

And although this manoeuvre requires $XT \neq 0$, we can dispense with this restriction when we have seen how the method proceeds.Thus we have

$$\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} = \lambda.$$

But $x$ and $t$ are, by definition, independent variables. They are assigned arbitrarily on the appropriate domains. it is clear, therefore, that any one of the choices

$\lambda$ is a function of only $x$; $\lambda$ is a function of only $t$; $\lambda$ is a function of $x$ and $t$.

Leads to an inconsistency. The only possible choice is $\lambda$=cte, and so we obtain

$$\frac{X''}{X} = \lambda,$$

and

$$\frac{T''}{T} = c^2 \lambda.$$

The problem for $X(x)$ now becomes, with suitable boundary conditions, an eigenvalue problem, which will have appropriate solutions only if:

$$\lambda = -\omega^2 < 0,$$

$$X'' + \omega^2 X = 0,$$

and then

$$T'' + c^2\omega^2 T = 0.$$

For specific values of $\lambda$, the eigenvalues. The general solution for both $X(x)$ and $T(t)$ involve trigonometric functions. Finally we observe, from the original separation of variables, namely.

$$XT'' - c^2X''T = 0$$

and so $T'' \propto T$: the equation is separable without the need to require $XT \neq 0$. A more general solution can now be obtained, in the familiar way, by summing over all the eigenvalues of the underlying Sturm-Liouville problem (permitted because the partial differential equation is linear).

This method goes ever directly to the other two elementary, standard equations of applied mathematics/theoretical physics.

**Example.** Apply the method of separation of variables to the heat conduction (diffusion) equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}.$$

($k > 0$,cte).

**Solution.** We set $U(x,t) = X(x)T(t)$, which gives $XT' = kX''T$ and we require for separability and consistency that $X'' \propto X$: Let $X'' = \lambda X$. The Sturm-Liouville problem, with suitable boundary conditions, then requires that $\lambda = -\omega^2 < 0 \;\; X'' + \omega^2 X = 0$, which has trigonometric solutions. This leaves $T' = -k\omega^2 T$, so that

$$T(t) = Aexp(-k\omega^2 t).$$

Where $A$ is an arbitrary constant: the $x$ dependence is oscillatory, but the $t$ dependence is a decaying exponential (because $\omega \neq 0$ and is real, and $k > 0$).

**Example.** Apply the method of separation of variables to the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

**Solution.** We write $U(x,y) = X(x)Y(y)$, and so obtain $X''Y + XY'' = 0$, and then with $X'' \propto X$ (or, indeed, $Y'' \propto Y$). we have $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$.

Depending on the boundary conditions, either $X(x)$ or $Y(y)$ will be described by a Sturm-Liouville problem. If this is $X(x)$,then $\lambda = \omega^2 > 0$ and $X(x)$ is a trigonometric function, but then $Y(y)$ is a hyperbolic (exponential) function; on the other hand, if $Y(y)$ is the Sturm-Liouville problem, the roles of $X(x)$ and $Y(y)$ are reversed. However, there will always be one of this pair of function that is trigonometric and the other hyperbolic.

### 3.6.2 Heat equation

We consider a thin bar of length **L**, made of a homogeneous material (iron, copper, aluminum, concrete, glass, etc.), thermally insulated along its length and its right end. We will first consider the case of an iron bar. We seek to determine how the temperature $U$ in this bar evolves over time. At the initial time $(t = 0)$, the bar is at ambient temperature $T_a$. The left end, $x_0$, is heated to a certain temperature $T_c$ for a determined duration $t_c$.

We want to determine how the temperature is distributed in the bar over time.

#### Mathematical modeling

We assume that due to the insulation, the temperature is uniform across the cross-section of the bar.

The function $U(x,t)$ represents the temperature at point $x$ at time $t$. We also denote the final measurement time by $t_f$.

If the bar has a homogeneous thermal conductivity and there are no heat sources, $u$ satisfies:

$$\frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} = 0, \quad x \in ]0, \mathbf{L}[, \quad t > 0,$$
$$k = \frac{\kappa}{\rho c}$$

with $\kappa$ being the thermal conductivity coefficient, $\rho$ the material's density, and $c$ its specific heat capacity.

The first term $\frac{\partial U}{\partial t}$ represents the evolution of temperature over time, showing that the temperature at a given moment depends on previous temperatures. The second term $-k\frac{\partial^2 U}{\partial x^2}$ represents heat diffusion or conduction, indicating that the temperature at one point depends on neighboring points.

At the left end $x_0$, the boundary condition is a **Dirichlet condition:**

$$U(0,t) = g(t),$$
$$g(t) = (T_c - T_a)1_{]0,t_c[} + T_a.$$

At the right end $x_L$, there is a **Neumann condition:**

$$\frac{\partial U}{\partial x}(L,t) = 0, \quad t > 0,$$

The initial condition is expressed as:

$$U(x,0) = T_a, \quad \forall x \in [0,L],$$

### Problem statement

We seek a function that provides a unique solution to the following problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} - k\dfrac{\partial^2 U}{\partial x^2} & = 0, \quad x \in ]0,L[; t \in ]0,t_f[, \\ U(x,0) & = T_a, \quad \forall x \in [0,L], \\ U(0,t) & = g(t), \quad \forall t \in ]0,t_f[, \\ \dfrac{\partial U}{\partial x}(L,t) & = 0. \quad \forall t \in ]0,t_f[. \end{cases}$$

### Heat equation: homogeneous boundary conditions

Consider the following heat conduction problem in a finite interval:

$$U_t - kU_{xx} = 0, \qquad 0 < x < L, \ t > 0, \tag{3.17}$$

$$U(0,t) = U(L,t) = 0, \qquad t \geq 0, \tag{3.18}$$

$$U(x,0) = f(x). \qquad 0 \leq x \leq L, \tag{3.19}$$

where $f$ is a given initial condition, and $k$ is a positive constant. In order to make (3.18) consistent with (3.19), we assume the compatibility condition

$$f(0) = f(L) = 0.$$

The problem defined above corresponds to the evolution of the temperature $u(x,t)$ in a homogeneous one-dimensional heat conducting rod of length $L$(i.e. the rod is narrow and is laterally insulated) whose initial temperature (at time $t = 0$) is known and is such that its two ends are immersed in a zero temperature bath.

We assume that there is no internal source that heats (or cools) the system. Note that the problem (3.17) and (3.19) is an initial boundary value problem that is linear and homogeneous. Recall also that the boundary condition (3.18) is called *the Dirichlet condition.* At the end of the present section, we shall also discuss other boundary conditions. We start by looking for solutions of the partial differential equation (3.17) that satisfy the boundary conditions (3.18), and have the special form

$$U(x,t) = X(x)T(t).$$

Where $X$ and $T$ are functions of the variables $x$ and $t$, respectively. At this step we do not take into account the initial condition (3.19). Obviously, we are not interested in the zero solution $U(x,t) = 0$. Therefore, we seek functions $X$ and $T$ that do not vanish identically. Differentiate the separated solution (3.20) once with respect to $t$ and twice with respect to $x$ and substitute these derivatives into the partial differential equation. We then obtain

$$XT_t = kX_{xx}T. \tag{3.20}$$

Now, we carry out a simple but decisive step *the separation of variables step.* We move to one side of the partial differential equation all the functions that depend only on $x$ and to the other side the functions that depend only on $t$. We thus write

$$\frac{T_t}{kT} = \frac{X_{xx}}{X}. \tag{3.21}$$

Since $x$ and $t$ are independent variables, differentiating (3.21) with respect to $t$ implies that there exists a constant denoted by $\lambda$ (which is called the separation constant) such that

$$\frac{T_t}{kT} = \frac{X_{xx}}{X} = -\lambda. \tag{3.22}$$

Equation (3.22) leads to the following system of ordinary differential equation's:

$$\frac{\partial^2 X}{\partial x^2} = -\lambda X, \quad 0 < x < L, \tag{3.23}$$

$$\frac{\partial T}{\partial t} = -\lambda kT, \quad t > 0, \tag{3.24}$$

which are coupled only by the separation constant $\lambda$. The function $U$ satisfies the boundary conditions (3.18) if and only if

$$U(0,t) = X(0)T(t) = 0,$$

$$U(L,t) = X(L)T(t) = 0.$$

Since $U$ is not the trivial solution $U = 0$, it follows that

$$X(0) = X(L) = 0.$$

Therefore, the function $X$ should be a solution of the boundary value problem

$$\frac{\partial^2 X}{\partial x^2} + \lambda X = 0, \quad 0 < x < L, \tag{3.25}$$

$$X(0) = X(L) = 0. \tag{3.26}$$

Consider the system (3.25) (3.26). A nontrivial solution of this system is called an eigenfunction of the problem with an eigenvalue $\lambda$. The problem (3.25)(3.26) is called an eigenvalue problem. The boundary condition (3.26) is called (as in the partial differential equation case) the Dirichlet boundary condition.

Note that the problem (3.25) - (3.26) is not an initial boundary problem for an ODE (for which it is known that there exists a unique solution). Rather, it is a boundary value problem for an ordinary differential equation. It is not clear a priori that there exists a solution for any value of $\lambda$. On the other hand, if we can write the general solution of the ordinary differential equation for every $\lambda$, then we need only to check for which $\lambda$ there exists a solution that also satisfies the boundary conditions. Fortunately, (3.25) is quite elementary. It is a second-order linear ordinary differential equation with constant coefficients, and its general solution (which depends on $\lambda$) has the following form:

1. if $\lambda < 0$, then $X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$,
2. if $\lambda = 0$, then $X(x) = \alpha + \beta x$,
3. if $\lambda > 0$, then $X(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$,

where $\alpha$, $\beta$ are arbitrary real numbers.

We implicitly assume that $\lambda$ is real, and we do not consider the complex case (although this case can, in fact, be treated similarly). We show that the system (3.25) - (3.26) does not admit a solution with a nonreal $\lambda$. In other words, all the eigenvalues of the problem are real numbers.

**Negative eigenvalue** ($\lambda < 0$)**:** The general solution can be written in a more convenient form: instead of choosing the two exponential functions as the fundamental system of solutions, we use the basis $\sinh(\sqrt{-\lambda}x), \cosh(\sqrt{-\lambda}x)$ In this basis, the general solution for $\lambda < 0$ has the form

$$X(x) = \tilde{\alpha}\cosh(\sqrt{-\lambda}x) + \tilde{\beta}\sinh(\sqrt{-\lambda}x). \tag{3.27}$$

The function $\sinh s$ has a unique root at $s = 0$, while $\cosh s$ is a strictly positive function. Since $X(x)$ should satisfy $X(0) = 0$, it follows $\tilde{\alpha} = 0$. The second boundary condition $X(L) = 0$ implies that $\tilde{\beta} = 0$. Hence, $X(x) \equiv 0$ is the trivial solution. In other words, the system (3.25) - (3.26) does not admit a negative eigenvalue.

**Zero eigenvalue** ($\lambda = 0$)**:** We claim that $\lambda = 0$ is also not an eigenvalue. Indeed, in this case the general solution is a linear function $X(x) = \alpha + \beta x$ that (in the nontrivial case $X \neq 0$) vanishes at most at one point; thus it cannot satisfy the boundary conditions (3.26).

**Positive eigenvalue** $(\lambda > 0)$**:** The general solution for $\lambda > 0$ is

$$X(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x). \tag{3.28}$$

Substituting this solution into the boundary condition $X(0) = 0$, we obtain $\alpha = 0$. The boundary condition $X(L) = 0$ implies $\sin(\sqrt{\lambda}L) = 0$. Therefore, $\sqrt{\lambda}L = n\pi$, where $n$ a positive integer. We do not have to consider the case $n < 0$, since it corresponds to the same set of eigenvalues and eigenfunctions. Hence, $\lambda$ is an eigenvalue if and only if

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions are

$$X(x) = \sin\frac{n\pi x}{L},$$

and they are uniquely defined up to a multiplicative constant.

In conclusion, the set of all solutions of problem (3.25)-(3.26) is an infinite sequence of eigenfunctions, each associated with a positive eigenvalue. It is convenient to use the notation

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

Recall from linear algebra that an eigenvalue has *multiplicity* $m$ if the space consisting of its eigenvectors is $m$-dimensional. An eigenvalue with multiplicity 1 is called *simple*. Using the same terminology, we see that the eigenvalues $\lambda_n$ for the eigenvalue problem (3.25)-(3.26) are all simple. Let us deal now with the ordinary differential equation (3.24). The general solution has the form

$$T(t) = Be^{-k\lambda t}.$$

Substituting $\lambda_n$, we obtain

$$T_n(t) = B_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right), \quad n = 1, 2, 3, \ldots \tag{3.29}$$

From the physical point of view, it is clear that the solution of (3.24) must decay in time, hence, we must have $\lambda > 0$. Therefore, we could have guessed a priori that the problem (3.25) - (3.26) would admit only positive eigenvalues.

We have thus obtained the following sequence of separated solutions

$$\begin{aligned} u_n(x,t) =& X_n(x)T_n(t), && (3.30)\\ =& B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, \ldots && (3.31)\end{aligned}$$

The superposition principle implies that any linear combination

$$u(x,t) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}. \tag{3.32}$$

of separated solutions is also a solution of the heat equation that satisfies the Dirichlet boundary conditions.

Consider now the initial condition. Suppose it has the form

$$f(x) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right),$$

i.e., it is a linear combination of the eigenfunctions. Then a solution of the heat problem (3.17) (3.19) is given by

$$u(x,t) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Hence, we are able to solve the problem for a certain family of initial conditions. It is natural to ask at this point how to solve for more general initial conditions?

The brilliant (although not fully justified at that time) idea of Fourier was that it is possible to represent an arbitrary function $f$ that satisfies the boundary conditions (3.18) as a unique infinite linear combination of the eigenfunctions $\sin(n\pi x/L)$.

In other words, it is possible to find constants $B_n$such that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right). \tag{3.33}$$

Such a series is called a *(generalized) Fourier series* (or expansion) of the function $f$ with respect to the eigenfunctions of the problem, and $B_n, n = 1, 2, \ldots$ are called the *(generalized) Fourier coefficients* of the series.

The last ingredient that is needed for solving the problem is called the *generalized superposition principle*. We generalize the superposition principle and apply it also to an infinite series of separated solutions. We call such a series a *generalized solution* of the partial differential equation if the series is uniformly converging in every subrectangle that is contained in the domain where the solution is defined.

In our case, the generalized superposition principle implies that the formal expression

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}. \tag{3.34}$$

is a natural candidate for a generalized solution of problem (3.17) (3.19). By a formal solution we mean that if we ignore questions concerning convergence, continuity, and smoothness, and carry out term-by-term differentiations and substitutions, then we see that all the required conditions of the problem (3.17)(3.19) are satisfied.

Before proving that under certain conditions (3.34) is indeed a solution, we need to explain how to represent an arbitrary function $f$ as a Fourier series. In other words, we need a method of finding the Fourier coefficients of a given function $f$.

Surprisingly, this question can easily be answered under the assumption that the Fourier series of $f$ converges uniformly. Fix $m \in \mathbb{N}$, multiply the Fourier expansion (3.33) by the eigenfunction $\sin(m\pi x/L)$, and then integrate the equation term-by-term over $[0, L]$. We get

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x)\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{3.35}$$

It is easily checked that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L/2, & m = n. \end{cases} \tag{3.36}$$

Therefore, the Fourier coefficients are given by

$$B_m = \frac{\int_0^L \sin(m\pi x/L) f(x) dx}{\int_0^L \sin^2(m\pi x/L) dx} = \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x)\,dx, \quad m = 1, 2, \ldots. \tag{3.37}$$

In particular, it follows that the Fourier coefficients and the Fourier expansion of $f$ are uniquely determined. Therefore, (3.34) together with (3.37) provides an explicit formula for a (formal) solution of the heat problem. For a given initial condition $f$, one only has to compute the corresponding Fourier coefficients in order to obtain an explicit solution.

### Wave equation

### Introduction

We motivate the more general analysis of an important class of second order equations (in two independent variables ) by considering the equation:

$$U_{tt} - c^2 U_{xx} = 0.$$

The classical wave equation, This equation arises in many elementary studies of wave propagation, it describes the amplitude, $U(x, t)$, of a wave as it propagates in one dimension, (Because we prefer a natural choice of notation for distance $(x)$ and time $(t)$, these we have been used here, rather than the more conventional $x$ and $y$, although we shall revert to these in our analysis of the general equation), The equation contains a positive constant, $c$ which will play a significant role in the interpretation of the resulting solution

### Separation of variables for the wave equation

We now apply the method of separation of variables to solve the problem of a vibrating string without external forces and with two clamped but free ends. Let $U(x, t)$ be the amplitude of the string at the point $x$ and time $t$, and let $f$ and $g$ be the amplitude and the velocity of the string at time $t = 0$. We need to solve the problem

$$\begin{aligned} U_{tt} - c^2U_{xx} =&0, \qquad\qquad 0 < x < L, \quad t > 0 && (3.38)\\ U_x(0,t) =&U_x(L,t) = 0, \qquad t \geq 0 && (3.39)\\ U(x,0) =&f(x), \qquad\qquad 0 \leq x \leq L && (3.40)\\ U_t(x,0) =&g(x). \qquad\qquad 0 \leq x \leq L && (3.41) \end{aligned}$$

where $f$, $g$ are given functions and $c$ is a positive constant. The compatibility conditions are given by

$$f'(0) = f'(L) = g'(0) = g'(L) = 0.$$

The problem (3.38) (3.41) is a linear homogeneous initial boundary value problem. As mentioned above, the conditions (3.39) are called Neumann boundary conditions. Recall that at the first stage of the method, we compute nontrivial separated solutions of the partial differential equation (3.38), i.e. solutions of the form

$$U(x,t) = X(x)T(t). \tag{3.42}$$

that also satisfy the boundary conditions (3.39). Here, as usual, $X$,$T$ are functions of the variables $x$ and $t$ respectively. At this stage, we do not take into account the initial conditions (3.40)(3.41). Differentiating the separated solution (3.42) twice in $x$ and twice in $t$, and then substituting these derivatives into the wave equation, we infer

$$XT_{tt} = c^2X_{xx}T.$$

By separating the variables, we see that

$$\frac{T_{tt}}{c^2T} = \frac{X_{xx}}{X}. \tag{3.43}$$

It follows that there exists a constant $\lambda$ such that

$$\frac{T_{tt}}{c^2T} = \frac{X_{xx}}{X} = -\lambda. \tag{3.44}$$

Equation (3.44) implies

$$\begin{aligned} \frac{d^2X}{dx^2} =& -\lambda X, \qquad\qquad 0 < x < L && (3.45)\\ \frac{d^2T}{dt^2} =& -\lambda c^2T. \qquad\qquad t > 0 && (3.46) \end{aligned}$$

The boundary conditions (3.39) for $U$ imply

$$U_x(0,t) = \frac{dX}{dx}(0)T(t) = 0, \quad U_x(L,t) = \frac{dX}{dx}(L)T(t) = 0.$$

Since $U$ is nontrivial it follows that

$$\frac{dX}{dx}(0) = \frac{dX}{dx}(L) = 0.$$

Therefore, the function $X$ should be a solution of the eigenvalue problem

$$\frac{d^2X}{dx^2} + \lambda X = 0. \quad 0 < x < L \tag{3.47}$$

$$\frac{dX}{dx}(0) = \frac{dX}{dx}(L) = 0. \tag{3.48}$$

This eigenvalue problem is also called the Neumann problem. We have already written the general solution of the ordinary differential equation (3.47):

**1** if $\lambda < 0$, then

$$X(x) = \alpha \cosh(\sqrt{-\lambda x}) + \beta \sinh(\sqrt{-\lambda x}).$$

**2** if $\lambda = 0$, then

$$X(x) = \alpha + \beta x.$$

**3** if $\lambda > 0$, then

$$X(x) = \alpha \cos(\sqrt{\lambda x}) + \beta \sin(\sqrt{\lambda x}).$$

where $\alpha$, $\beta$ are arbitrary real numbers.

**Negative eigenvalue** $\lambda < 0$ The first boundary condition $(dX/dx)(0) = 0$ implies that $\beta = 0$. Then $(dX/dx)(L) = 0$ implies that $\sinh(\sqrt{-\lambda L} = 0)$. Therefore, $X(x) = 0$ and the eigenvalue problem (3.47)(3.48) does not admit negative eigenvalues.

**Zero eigenvalue** $\lambda = 0$ The general solution is a linear function $X(x) = \alpha + \beta x$. Substituting this solution into the boundary conditions (3.48) implies that $\lambda = 0$ is an eigenvalue with a unique eigenfunction $X_0(x) = 1$ (the eigenfunction is unique up to a multiplicative factor).

**Positive eigenvalue** $\lambda > 0$ The general solution for $\lambda > 0$ has the form

$$X(x) = \alpha \cos(\sqrt{\lambda x}) + \beta \sin(\sqrt{\lambda x}). \tag{3.49}$$

Substituting it in $(dX/dx)(0) = 0$, we obtain $\beta = 0$. The boundary condition $(dX/dx)(L) = 0$ implies now that $\sin(\sqrt{\lambda L}) = 0$ Thus $\sqrt{\lambda L} = n\pi$, where $n \in \mathbb{N}$ Consequently,$\lambda > 0$ is an eigenvalue if and only if:

$$\lambda = (\frac{n\pi}{L})^2 \quad n = 1, 2, \ldots$$

The associated eigenfunction is

$$X(x) = \cos(\frac{n\pi x}{L}).$$

and it is uniquely determined up to a multiplicative factor. Therefore, the solution of the eigenvalue problem (3.47)(3.48) is an infinite sequence of nonnegative simple eigenvalues and their associated eigenfunctions. We use the convenient notation:

$$X_n(x) = \cos(\frac{n\pi x}{L}), \quad \lambda_n = (\frac{n\pi}{L})^2 \quad n = 0, 1, 2 \dots$$

Consider now the ordinary differential equation (3.46) for $\lambda = \lambda_n$. The solutions are

$$T_0(t) = \gamma_0 + \delta_0 t, \tag{3.50}$$

$$T_n(t) = \gamma_n \cos(\sqrt{\lambda_n c^2} t) + \delta \sin(\sqrt{\lambda_n c^2} t) \quad n = 1, 2, 3 \dots \tag{3.51}$$

Thus, the product solutions of the initial boundary value problem are given by

$$U_0(x,t) = X_0(x)T_0(t) = \frac{A_0 + B_0 t}{2}. \tag{3.52}$$

$$U_n(x,t) = X_n(x)T_n(t) = \cos(\frac{n\pi x}{L})(A_n \cos(\frac{cn\pi t}{L}) + B_n \sin(\frac{cn\pi t}{L}) \quad , \quad n = 0, 1, 2 \dots \tag{3.53}$$

### 3.6.3 The Fourier transform

**Definition 3.6.1.** *The Fourier transform of the function $f(x)$ for $-\infty < x < \infty$ is given by the formula*

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx.$$

That is, we start with a function $f(x)$ defined on the real x-axis, substitute it into equation, and arrive at the new function $F(\xi)$ for $-\infty < \xi < \infty$. For example, table lists some common Fourier transforms.

| **Function** $f(x)$ | **Fourier Transform** $F(\xi)$ |
|---|---|
| $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ -e^{x}, & x < 0 \end{cases}$ | $F(\xi) = -i\sqrt{\frac{2}{\pi}} \frac{\xi}{1+\xi^2}$ |
| $f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$ | $F(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi}$ |
| $f(x) = e^{-x^2}$ | $F(\xi) = \frac{1}{\sqrt{2}} e^{-(\xi/2)^2}$ |

Table 3.1: Some Common Fourier Transforms

The reader can refer to the tables in the appendix for additional transforms. We can see from the examples that the transformed function $F(\xi)$ may or may not be a complex-valued function of $\xi$. In the first example, the transformed function $F(\xi)$ contains the complex number $i$, so we call it a **complex-valued function** of the real variable $\xi$ ($\xi$ ranges from $-\infty$ to $\infty$). In other words, the argument $\xi$ is real, but the value of the function is complex.

The usefulness of the Fourier transform (as with most integral transforms) comes from the fact that it changes the operation of differentiation into multiplication; that is, differential equations are changed into algebraic equations. There are also a host of other properties that make the Fourier transform a useful operational tool; we list a few of the more important ones.

### Properties of the Fourier transform

**1** **Property 1** (Fourier Transform Pair) The Fourier transform of $f(x)$, $-\infty < x < \infty$, produces a new function $F(\xi)$ defined by the formula

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx.$$

and the inverse Fourier transform of $F(\xi)$, $-\infty < x < \infty$ will reproduce the original function $f(x)$ according to

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi.$$

For example,

$$e^{-|x|} \xrightarrow{\mathcal{F}} \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\xi^2} \xrightarrow{\mathcal{F}^{-1}} e^{-|x|}.$$

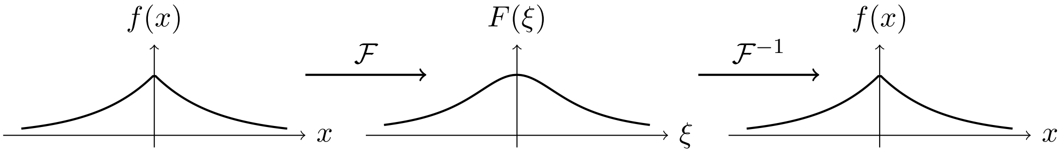


Figure 3.1: Graph of a function and its transform.

**2** **Property 2** (Linear Transformation)

The Fourier transform is a linear transformations; that is

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g].$$

For example, the Fourier transform of the expression

$$\frac{1}{x^2+1} + 3e^{-x^2}.$$

would be

$$\mathcal{F}[\frac{1}{x^2+1}] + 3\mathcal{F}[e^{-x^2}].$$

3 **Property 3** (Transformation Of Partial Derivatives)

when we discuss how derivatives transform, we must distinguish partial derivatives with respect to various variables. For instance, if the Fourier transforms the $x$-variable (the variable of integration in the transform) and if the function being transformed is a partial derivative of a function $U(x,t)$ with respect to $x$,then the rules of transformation are:

$$\mathcal{F}[U_x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_x(x,t)e^{-i\xi x}dx = i\xi\mathcal{F}[U].$$

$$\mathcal{F}[U_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_{xx}(x,t)e^{-i\xi x}dx = -\xi^2\mathcal{F}[U].$$

On the other hand, if we transform the partial derivative $U_t(x,t)$ (and if the variable of integration in the transform is $x$), then the transform is given by

$$\mathcal{F}[U_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_t(x,t)e^{-i\xi x}dx = \frac{\partial\mathcal{F}[U]}{\partial t}.$$

$$\mathcal{F}[U_{tt}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_{tt}(x,t)e^{-i\xi x}dx = \frac{\partial^2\mathcal{F}[U]}{\partial t^2}.$$

4 **Property 4** (Convolution Property)

Every integral transform has what is called a **convolution property**. The general idea is that the transform of a product of two function $f(x)g(x)$ is not the product of the individual transforms; that is,

$$\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f]\mathcal{F}[g].$$

However, in transform theory there is something called the convolution $(f * g)$ of two functions that more or less plays the role of the product. What is true about this convolution $(f * g)$ is that

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g].$$

So what is this mysterious convolution $f * g)$, it's given by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

And it can be shown without too much trouble that $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$ holds. We see from the definition of the convolution that given two functions $f(x)$ and $g(x)$, the convolution $(f * g)(x)$ is a new function.

| $f(x)$ | $F(w)$ |
|---|---|
| $e^{-ax^2}$ | $\frac{1}{\sqrt{4\pi a}}e^{-\frac{w^2}{4a}}$ |
| $\frac{\partial f}{\partial t}$ | $\frac{\partial F}{\partial t}$ |
| $\frac{\partial f}{\partial x}$ | $iwF(w)$ |
| $\frac{\partial^2 f}{\partial x^2}$ | $-w^2F(w)$ |
| $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi$ | $F(w)G(w)$ |
| $\delta(x-x_0)$<br>$f(x-\beta)$ | $\frac{1}{2\pi}e^{-iwx_0}$<br>$e^{-iw\beta}F(w)$ |
| $xf(x)$ | $i\frac{dF}{dw}$ |
| $\frac{2\alpha}{x^2+\alpha^2}$ | $e^{-\alpha\|w\|}$ |
| $\int_0^x \phi(t)dt$ | $\frac{1}{iw}F(\phi(x))$ |
| $\begin{cases} 0, & \|x\| < \alpha \\ 1, & \|x\| > \alpha \end{cases}$ | $\frac{1}{\pi}\frac{\sin(aw)}{w}$ |

Table 3.2: Table of some Fourier transforms

**Table of some Fourier transforms**

**Application**

**Example.**

$$U_{xx} + U_{yy} = 0, \quad -\infty < x < +\infty, \quad 0 < x < +\infty$$

Taking into account the boundary conditions:

$$U(x,0) = f(x).$$

Fourier transform of $x$ and boundary conditions:

$$U_{yy}(w,y) - w^2U(w,y) = 0.$$

$$U(w,0) = F(w).$$

The solution is:

$$U(w,y) = A(w)e^{wy} + B(w)e^{-wy}.$$

To ensure the boundary conditions must be:

$$A(w) = 0, \quad w > 0$$

$$B(w) = 0, \quad w < 0$$

And the solution is:

$$U(w, y) = C(w)e^{-|w|y}.$$

Using the boundary conditions:

$$C(w) = F(w).$$

We have also:

$$g(x, y) = \int_{-\infty}^{+\infty} e^{wy}.e^{iwx}dw + \int_{0}^{+\infty} e{-}wy.e^{iwx}dw.$$

$$g(x, y) = \frac{1}{y + ix} + \frac{1}{y - ix},$$

$$g(x, y) = \frac{2y}{x^2 + y^2}.$$

Therefore,

$$U(x, y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\frac{2}{(x - \xi)^2 + y^2}d\xi.$$

### 3.6.4 The method of D'Alembert

Jean-Baptiste le Rond D'Alembert, a prominent 18th-century French mathematician, introduced a significant method for solving the one-dimensional wave equation, now known as *D'Alembert's solution*, This approach provides a clear and explicit formula for understanding wave propagation in various physical contexts, such as vibrating strings and sound waves. The one-dimensional wave equation is expressed as:

$$U_{tt} = c^2U_{xx}.$$

Here $U(x, t)$ represents the displacement at position $x$ and time $t$, while $c$ denotes the wave propagation speed. D'Alembert insight was to transform this partial differential equation by introducing new variables: $\xi = x - ct$ end $\eta = x + ct$

This change of variables simplifies the wave equation, allowing it to be solved more straightforwardly. The general solution, known as *D'Alembert formula* is

$$U(x, t) = f(x + ct) + g(x - ct).$$

In this formulation, $f$ and $g$ are arbitrary functions determined by initial conditions, representing waves traveling to the right and left, respectively. This decomposition elucidates how initial disturbances evolve over time as two separate waves moving in opposite directions without changing shape. D'Alembert first presented this solution in 1747 while studying the problem of a vibrating string, marking a foundational moment in the mathematical analysis of wave phenomena. His method not only offered a practical solution technique but also enhanced the conceptual understanding of wave propagation, influencing subsequent developments in mathematical physics.

### The wave equation

We write the wave equation as

$$U_{tt} = c^2 U_{xx}. \quad -\infty < x < +\infty \tag{3.54}$$

(Physically, you can imagine a very long string.) This is the simplest second order equation. The reason is that the operator factors nicely:

$$U_{tt} - c^2 U_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) U = 0. \tag{3.55}$$

This means that, starting from a function $u(x,t)$, you compute $U_t + cU_x$, call the result $V$, then you compute $V_t - cV_x$, and you ought to get the zero function. The general solution is

$$U(x,t) = f(x+ct) + g(x-ct). \tag{3.56}$$

where $f$ and $g$ are two arbitrary(twice differentiable) functions of a single variable.

*Proof.* Because of (3.18), if we let $V = U_t + cU_x$, we must have $V_t - cV_x = 0$. Thus we have two first-order equations

$$V_t - cV_x = 0. \tag{4a}$$

and

$$U_t + cU_x = V. \tag{4b}$$

These two first-order equations are equivalent to (3.17) itself. Lets solve them one at a time. Equation (4a) has the solution $V(x,t) = h(x+ct)$ , where his any function. So we must solve the other equation, which now takes the form

$$U_t + cU_x = h(x+ct). \tag{4c}$$

for the unknown function $U(x,t)$. It is easy to check directly by differentiation that one solution is $U(x,t) = f(x+ct)$, where $f'(s) = h(s)/2c$ [A prime (') denotes the derivative of a function of one variable.] To the solution$f(x+ct)$ we can add $g(x-ct)$ to get another solution (since the equation is linear). The most general solution of (4b) in fact turns out to be a particular solution plus any solution of the homogeneous equation; that is,

$$u(x,t) = f(x+ct) + g(x-ct).$$

as asserted by the theorem. A different method to derive the solution formula (3.19) is to introduce the characteristic coordinates

$$\xi = x + ct \quad \eta = x - ct.$$

. By the chain rule, we have $\partial_x = \partial\xi + \partial\eta$ and $\partial t = c\partial\xi + c\partial\eta$. Therefore, $\partial t - c\partial x = -2c\partial\eta$and $\partial t + c\partial x = 2c\partial\xi$. So equation (3.17) takes the form

$$(\partial t - c\partial x)(\partial t + c\partial x)U = (-2c\partial\xi)(2c\partial\eta)U = 0.$$

which means that $U\xi\eta = 0$ since $c \neq 0$. The solution of this transformed equation is

$$U = f(\xi) + g(\eta).$$

The wave equation has a nice simple geometry. There are two families of characteristic lines, $x \pm ct = constant$. The most general solution is the sum of two functions. One,$g(x-ct)$,is a wave of arbitrary shape traveling to the right at speed $c$.The other,$f(x+ct)$, is another shape traveling to the left at speed $c$. ■

## Chapter 4

# Applications of Partial Differential Equations

## Introduction

Since semi-analytical computations are generally complex in numerical analysis, and in our research we are particularly concerned with finding numerical solutions for differential equations that we discussed previously in chapters three, we will therefore use programs that facilitate complex calculations and save time in studying and solving equations accurately. We will use the MATLAB program.

## 4.1 Implementation of Numerical Algorithms in MATLAB

### 4.1.1 Programming in MATLAB for Single-Step Methods

#### Euler's Method Implementation

```
clc;
clear all;
syms f x y;
h = input( Enter step size h =  );
f = input( Enter the function f(x,y) =  );
X(1) = input( Enter initial x0 =  );
Y(1) = input( Enter initial y0 =  );
xf = input( Enter final xf =  );

for i = 1:(xf-X(1))/h
X(i+1) = X(i) + h;
y = Y(i);
x = X(i);
```

```
k1 = subs(f);
Y(i+1) = Y(i) + k1*h;
plot(X,Y, bo- , markerfacecolor , r );
title( Numerical Solution of ODE );
xlabel( X axis );
ylabel( Y axis );
hold on;
shg;
pause(h);
end
disp( Final Y values: );
Y
```

**Example.** Find the solution at $x = 1$ using **Euler's method** with step size $h = 0.2$:

$$\begin{cases} y' = x - y, \\ y(0) = 1. \quad h = 0.2 \end{cases}$$

### Input data

```
step size = 0.2
the function f(x, y) = x - y
x0 = 0
y0 = 1
xf = 1

```

### Output

```
Y =

| 1.0000 | 0.8000 | 0.6800 | 0.6240 | 0.6192 | 0.6554 |

```

The computed values of $y(x)$ simplify calculations, saving time and effort. The results are visualized in the following figure. We vary the step size to examine its effect on the obtained solution. Three different values are considered: $h = 0.2$ ,$h = 0.1$ and $h = 0.05$ . The corresponding solutions are computed and plotted on the same graph, as shown in figure (4.2). It is observed that the smaller the step size , the closer the numerical solution is to the exact solution. Conversely,

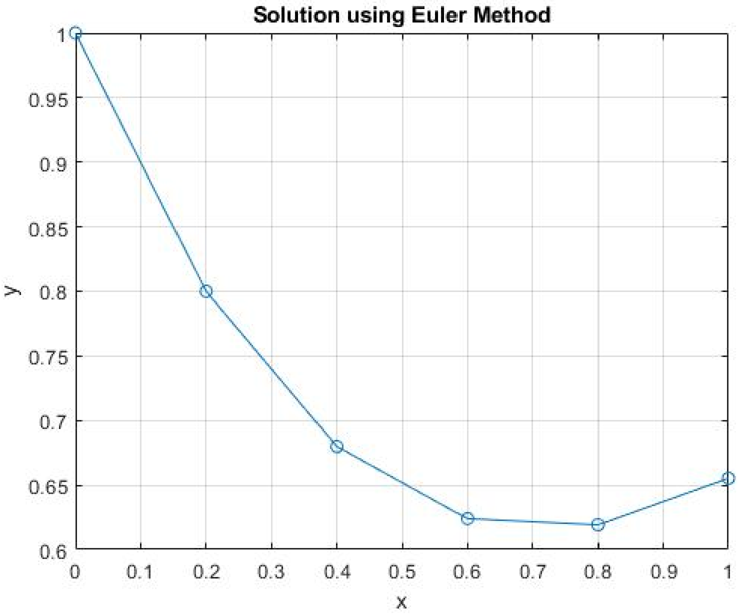


Figure 4.1: Graphical representation of the function $y$ using MATLAB

the solution for $h = 0.2$ shows the greatest deviation. This indicates that as $h$ approaches zero, the numerical solution converges to the exact solution.

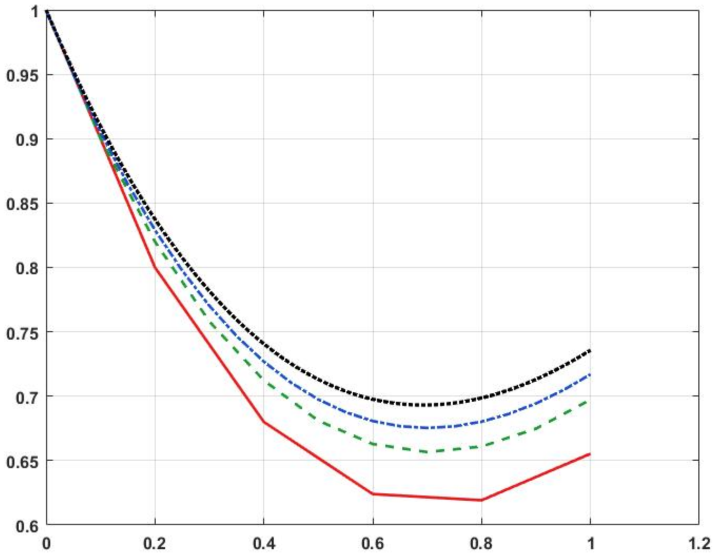


Figure 4.2: Graph showing the effect of the step size on the approximate solution

## 4.2 Solution of an Elliptic PDE

**Example.** We can consider solving Laplace's equation using the direct method while varying $h$, in order to study its effect on the solution

$$\begin{cases} \Delta u = 0 \text{ in the domain } (x,y) \in [0,20] \times [0,10] \\ u(x,0) = u(x,10) = u(0,y) = 0 \text{ and } u(20,y) = 100 \\ h_x = h_y = h \in \{5, 2.5, 1.25, 0.625, 0.3125\} \end{cases}$$

**Case where $h = 5$** We have: $h_x = \frac{b-a}{n_x} \Rightarrow n_x = \frac{b-a}{h_x} = \frac{20-0}{5} = 4$ and $n_y = \frac{d-c}{h_y} = \frac{10-0}{5} = 2$. The discretized grid contains $(n_x+1) \times (n_y+1)$ cells since we need to add the points where $x_i = 0$ and those where $y_j = 0$ ,i.e., the points where the curve intersects the axes. However, since the boundary conditions provide the values on the edges, the unknown points are only those inside the grid. Thus, the number of unknowns is $(n_x - 1) \times (n_y - 1) = 3 \times 1 = 3$ We obtain the following system of three equations with three unknowns

$$\begin{cases} -4u_{1,1} + u_{2,1} + 0u_{3,1} = 0 \\ u_{1,1} - 4u_{2,1} + u_{3,1} = 0 \\ 0u_{1,1} + u_{2,1} - 4u_{3,1} = -100 \end{cases}$$

Now, we need to solve the matrix system $A \times U = B$ where

$$A = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ -100 \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix}$$

Using one of the methods studied in Numerical Analysis II (solving linear systems), we obtain the solution

$$U = \begin{pmatrix} 1.786 \\ 7.143 \\ 26.786 \end{pmatrix}$$

**Case where $h = 2.5$** We also have

$$n_x = \frac{b-a}{h} = \frac{20-0}{2.5} = 8 \text{ and } n_y = \frac{d-c}{h} = \frac{10-0}{2.5} = 4,$$

which gives us a system of $n = (n_x - 1) \times (n_y - 1) = 7 \times 3 = 21$ equations with 21 unknowns of the form

$$\begin{cases} -4u_{1,1} + u_{2,1} + \cdots + u_{1,2} + \cdots = 0 \\ u_{1,1} - 4u_{2,1} + u_{3,1} + \cdots + u_{2,2} + \cdots = 0 \\ \vdots \\ \cdots + u_{6,1} - 4u_{7,1} + \cdots = -100 \\ u_{1,1} + \cdots - 4u_{2,1} + u_{2,2} + \cdots = 0 \\ \vdots \end{cases}$$

Using a MATLAB program, we transform the matrix $U$ into a vector $\vec{v}$ to solve the system $AU = B$ correctly, as the solution requires $U$ to be in vector form. Here is the transformation code:

```
k = 1;
for j = 1:ny-1
for i = 1:nx-1
v(k) = u(i,j);
k = k + 1;
end
end

```

This gives us the following system

$$\begin{cases} -4v_1 + v_2 + \cdots + v_8 + \cdots = 0 \\ v_1 - 4v_2 + v_3 + \cdots + v_9 + \cdots = 0 \\ \vdots \\ \cdots + v_6 - 4v_7 + \cdots = -100 \\ v_1 + \cdots - 4v_8 + v_9 + \cdots = 0 \end{cases}$$

We now need to solve the following matrix system: $A \times \vec{v} = B$, where

$$A = \begin{pmatrix} -4 & 1 & 1 & & & & & & \\ 1 & -4 & 1 & 1 & & & & & \\ 0 & 1 & -4 & 1 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & 1 & -4 & 0 & 1 & & \\ 1 & & & & 0 & -4 & 1 & & \\ 0 & \ddots & \ddots & & 1 & \ddots & \ddots & \ddots & \\ 0 & & & & 1 & & & 1 & -4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ -100 \\ \vdots \\ \vdots \\ -100 \end{pmatrix}$$

Using the Jacobi or Gauss-Seidel method could provide solutions to this system. However, as the title suggests, we will adapt the direct resolution method to solve this system. Here is the complete, explained MATLAB program that allowed us to obtain the matrix $U$ from the calculations of the elements of vector $\vec{v}$.

```
clc; clear
h = 2.5;
a = 0;
b = 20;
c = 0; d = 10;
nx = (b - a) / h;
ny = (d - c) / h;
n = (nx - 1) * (ny - 1);

%%% Filling matrix A
A = zeros(n);
for i = 1:(n - 1)
A(i, i) = -4;
A(i + 1, i) = 1;
A(i, i + 1) = 1;
if (mod(i, (nx - 1)) == 0)
A(i + 1, i) = 0;
A(i, i + 1) = 0;
end
end

for i = 1:n - nx + 1
A(nx - 1 + i, i) = 1;
A(i, nx - 1 + i) = 1;
end
A(n, n) = -4;

```

```
%%% Filling matrix B
for i = 1:n
B(i) = 0;
if (mod(i, (nx - 1)) == 0)
B(i) = -100;
end
end

%%% Solving the system Av = B and transforming the vector...
v into matrix U
V = A \ B ;
k = 1;
for j = 1:ny - 1
for i = 1:nx - 1
u(j, i) = V(k);
k = k + 1;
end
end

%%% Shifting elements to insert boundary conditions
for j = ny:-1:2
for i = nx:-1:2
u(j, i) = u((j - 1), i - 1);
end
end

for i = 1:nx
for j = 1:ny
u(1, i) = 0;
u(j, 1) = 0;
u(ny + 1, j) = 0;
u(j, nx + 1) = 100;
end
end

u(1, nx + 1) = 0;
%%% vector x and y
x=0:h:b;y=0:h:d;
mesh(x,y,u)

```

**Remark.** To evaluate the system with other values of $h$, simply replace 2.5 with the desired values and compile the program. This will generate the corresponding curve for each value of $h$.

Here is a summary of the curves obtained by varying $h$ Although this method pro-

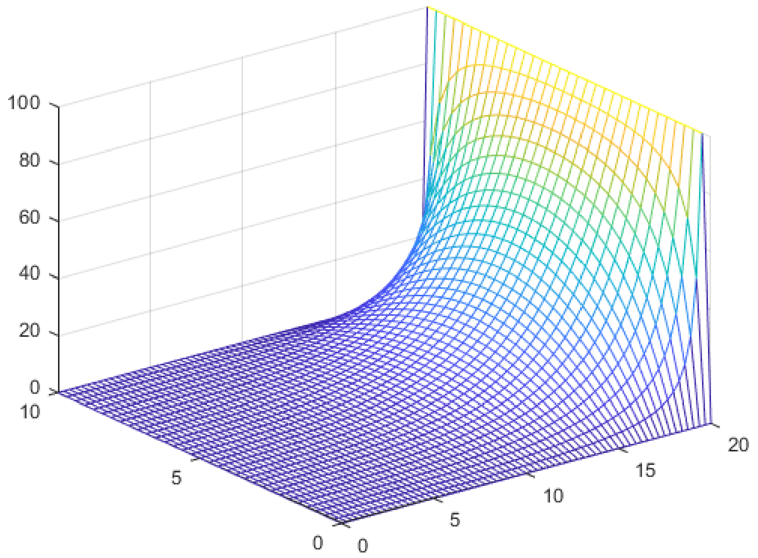


Figure 4.3: The evolution of the curve as a function of $h = 0.2125$ using the direct method

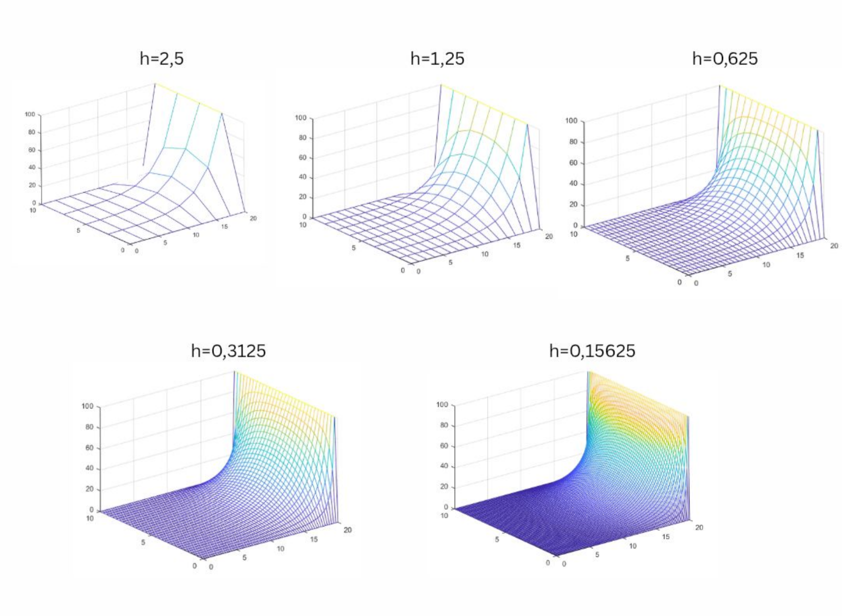


Figure 4.4: Variation of the $\Delta u = 0$ curve as a function of $h$ using the direct method

vided us with an approximate solution to the PDE $\Delta u = 0$, it was obtained by approximating $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ using a second-order truncated Taylor series. Consequently, we introduced an error on the order of $h^2$.

Not only is this error significant, but the computation of the elements of the vector $\vec{v}$ also becomes increasingly memory-intensive. For example, with $h_x = h_y = h = 1.25$, the matrix $A \in \mathfrak{M}(105)$. Solving this matrix system, where $A$ is a square matrix of size $(105, 105)$, requires approximately 14,000 bytes of memory in the machine. However, to refine the numerical solution further and bring it closer to the analytical solution, we must let $h$ tend to zero so that it approaches the limit approximated by the Taylor method. This will further increase memory demands and risks crashing the machine unless we have access to machines with enormous memory capacities (which are not accessible to everyone!).

## 4.3 Solution of an Parabolic PDE

### 4.3.1 Discretization of Space and Time

**Discretization of Space**

We need to solve our partial differential equation in the space between 0 and 2 with a step size $\Delta x$. To find the point $x_i$ from the point $x_{i-1}$, we must add $\Delta x$ to the latter, so we have

$$\Rightarrow \begin{cases} x_i = x_{i-1} + \Delta x \\ x_{i-1} = x_{i-2} + \Delta x \\ \vdots \\ x_2 = x_1 + \Delta x \\ x_1 = x_0 + \Delta x \end{cases}$$

By summing the equations obtained and after simplification, we find $x_i = x_0 + (i - 1 + 1)\Delta x = x_0 + i\Delta x$ where $x_0$ is none other than the point $a$ of the interval $[a, b]$. Therefore, finally

$$x_i = a + i\Delta x, \quad 1 \le i \le n_x$$

with $n_x = \frac{b-a}{\Delta x} = \frac{2}{\Delta x}$

**Discretization of Time**

Our equation, i.e., the solution process, must be repeated in a time interval equal to $\Delta t$ until reaching $T_{max}$. To go from $t_j$ to $t_{j+1}$, we must add $\Delta t$ to $t_j$ each time. Using a reasoning by induction as before, we derive: $t_j = j\Delta t$ because $t_0 = 0$ and corresponds to the instant $t = 0$. $t_j = j\Delta t, \quad 1 \le j \le n_t$ with $n_t = \frac{T_{max}}{\Delta t}$

### 4.3.2 Programming the Analytical Solution

Let:

$$u_{\text{exact}}(x,t) = 800 \times \sum_{n=0}^{\infty} \frac{1}{\pi^2(2n+1)^2} \times \cos\frac{\pi(2n+1)(x-1)}{2} \times e^{-0.3738(2n+1)^2 t}$$

Since we are in numerical analysis and not fundamental analysis, we must understand that we only work with points on a grid and not in the entire domain as one might think. To evaluate the numerical value of $u_{\text{exact}}$ for any point belonging to the grid, we must create a loop that, for each point $(x_i, t_j)$, calculates $u_{\text{exact}}(x_i, t_j)$. Let $\infty = 100$ be acceptable in numerical analysis. We then have the following program that calculates the elements of the matrix $v$ representing exactly the values of $u(x_i, t_j)$ on the grid.

```
clc; clear;
k = 0.13; c = 0.11; p = 7.8; dx = 0.25;
r = 1/4;
dt = dx*dx*c*p*r/k;
Tmax = 100*dt;
cla = 0; clb = 0;
a = 0; b = 2;
nx = (b-a)/dx;
nt = Tmax/dt;
x = 0:dx:b; i = 0:dt:Tmax;
v = zeros(nx+1, nt+1);
n = 0;
while (n ≤ 100)
for i = 1:nx+1
for j = 1:nt+1
u(i,j) = v(i,j) + 800 * 1/(pi^2 * (2*n + 1)^2) * ...
cos(pi * (2*n + 1) * (x(i) - 1)/2) * ...
exp(-0.3738 * (2*n + 1)^2 * t(j));
v(i,j) = u(i,j);
end
end
n = n+1;
end
mesh(t, x, v)
```

### 4.3.3 Numerical Resolution of the System

When we fix $j$ and vary $i$ from 1 to $n_x - 1$, we obtain the following linear system of $n_x - 1$ equations with $n_x - 1$ unknowns

$$\begin{cases} u_1^{j+1} = r u_2^j + (1-2r) u_1^j + r u_{i+1}^j \\ u_2^{j+1} = r u_3^j + (1-2r) u_2^j + r u_{j+1}^j \\ \vdots \\ u_{n_x-1}^{j+1} = r u_{n_x}^j + (1-2r) u_{n_x-1}^j + r u_{n_x-2}^j \end{cases}$$

$$\Longleftrightarrow \begin{cases} U^{j+1} = M \times U^j + N \\ 0 \leq j \leq n_t - 1 \end{cases}$$

Where $u_0^j$ is the boundary condition at $a$, denoted as $u_0^j = da$ in the program, and $u_{n_x}^j$ is the boundary condition at $b$, denoted as $u_{n_x}^j = clb$. Thus, we have

$$M = \begin{pmatrix} 1-2r & r & 0 & \cdots & \cdots & 0 \\ r & 1-2r & r & \ddots & \cdots & \cdots \\ 0 & r & 1-2r & r & \ddots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & r & 1-2r \end{pmatrix} \text{ and } N = \begin{pmatrix} rU_0^j \\ 0 \\ \vdots \\ \vdots \\ 0 \\ rU_{n_x}^j \end{pmatrix}$$

The matrix $M$ is a triangular or tridiagonal matrix, and the resolution is simpler to program using one of the iterative or direct methods covered in Numerical Analysis II. However, once again, we will suffice with the direct resolution using matrix division. That is, if we need to solve the system $Au = B$, we will write in Matlab $x = A \setminus B$, and the solution will be obtained automatically.

**Remark.** $B$ must be a column vector with the same number of elements as the number of rows in $A$, and the vector $I$ should be given as a column vector. It is not new to us to provide the transpose of a matrix to achieve the desired result.

We took $r = \frac{1}{4}$ for the case where $r < \frac{1}{2}$ and $r = 0.625$ for the case where $r > \frac{1}{2}$. The following program calculates the values of the vector $ii$, replaced by the vector $h$ in the program to link it with the first program calculating the elements of the matrix $v$ for the exact solution. At each time step $j$, the values are stored in a matrix $w$, initialized by the initial conditions. To obtain the curve for the second case, it suffices to replace $r$ with 0.625.

```
clc; clear;
k = 0.13; c = 0.11;
p = 7.8; dx = 0.25;
```

```
r = 1/4; dt = dx * dx * c * p * r / k;
Tmax = 100 * dt;
a = 0; b = 2;
cla = 0; clb = 0;
nx = (b - a) / dx;
nt = Tmax / dt;
x = a : dx : b; t = 0 : dt : Tmax;
for i = 1 : nx - 1
N(i) = 0;
end
N(1) = r * cla;
N(nx - 1) = r * clb;
for i = 1 : nx - 2
M(i, i) = 1 - 2 * r;
M(i, i + 1) = r;
M(i + 1, i) = r;
end
M(nx - 1, nx - 1) = 1 - 2 * r;
for i = 1 : nx + 1
if x(i) < 1
C(i) = 100 * x(i);
else
C(i) = 100 * (2 - x(i));
end
end
for i = 1 : nx - 1
h(i) = C(i + 1);
end
j = 1;
  h = h ;
  while (j < nt + 2)
  for i = 1 : nx - 1
  w(i, j) = h(i);
  end
  h = M * h + N ;
  j = j + 1;
  end
  for i = nx - 1 : -1 : 2
  for j = nt + 1 : -1 : 1
  w(i, j) = w(i - 1, j);
  end
  end
  for j = 1 : nt + 1
  w(1, j) = 0;
  w(nx + 1, j) = 0;
  end
```

```
    mesh(t, x, w);

```

Here is the curve obtained after compilation for the case

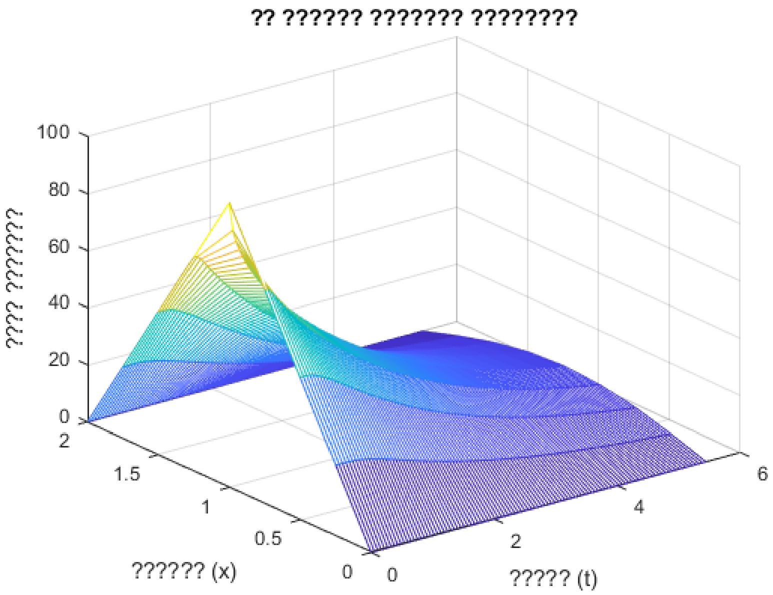


Figure 4.5: Curve of the numerical solution obtained using the explicit method for $r < \frac{1}{2}$

## 4.4 Solution of an Hyperbolic PDE

### 4.4.1 Problem Statement

We aim to solve a hyperbolic equation given by

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial x^2} + E = 0 \tag{4.1}$$

### 4.4.2 Solution by Finite Difference Method

**Simplified numerical scheme**

Let $a = c_1$ (the wave speed), $b = 0$, $c = -1$ and $e = 0$, which gives

$$\Delta = b^2 - 4ac = 0 - 4(c_1)(-1) = 4c_1 > 0 \text{ since } c_1 > 0.$$

This is indeed a hyperbolic equation which is none other than the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c_1\frac{\partial^2 u}{\partial x^2} = 0.$$

As both derivatives are second-order derivatives, we use the Taylor approximations found in equations (4.1) to establish the numerical scheme for this PDE. We therefore have

$$\frac{u_i^{t+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} = c_1 \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2}$$

$$\Rightarrow u_i^{j+1} = c_1 \frac{(\Delta t)^2}{(\Delta x)^2}(u_{i+1}^j + u_{i-1}^j) - 2\left(1 - c_1 \frac{(\Delta t)^2}{(\Delta x)^2}\right) u_i^j - u_i^{j-1}.$$

This is the numerical equation for the wave equation. To simplify the scheme, let

$$c_1 \frac{(\Delta t)^2}{(\Delta x)^2} = 1 \Rightarrow \Delta t = \frac{\Delta x}{\sqrt{c_1}}$$

$$\Rightarrow u_i^{j+1} = u_{i+1}^j + u_{i-1}^j - u_i^{j-1}. \tag{4.2}$$

**Remark.** With finite differences, equation (4.2) is the numerical scheme used to solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c_1 \frac{\partial^2 u}{\partial x^2}$. However, there is an implementation problem with this scheme since $u$ is known at $t = t_0 = 0$ which is the initial condition. But to calculate $u$ at $t = \Delta t = t_1$, we need to know the value of $u$ at $t = t_{-1} = -\Delta t$.

### Mixed conditions

Knowing the values of $u$ at $t = -\Delta t$ is no longer a problem when we have Neumann-type conditions (a condition on the propagation speed) $\frac{\partial u}{\partial t}(x, 0) = g(x)$, at $t = 0$

$$\frac{\partial u(x,0)}{\partial t} = \frac{u(x, \Delta t) - u(x, -\Delta t)}{2\Delta t} = g(x)$$

$$\Rightarrow \frac{u_i^1 - u_i^{-1}}{2\Delta t} = g(x_i) \Rightarrow u_i^{-1} = u_i^1 - 2\Delta t \times g(x_i).$$

Substituting this into equation (4.2) yields

$$u_i^1 = u_{i+1}^0 + u_{i-1}^0 - u_i^1 + 2\Delta t \times g(x_i) \Rightarrow u_i^1 = \frac{1}{2}(u_{i+1}^0 + u_{i-1}^0) + \Delta t \times g(x_i).$$

### Limitations of the finite difference method

With a condition on the speed, an approximation of some variables like $b = 0$ and $e = 0$, a simplification of the expression $c_1 \frac{\Delta t}{\Delta x}$ to 1, we were able to solve this hyperbolic equation. However, unfortunately, this is not always the case, and the availability of such a Neumann-type condition is not usual. We will therefore often be stuck in solving a hyperbolic equation if we rely solely on the finite difference method.

**Example.** Consider the following wave equation with boundary and initial conditions

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & x \in [0,2],\ t > 0 \\ u(0,t) = u(2,t) = 0, & t \geq 0 \\ \frac{\partial u}{\partial t}(x,0) = x(1-x), & 0 < x < 2 \\ u(x,0) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases} & \end{cases}$$

The exact solution is given by the Fourier series

$$u(x,t) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin\left(\frac{(2n-1)\pi x}{2}\right) \cos\left(\frac{(2n-1)\pi c t}{2}\right)}{(2n-1)^2}$$

### Numerical Implementation

The following MATLAB code implements a numerical solution

```
clear all; clc;
syms a b c e x k;
cla = 0; clb = 2;

% User inputs
k = input( Enter the value of k:  );
a = input( Enter the value of a:  );
b = input( Enter the value of b:  );
c = input( Enter the value of c:  );
e = input( Enter the value of e:  );
dx = input( Enter the discretization step:  );

% Initialization
n = (clb-cla)/dx;
N = n;
△ = b^2 - 4*a*c;
m = (b-sqrt(△))/(2*a);

d = zeros(N+1,n+1);
t = zeros(N+1,n+1);
p = zeros(N+1,n+1);
q = zeros(N+1,n+1);
y = 0:dx:2;

% Initial conditions
u = 2*x;
```

```
f = diff(u);
u = 2*(2-x);
g = diff(u);

% First iteration
for j = 1:n+1
d(1,j) = y(j);
if y(j) > 1
p(1,j) = subs(g, x, d(1,j));
u(1,j) = 2*(2-d(1,j));
q(1,j) = 0;
else
p(1,j) = subs(f, x, d(1,j));
u(1,j) = 2*d(1,j);
q(1,j) = 0;
end
end

% Boundary conditions
for i = 2:N+1
u(i,1) = 0;
q(i,1) = 0;
u(i,n+1) = 0;
q(i,n+1) = 0;
end

% Main computation loop
for i = 2:N+1
for j = 2:n
d(i,j) = (d(i-1,j-1) + d(i-1,j+1))/2;
t(i,j) = t(i-1,j-1) + m*(d(i,j)-d(i-1,j-1));

A = [a*m c ; -a*m c];
B = [a*m*p(i-1,j-1)+c*q(i-1,j-1)-e*(t(i,j)-t(i-1,j-1));
-a*m*p(i-1,j+1)+c*q(i-1,j+1)-e*(t(i,j)-t(i-1,j+1))];
V = A\B;

p(i,j) = V(1);
q(i,j) = V(2);
u(i,j) = ((p(i,j)+p(i-1,j-1))*(d(i,j)-d(i-1,j-1))/2 ...
+ ((q(i,j)+q(i-1,j-1))*(t(i,j)-t(i-1,j-1))/2 ...
+ u(i-1,j-1);
end

% Boundary treatments
for j = [1, n+1]
```

```
if j == 1
d(i,j) = 0;
t(i,j) = t(i-1,j+1) - m*(d(i,j)-d(i-1,j+1));
A = -a*m;
B = -a*m*p(i-1,j+1) + c*q(i-1,j+1) - e*(t(i,j)-t(i-1,j+1)...
);
else
d(i,j) = 2;
t(i,j) = t(i-1,j-1) + m*(d(i,j)-d(i-1,j-1));
A = a*m;
B = a*m*p(i-1,j-1) + c*q(i-1,j-1) - e*(t(i,j)-t(i-1,j-1))...
;
end
p(i,j) = A\B;
end
end

% Visualization
mesh(t, d, double(u))
xlabel( Time (t) );
ylabel( Position (x) );
zlabel( Displacement u(x,t) );
title( Numerical Solution of Wave Equation );

```

### Execution Results

Numerical solution values at selected points

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 1.0000 | 1.5000 | 2.0000 | 1.5000 | 1.0000 | 0.5000 | 0.0 |
| 0.0 | 0.5000 | 1.0000 | 1.0000 | 1.5000 | 1.5000 | 1.0000 | 0.5000 | 0.0 |
| 0.0 | 0.5000 | 0.5000 | 1.0000 | 0.5000 | 1.0000 | 1.0000 | 0.5000 | 0.0 |
| 0.0 | 0.0 | 0.5000 | 0.0 | 0.5000 | 0.0 | 0.5000 | 0.5000 | 0.0 |
| 0.0 | 0.0 | −0.5000 | 0 | −0.5000 | 0.0 | −0.5000 | 0.0 | 0.0 |
| 0.0 | −0.5000 | −0.5000 | −1.0000 | −0.5000 | −1.0000 | −0.5000 | −1.0000 | 0.0 |
| 0.0 | −0.5000 | −1.0000 | −1.0000 | −1.5000 | −1.0000 | −1.5000 | −0.5000 | 0.0 |
| 0.0 | −0.5000 | −1.0000 | −1.5000 | −1.5000 | −2.0000 | −1.0000 | −1.5000 | 0.0 |
| 0.0 | −0.5000 | −1.0000 | −1.5000 | −2.0000 | −1.5000 | −2.0000 | −0.5000 | 0.0 |

**Table 3.5:** Numerical values of the solution $u(x,t)$ at different space-time points.

# Conclusion

In conclusion, this study has demonstrated that differential equationsboth ordinary and partialare indispensable tools for describing and understanding a wide range of natural and engineering phenomena. Throughout this thesis, we have aimed to shed light on the essential theoretical foundations, while also offering insight into numerical solutions and practical applications using mathematical software, particularly MATLAB.

Despite significant advancements in the field, finding exact or closed-form solutions to many differential equationsespecially those with complex structures or non-standard boundary conditionsremains a challenging task. This limitation continues to drive researchers to develop more advanced and flexible numerical methods and approximation techniques.

This reality underscores that differential equations are not merely a classical mathematical topic, but rather an active and evolving area of research. They offer vast opportunities for exploring complex systems and call for continued efforts to bridge the gap between theory and application.

We hope that this thesis represents a modest contribution to this ongoing journey and serves as a source of inspiration for those interested in further exploring this rich and fascinating field.

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# ملخص

تقدم هذه المذكرة، التي تحمل عنوان **"المعادلات التفاضلية: دراسة نظرية وعددية"**، نظرة شاملة على المفاهيم الأساسية المتعلقة بالمعادلات التفاضلية العادية والجزئية. تبدأ بدراسة المعادلات التفاضلية العادية، ثم تنتقل إلى المعادلات التفاضلية الجزئية الخطية من الرتبة الأولى، مع التركيز على الجوانب النظرية وطرق الحل الأساسية. كما تتناول تصنيف المعادلات التفاضلية الجزئية من الرتبة الثانية إلى أنواع إهليجية وقطع زائد وقطع مكافئ، مما يساعد على فهم سلوكها. وتُختتم بعرض أهم التطبيقات الفيزيائية لهذه المعادلات، مع الإشارة إلى أهمية الطرق العددية في حال تعذر الحصول على حلول تحليلية.

# Résumé

Ce mémoire, intitulé **"Équations différentielles : étude théorique et numérique"**, présente une vue d'ensemble des concepts fondamentaux liés aux équations différentielles ordinaires et aux équations aux dérivées partielles. Il commence par l’étude des équations différentielles ordinaires, puis aborde les équations aux dérivées partielles linéaires du premier ordre, en mettant l’accent sur les bases théoriques et les méthodes de résolution. Le travail inclut également la classification des équations aux dérivées partielles du second ordre en types elliptiques, paraboliques et hyperboliques, essentielle pour comprendre leur comportement. Enfin, il met en évidence les principales applications physiques de ces équations, tout en soulignant l’importance des méthodes numériques lorsque les solutions analytiques sont inaccessibles.

# Abstract

This memory, entitled **"Differential Equations : Theoretical and Numerical Study"**, provides a general overview of the fundamental concepts related to both ordinary and partial differential equations. It begins with the study of ordinary differential equations and progresses to linear first-order partial differential equations, emphasizing theoretical foundations and basic solution methods. The work also includes a classification of second-order PDEs, distinguishing between elliptic, parabolic, and hyperbolic types, which is essential for understanding their behavior. Finally, the thesis highlights key applications of partial differential equations in various physical contexts and touches on the importance of numerical methods in cases where analytical solutions are difficult or impossible to obtain.