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A numerical study of coupled two-dimensional time fractional Burgers equation

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Bezziou Hind & Boulahia Yousra.✍

Dedication

First and foremost, I thank Allah Almighty for his guidance and countless blessings.

I would also thank me, for my patience and all the effort I put in to reach my goal.

I dedicate this work to my beloved parents, who were the main reason behind my success.

To my sisters Aya and Nesrine, and my brother Yahia.

To my small but precious family.

To my dear friends and faithful companions along the way.

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To everyone who stood by me on this journey, with a prayer, a piece of advice, or even just a kind word.

To all my loved ones, both present and absent,

this work is for you.

Bezziou Hind.✍

إهداء

بِسْمِ اللَـهِ الرَحمَـن الرَحيم ، وَآخِرُ دَعْواهُمْ أَنِ الْحَمْدُ لِلَـهِ رَبِ الْعَالَمِينَ.

لَمْ تَكُنِ الرِحلَةُ قَصِيرَةً، وَلَا الطَرِيقُ مَحْفُورَةً بِالتَسهِيلَاتِ، وَلَكِنَنِي فَعَلْتُهَا. فَالْحَمْدُ لِلَـهِ الَذي يَسَر الْبِدَايَاتِ، وَأَكْمَلَ النِهايَاتِ، وَبَلَغنَا الْغَايَاتِ. الْحَمْدُ لِلَـهِ الَذي بِنِعْمَتِهِ تَتِمُ الصَالِحَاتُ.

وَبِكُلِ حُبٍ أُهْدِي ثَمرَةَ نَجَاحِي:

إِلَى نَفْسِي الطَموحَةِ أَوَلا: ابْتَدَأَت بِطُمُوحٍ، وَانْتَهَت بِنَجَاحٍ.

إِلَى الَذي زَيَن اسْمِي بِأَجْمَلِ الْأَلْقَابِ: مَنْ دَعَمَنِي بِحُبٍ لَا يَعْرِفُ الْحُدُود، وَأَعْطَانِي بِلا مُقَابِلٍ. إِلَى مَنْ عَلَمني أَنَ الْحَيَاةَ كِفَاحٌ، وَسِلَاحُهَا الْعِلْمُ وَالْمَعْرِفَة، دَاعِمِي الْأَوَل فِي مَسِيرَتِي، وَسَنْدِي وَقُوَتي وَمَلَاذِي بَعْدَ اللَـه، فَخْرِي وَاعْتِزَازِي... أَبِي.

إِلَى مَنْ جعَلَ اللَـهُ الْجَنَة تَحْتَ أَقْدَامِهَا: وَاحْتَضَنَنِي قَلْبُهَا قَبْلَ يَدَيْهَا، وَسَهَلتْ لِيَ الشَدائِدَ بِدُعَائِهَا. إِلَى الْقَلْبِ الْحَنُونِ وَ الشَمعَةِ الَتي أَضَاءَتْ لِي فِي اللَيالِي الْمُظْلِمَات، سِرُ قُوَتِي وَنَجَاحِي، جَنَتِي... أُمِيْ.

إِلَى مَنْ سَانَدُونِي بِكُلِ حُبٍ وقت ضَعْفِي: زَارعين الثِقةَ وَالْإِصْرَارَ بداخلي، السَنْد وَالْكَتِف الَذي اتكأْ عَلَيْهِ دَائِمًا... عَمِيْ وَأَخِي.

إِلَى أَصْدِقَاءِ الْعُمْر: الَذينَ غَمَرُونِي بِالْحُبِ وَالتَوجِيهِ، وَأَمَدُوْنِي بِالْقُوَة فِي كُلِ عَثْرَةٍ، وَرَزَقَنِيَ اللَـهُ بِهِمْ لِأَعْرِفَ مَعْنَى الْحَيَاة.

إِلَى جَمِيعِ مَنْ آمَنَ بِي وَدَعَمَنِي: فِي اللَحظَاتِ الصَعبَةِ، حَتَىْ بَلَغْتُ مَا أَنَا عَلَيْهِ الْيَوْمَ.

وَأَخِيرًا: مَنْ قَالَ أنَا لَهَا نَالَهَا، وَأَنَاْ لَهَا إِنِ أبْتَ رَغْمًا عَنْهَا أَتَيْتُ بِهَا... مَا كُنْتُ لِأَفْعَلَ ذَلِكَ لَوْلَا تَوْفِيقُ اللَه.

هَا هُوَ الْيَوْمُ الْعَظِيمُ... هَذَا هُوَ الْيَوْمُ الَذي جَاهَدْتُ فِيهِ سِنِينَ الدِراسَةِ الشَاْقَة، حَتَىْ أَتَتْ ثِمَارُهَا بِفَضْلِ اللَهِ وَكَرَمِهِ. الْحَمْدُ لِلَهِ الَذي مَا تَيَقَنتُ بِهِ خَيْرًا إِلَاْ وَأَغْرَقَنِي فَرَحًا يُذْهِبُ التَعبَ.

الْحَمْدُ لِلَهِ رَبِ الْعَالَمِينَ.

بو لحية يسرى.

ÉTUDE NUMÉRIQUE DES ÉQUATIONS COUPLÉES DE BURGERS À DÉRIVÉE FRACTIONNAIRE EN TEMPS EN D2.

Résumé

Cette mémoire propose l'étude numérique d'un système couplé d'équations de Burgers aux dérivées conformes fractionnaires en dimension 2, qui est considéré comme difficile à résoudre en raison du chevauchement de la dérivée fractionnaire du temps et des termes non linéaires dans ce système.

Donc tout d'abord, nous avons appliqué la transformation de Cole-Hopf pour se ramener à une équation linéaire de type équation de chaleur à dérivées conformes fractionnaires en dimension 2. Ensuite, nous avons adopté les méthodes explicite et implicite pour calculer la solution numérique de cette équation linéaire. Enfin, nous pouvons alors obtenir une solution numérique des équations de Burgers aux dérivées conformes fractionnaires via la combinaison de la solution de l'équation de la chaleur D2 et la transformation inverse de Cole-Hopf.

Cette étude a démontré l'efficacité de cette méthodologie pour trouver des solutions numériques à de tels problèmes non linéaires de nature fractionnaire.

Mots Clés: Système couplé, équation de Burgers, transformée de Cole-Hopf, équation de la chaleur, dérivée fractionnaire du temps.

A NUMERICAL STUDY OF COUPLED TWO-DIMENSIONAL TIME FRACTIONAL BURGER'S EQUATION.

Abstract

This thesis presents a numerical study of a coupled system of Burgers equations with conformable fractional derivatives in two dimensions, which is considered difficult to solve due to the overlap between the time fractional derivative and the nonlinear terms in the system.

First, we applied the Cole-Hopf transformation to reduce the system to a linear equation of the heat equation type with conformable fractional derivatives in two dimensions. Then, we adopted both explicit and implicit methods to compute the numerical solution of this linear equation. Finally, we obtained a numerical solution of the Burgers equations with conformable fractional derivatives through the combination of the solution of the 2D heat equation and the inverse Cole-Hopf transformation.

This study demonstrated the effectiveness of this methodology in finding numerical solutions to such nonlinear problems of fractional nature.

Keywords: Coupled system, Burgers equation, Cole-Hopf Transformation, Heat equation, fractional time derivative.

دراسة عددية لنظام مزدوج من معادلات بورغر ثنائية البُعد ذات مشتقات كسرية بالنسبة للزمن.

ملخص

تهدف هذه المذكرة إلى دراسة الحل العددي لنظام مقترن من معادلات بورجرز ثنائية البعد الذي يحتوي على مشتقة زمنية كسرية من النوع القابل للتطابق، والذي يعتبر صعبا للحل نظرًا لتداخل كل من المشتقة الزمنية الكسرية والحدود غير الخطية في هذا النظام.

لذلك قمنا أولا، بتطبيق تحويل كول هوبف بتبسيط النظام إلى معادلة حرارة ذات البعد الثاني بصيغة مشتقة زمنية قابلة للتطابق، ثم قمنا باعتماد كل من الطريقتين الصريحة والضمنية لحساب الحل العددي لهذه المعادلة الخطية. وأخيرا قمنا بتطبيق التحويل العكسي لكولهوبف باستخدام الحل العددي الناتج من معادلة الحرارة لإيجاد الحل العددي للنظام الأصلي.

وقد بينت هذه الدراسة فعالية هذه المنهجية في إيجاد الحلول العددية لمثل هذه المسائل الغير الخطية ذات الطابع الكسري.

الكلمات المفتاحية: النظام المقترن، معادلة برجرز، تحويل كول هوبف، معادلة الحرارة، المشتقة الزمنية الكسرية.

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NOTATIONS

$\mathbb{R}^n$: The real vector space of dimension n with $n \geq 2$.

Ω : Open subset of order $\mathbb{R}^n$.

$\partial\Omega$: The boundary of Ω.

$M_n(\mathbb{R})$: The set of square matrices.

For $V = (v_1, v_2, ..., v_n)^t \in \mathbb{R}^n$, we define the norms $||.||_1, ||.||_2, ||.||_\infty$:

$$||V||_1 = \sum_{i=1}^{n} |v_i| \,.$$

$$||V||_2 = \left(\sum_{i=1}^{n} v_i^2 \right)^{\frac{1}{2}} .$$

$$||V||_\infty = \max_{1 \leq i \leq n} |v_i|,$$

and for a matrix $\mathcal{A} \in M_n(\mathbb{R})$ with coefficients (a_{ij}), we define:

$$||\mathcal{A}||_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |a_{ij}| \right) .$$

$$||\mathcal{A}||_2 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} .$$

$$||\mathcal{A}||_\infty = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |a_{ij}| \right).$$

Eq : Equation.

PDE: Partial differential equation.

ODE: Ordinary differential equation.

FDM: Finite difference method.

F.T: Fourier transform.

F.T^{-1}: The inverse Fourier transform.

2D: two dimensions.

General Introduction

GENERAL INTRODUCTION

PARTIAL differential equations are fundamental mathematical tools used in modeling and analyzing the behavior of complex physical and engineering systems. These equations are primarily employed to represent spatial and temporal changes within such systems, providing an accurate approach to understanding complex natural processes. Among these equations, the nonlinear Burgers' equation stands out as a simple yet powerful model, combining the effects of nonlinear convection and viscous diffusion. It is widely used to represent several important physical phenomena such as fluid flow, wave propagation, gas dynamics, and heat transfer.

The earliest form of the Burgers' equation appeared in the scientific literature in 1918 through the work of Bateman, but it gained significant recognition and academic adoption primarily through the work of the Dutch physicist Jan Martinus Burgers between 1939 and 1965. Burgers used this equation as a mathematical model to represent complex phenomena in various scientific fields such as number theory, gas dynamics, and elasticity theory. He observed similarities between this equation and the Navier-Stokes equations, owing to the presence of higher-order terms multiplied by small nonlinear coefficients, which gave the equation a realistic physical character.

Despite its relative simplicity, finding direct analytical solutions to the Burgers' equation posed a major challenge due to its nonlinear nature. However, in 1950, Julian David Cole and Eberhard Hopf independently proposed an innovative solution known as the Cole-Hopf transformation, which allowed the nonlinear equation to be converted into a linear heat equation. This transformation represented a breakthrough in the analysis of nonlinear equations and opened the door to obtaining precise solutions using traditional mathematical tools such as the Fourier transform.

As scientific advancements continued and the need for more accurate models to explain unconventional physical phenomena grew, fractional derivatives emerged as a powerful tool for modeling processes characterized by nonlinearity, which could not be adequately represented by classical models. Consequently, the Burgers' equation was generalized by replacing classical derivatives with fractional derivatives, leading to the formulation of the fractional Burgers' equation, which represents an important extension in the study of nonlinear systems.

In recent years, fractional calculus has attracted significant attention from scientists as an essential field in applied mathematics. Various fractional derivative definitions, such as the Riemann-Liouville definition, the Caputo definition, have been introduced and extensively studied. In this context, a new definition known as the Conformable Fractional Derivative was proposed by R. Khalil and his collaborators, which has led to numerous studies exploring this definition.

This thesis aims to explore analytical and numerical solutions of coupled two-dimensional conformable fractional Burgers equations. The objective of this study is to verify the applicability of the Cole-Hopf transformation to this type of conformable fractional problem, reducing it to a linear conformable heat equation. To solidify this study, numerical experiments will be proposed to test the effectiveness of this approach.

This thesis is structured into three main chapters, in addition to an introduction, a conclusion, and appendices.
Chapter one provides the foundational concepts in numerical analysis, serving as a theoretical basis for the later chapters.
Chapter two investigates a two-dimensional fractional Burgers' equation system, detailing its transformation into a heat equation via the Cole–Hopf transformation, along with an analytical study of the resulting system.
Chapter three presents a numerical investigation of a coupled system of two-dimensional partial differential equations involving time-fractional derivatives.

Chapter 1

Preliminaries

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce the fundamental concepts necessary for understanding and developing the numerical methods that will be employed throughout this thesis. It begins by covering key definitions and concepts related to partial differential equations (PDEs). Next, we discuss their classification based on structural properties. This classification helps in understanding the different types of PDEs. Then we present essential mathematical tools and transformations, including the Finite Difference Method, the Cole–Hopf Transformation and the Fourier Transform.
Additionally, a brief discussion on conformable fractional derivatives is included.

1.1 Definitions

Definition 1.1. (PDE)
In mathematics, particularly in differential calculus, a partial differential equation (also known as a PDE) is a differential equation whose solutions involve unknown functions based on several variables that satisfy certain conditions related to their partial derivatives. It is a mathematical equation that, in addition to the dependent variable (u in the following cases), involves independent variables (x, y, ...) belonging to $\mathbb{R}^n$ and one or more partial derivatives and can be written in the form:

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \ldots\right) = 0. \tag{1.1}$$

Definition 1.2. (Order of a PDE)
The order of a PDE is the highest derivative that appears in the equation.

Definition 1.3. (Linear and non linear PDE)
A PDE is said to be linear if it can be written in the form $Lu = f$, where $u \to Lu$ is a linear operator with respect to u. Otherwise, it is nonlinear.

For examples:

- **The Burgers' Equation:**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = r\frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

 where,

 - $u = u(x,t)$ is the unknown function of space x and time t.
 - $\frac{\partial u}{\partial t}$ represents the time derivative of u.
 - $u\frac{\partial u}{\partial x}$ is the convective term, describing the transport of u.
 - $\frac{\partial^2 u}{\partial x^2}$ is the spatial second derivative of u, representing diffusion.
 - r is a constant, often associated with the diffusion coefficient.

 This is a nonlinear partial differential equation because of the $u\frac{\partial u}{\partial x}$ term.

- **The Heat Equation:**

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}, \tag{1.3}$$

 where,

 - $u = u(x,t)$ is the temperature or the unknown function of space x and time t.
 - $\frac{\partial u}{\partial t}$ is the rate of change of temperature with respect to time.
 - $\frac{\partial^2 u}{\partial x^2}$ is the spatial second derivative, representing thermal diffusion.
 - α is the thermal diffusivity, a positive constant.

 This is a linear partial differential equation because there are no nonlinear terms in the equation.

1.2 Classification of PDEs

We have seen that there are very different types of partial differential equations, which can be classified as follows:

Elliptic, Parabolic and Hyperbolic PDEs.

Considering the general form of a second-order PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \tag{1.4}$$

Where, $u = u(x,y)$ is the unknown function A, B, C, D, E, F and G are assumed to be constants for simplicity. The type of PDE depends on the sign of $B^2 - 4AC$:

- If $B^2 - 4AC < 0$, the PDE is called: **Elliptic**.
- If $B^2 - 4AC = 0$, the PDE is called: **Parabolic**.
- If $B^2 - 4AC > 0$, the PDE is called: **Hyperbolic**.

Examples:

- Let consider the Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.5}$$

  We have, $B^2 - 4AC = -4 < 0$, so this equation is **elliptic**.

- The diffusion equation of the form:

$$\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} = 0, \text{ with } d > 0, \tag{1.6}$$

  is **parabolic** because $B^2 - 4AC = 0$.

- The wave equation of the form:

$$\frac{\partial^2 u}{\partial y^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \text{ with } c > 0, \tag{1.7}$$

  is **hyperbolic**, because $B^2 - 4AC = 4c^2 > 0$.

1.3 Finite Difference Method

1.3.1 Definition

The finite difference method is a common technique for finding approximate solutions to partial differential equations. It involves solving a system of relations (a numerical scheme) that links the values of the unknown functions at certain points that are sufficiently close to one another. This method is considered the simplest to implement because it proceeds in two steps: First, the discretization of the differential operators using finite differences and second, the convergence of the resulting numerical scheme as the distance between the points decreases.

1.3.2 Principle of the method

The principle of the method consists in replacing the derivatives appearing in partial differential equations by divided differences, which are obtained by performing Taylor series expansions or by combining pointwise values of the function at a finite number of discrete points or mesh nodes. This method is attributed to the work of several 18th-century mathematicians (Euler, Taylor, Leibniz,...). This technique makes it possible to develop schemes to replace the derivatives in PDEs, thereby enabling the consideration of a numerical solution. Let $u(x,t)$ be a function of space and time. According to the definition of a derivative, we have:

$$\frac{\partial u}{\partial x}(x,t) = \lim_{h\to 0} \frac{u(x+h,t)-u(x,t)}{h}. \tag{1.8}$$

$$\frac{\partial u}{\partial t}(x,t) = \lim_{k\to 0} \frac{u(x,t+k)-u(x,t)}{k}. \tag{1.9}$$

If h and k are small space and time steps, then the Taylor expansion of u around x is:

$$u(x+h,t) = u(x,t) + h\frac{\partial u}{\partial x}(x,t) + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3}(x,t) + \dots + \frac{h^n}{n!}\frac{\partial^n u}{\partial x^n}(x,t) + \mathcal{O}(h^{n+1}). \tag{1.10}$$

$$u(x,t+k) = u(x,t) + k\frac{\partial u}{\partial t}(x,t) + \frac{k^2}{2!}\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{k^3}{3!}\frac{\partial^3 u}{\partial t^3}(x,t) + \cdots + \frac{k^n}{n!}\frac{\partial^n u}{\partial t^n}(x,t) + \mathcal{O}(k^{n+1}). \tag{1.11}$$

The Taylor series expansion around time t is written as:

$$u(x,t+k) = u(x,t) + k\frac{\partial u}{\partial t}(x,t) + \frac{k^2}{2!}\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{k^3}{3!}\frac{\partial^3 u}{\partial t^3}(x,t) + \cdots + \frac{k^n}{n!}\frac{\partial^n u}{\partial t^n}(x,t) + \mathcal{O}(k^{n+1}). \quad (1.12)$$

$$u(x,t-k) = u(x,t) - k\frac{\partial u}{\partial t}(x,t) + \frac{k^2}{2!}\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{k^3}{3!}\frac{\partial^3 u}{\partial t^3}(x,t) + \cdots + (-1)^n\frac{k^n}{n!}\frac{\partial^n u}{\partial t^n}(x,t) + \mathcal{O}(k^{n+1}). \quad (1.13)$$

The terms $\mathcal{O}(h^{n+1})$ and $\mathcal{O}(k^{n+1})$ are called the remainder term or truncation error. Using these expansions, we can derive expressions for the first and second derivatives. By truncating the series at first order in h, we obtain:

$$\frac{u(x+h,t) - u(x,t)}{h} = \frac{\partial u}{\partial x}(x,t) + \mathcal{O}(h). \quad (1.14)$$

He approximation of the derivative $\frac{\partial u}{\partial x}(x)$ is of order 1, which shows that the truncation error $\theta(h)$ approaches zero, as it is considered proportional to the first power of h. The power of h with which the truncation error tends to zero is called the **order of the method**.

1.3.3 Finite Difference Schemes

The principle of finite difference schemes is based on Taylor's formula, allowing the derivatives of the unknown function to approximate "discrete derivatives".

We denote u_i as the discrete value of $u(x)$ at the point x_i, that is $u_i = u(x_i)$. Similarly, for the derivative of $u(x)$ at the node x_i, we denote:

$$\left(\frac{\partial u}{\partial x}\right)_{x=x_i} = \left(\frac{\partial u}{\partial x}\right)_i = u_i'. \quad (1.15)$$

This notation is used equivalently for all higher-order derivatives of the quantity u. The first-order finite difference scheme mentioned above is written, in index notation as:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_i}{h} + \theta(h). \quad (1.16)$$

This scheme is called forward or offset. We can also create another first-order scheme called backward and we write:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i - u_{i-1}}{h} + \theta(h). \tag{1.17}$$

We can also create another first-order scheme called centered and we write:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{h} + \theta(h). \tag{1.18}$$

1.3.4 Higher-Order Finite Difference Schemes

By manipulating Taylor expansions in the neighborhood of x_i, higher-order finite difference schemes can be constructed, we have:

$$u_{i+1} = u(x_i + h) = u_i + h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \mathcal{O}(h^3). \tag{1.19}$$

$$u_{i-1} = u(x_i - h) = u_i - h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \mathcal{O}(h^3). \tag{1.20}$$

By subtraction, we find:

$$u_{i+1} - u_{i-1} = 2h\left(\frac{\partial u}{\partial x}\right)_i + \mathcal{O}(h^3). \tag{1.21}$$

Hence:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2h} + \mathcal{O}(h^2). \tag{1.22}$$

This is called a second-order centered scheme.

It is necessary to use several adjacent nodes to x_i to achieve higher orders. The number of points required to write the scheme is called the stencil. For example, a third-order finite difference scheme for the first derivative is written as:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{-u_{i+2} + 6u_{i+1} - 3u_i - 2u_{i-1}}{6h} + \mathcal{O}(h^3). \tag{1.23}$$

1.3.5 Higher-Order Derivatives

The principle is identical and relies on Taylor expansions around x_i. For example, to construct an approximation scheme for the second derivative of u, we write:

$$u_{i+1} = u_i + h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{h^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \mathcal{O}(h^4). \tag{1.24}$$

Similarly, the expansion for u_{i-1} is:

$$u_{i-1} = u_i - h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{h^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \mathcal{O}(h^4). \tag{1.25}$$

By adding these two equations, we obtain:

$$u_{i+1} + u_{i-1} - 2u_i = h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \mathcal{O}(h^4). \tag{1.26}$$

This leads to the second-order centered scheme for approximating the second derivative of u:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mathcal{O}(h^2). \tag{1.27}$$

There are also forward and backward formulations for the second derivative, both of order 1:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+2} - 2u_{i+1} + u_i}{h^2} + \mathcal{O}(h), \quad \text{(forward)}. \tag{1.28}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_i - 2u_{i-1} + u_{i-2}}{h^2} + \mathcal{O}(h), \quad \text{(backward)}. \tag{1.29}$$

It is also possible to construct, using the same procedure, higher-order finite difference schemes for second, third, and higher derivatives.

1.4 Cole-Hopf Transformation

1.4.1 Definition

The Cole–Hopf transformation is a mathematical technique used to convert a special class of parabolic partial differential equations (PDEs), particularly those with nonlinear terms, into a linear heat equation. This transformation simplifies the analysis and solution of nonlinear PDEs by reducing them to a linear form.

1.4.2 Principle of the technique

The method uses a change of variables designed to eliminate the nonlinear terms from the original equation. Specifically, the Cole–Hopf transformation introduces a new dependent variable, typically by expressing the original unknown function as the logarithmic derivative of another function. This substitution converts the nonlinear PDE into a linear heat equation, which is well understood and significantly easier to solve.
Once the equation is linearized, the general solution can be obtained using the fundamental solution of the heat equation, followed by inverting the transformation to recover the original variable.

In the following chapter, we will examine this technique in detail, explore its applications, and demonstrate how it can be used.

1.5 Fourier Transform

The Fourier transform is a powerful mathematical tool used to analyze functions by transforming them from the spatial or temporal domain into the frequency domain. Its importance lies in its ability to simplify the treatment of differential equations by converting derivatives into algebraic operations in the frequency space. This significantly facilitates the solution process, especially for linear systems or equations with constant coefficients. The Fourier transform is widely used in mathematical physics, wave propagation, heat transfer and signal processing.

Direct Fourier Transform

Let $\phi(x, y, t)$ be a well-behaved function (integrable in the suitable sense). The two-dimensional Fourier transform with respect to the spatial variables x and y is defined as:

$$\mathcal{F}(\phi(x,y,t)) = \hat{\phi}(k_x,k_y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi(x,y,t)\, e^{-2\pi i(k_x x+k_y y)}\, dx\, dy. \tag{1.30}$$

where, $\hat{\phi}(k_x,k_y,t)$ represents the spectral (frequency-domain) representation of the original function.

Inverse Fourier Transform

To recover the original function from its frequency representation, the inverse Fourier transform is used:

$$\phi(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{\phi}(k_x,k_y,t)\, e^{2\pi i(k_x x+k_y y)}\, dk_x\, dk_y. \tag{1.31}$$

This inverse transform is essential in practical applications, as it allows one to return from the frequency domain to the original spatial or temporal domain.

1.6 Basic notions on fractional conformable derevatives

Let us introduce a definition and some important properties of fractional conformable derivatives that will be used throughout the rest of the problem.

Definition 1.4. ([6, 13])
Given a function $f : [0 : \infty) \longrightarrow \mathbb{R}$, then the conformable fractional derivative of f with order α is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1.32}$$

for all $t > 0$, $\alpha \in (0.1)$.
If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exist, then define:

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t). \tag{1.33}$$

The properties of this definition are summarized in the theorem below.

Theorem 1.1. *([6, 13])*
Let $0 < \alpha \leq 1$, f and g be α-differentiable at a point $t > 0$. then,

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha(f/g) = (gT_\alpha(f) - fT_\alpha(g))/g^2$.
6. *In addition if f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}df/dt$.

CHAPITRE 2

ANALYTICAL SOLUTION OF COUPLED TWO DIMENSIONAL FRACTIONAL BURGERS EQUATION

CHAPTER 2

ANALYTICAL SOLUTION OF COUPLED TWO DIMENSIONAL FRACTIONAL BURGERS EQUATION

IN this chapter, we will attempt to establish an analytical solution for a coupled system of fractional conformable derivative equations. To do this, we focus on the transformation of a coupled system of fractional Burgers equations into a heat equation with fractional conformable derivatives using the Cole-Hopf transformation. We can then obtain an analytical solution to the Burgers equations with fractional conformable derivatives through a combination of the solution of the $2D$ heat equation and the inverse Cole-Hopf transformation.
We schematize this approach through the figure 2.1.

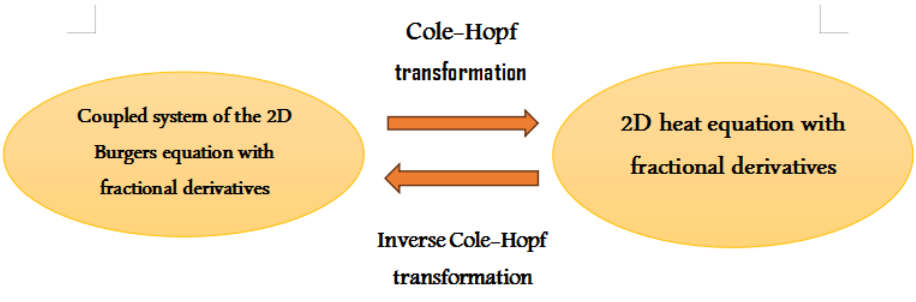

Figure 2.1: Transformation of the $2D$ Burgers system of equations into a $2D$ heat equation.

2.1 Position of problem

Let us consider the following coupling system of 2D Burgers' equations:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\ \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right), \end{cases} \tag{2.1}$$

where: $u = u(x,y,t)$, $v = v(x,y,t)$, $r > 0$ is the diffusion coefficient, $(x,y) \in [0,b] \times [0,b], t > 0$, $\partial^\alpha u/\partial t^\alpha$, $\partial^\alpha v/\partial t^\alpha$ are the conformable fractional derivatives of order $\alpha \in]0,1]$, respectively for u and v.

Subject to the initial conditions:

$$\begin{cases} u(x,y,0) = u_0(x,y), \text{ for any } (x,y) \in \Omega, \\ \\ v(x,y,0) = v_0(x,y), \text{ for any } (x,y) \in \Omega, \end{cases} \tag{2.2}$$

and the boundary conditions:

$$\begin{cases} u(x,y,t) = f(x,y,t), \text{ for any } (x,y) \in \partial\Omega,\ t > 0, \\ \\ v(x,y,t) = g(x,y,t), \text{ for any } (x,y,t) \in \partial\Omega,\ t > 0, \end{cases} \tag{2.3}$$

where, f and g are two given functions.

And the following potential symmetry condition.

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \tag{2.4}$$

which is necessary for the rest of this study [18].

2.2 Linearization of the problem using the Cole-Hopf transformation

Let us recall that The Cole-Hopf transformation introduced by Eberhard Hopf in (1950) and Julian Cole (1951) which allows us to transform the non-linear viscous Burgers equation into a linear heat-type equation.

By using property 6 of theorem 1.1, we can rewrite system (2.1) as follows:

$$\begin{cases} t^{(1-\alpha)}\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\ \\ t^{(1-\alpha)}\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right). \end{cases} \tag{2.5}$$

2.2.1 Linearisation of a coupled system

This operation is performed in two steps.

First step

Suppose that $u = \psi_x$ and $v = \psi_y$, the system (2.5) becomes:

$$\begin{cases} t^{(1-\alpha)}\psi_{xt} + \psi_x\psi_{xx} + \psi_y\psi_{xy} = r\left(\psi_{xxx} + \psi_{xyy}\right), \\ \\ t^{(1-\alpha)}\psi_{yt} + \psi_x\psi_{yx} + \psi_y\psi_{yy} = r\left(\psi_{yxx} + \psi_{yyy}\right), \end{cases} \tag{2.6}$$

we can also write:

$$\begin{cases} t^{(1-\alpha)}\psi_{xt} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\psi_x^2\right) + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xxx} + \psi_{xyy}\right), \\ \\ t^{(1-\alpha)}\psi_{yt} + \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\psi_x^2\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{yxx} + \psi_{yyy}\right). \end{cases} \tag{2.7}$$

By integrating respectively the first equation of system (2.7) with respect to x and the second with respect to y, we find:

$$\begin{cases} t^{(1-\alpha)}\psi_t + \left(\dfrac{1}{2}\psi_x^2\right) + \left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta_1(y,t), \\ \\ t^{(1-\alpha)}\psi_t + \left(\dfrac{1}{2}\psi_x^2\right) + \left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta_2(x,t), \end{cases} \tag{2.8}$$

where, $\eta_1(y,t)$ and $\eta_2(x,t)$ are arbitrary functions depending respectively of y and x.

By combining the two equation of system (2.8) and using condition (2.4), we can conclude that ψ satisfies the following equation (for more details see [17]):

$$t^{(1-\alpha)}\psi_t + \left(\frac{1}{2}\psi_x^2\right) + \left(\frac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta(t). \tag{2.9}$$

Second step

Introducing the transformation as $\psi = -2r\ln\phi$, so we have:

$$u = -2r\frac{\phi_x}{\phi} \quad \text{and} \quad v = -2r\frac{\phi_y}{\phi}. \tag{2.10}$$

The computation of the derivatives of the function ψ gives:

$$\psi_t = -2r\frac{\phi_t}{\phi}, \quad \psi_x = -2r\frac{\phi_x}{\phi}, \quad \psi_y = -2r\frac{\phi_y}{\phi}. \tag{2.11}$$

$$\psi_{xx} = -2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}, \quad \psi_{yy} = -2r\frac{\phi_{yy}}{\phi} + 2r\frac{\phi_y^2}{\phi^2}. \tag{2.12}$$

Inserting the derivatives ψ_t, ψ_x and ψ_y in the left side respectively ψ_{xx} and ψ_{yy} in the right side of the equation (2.9), we obtain:

$$\begin{gathered} -2rt^{(1-\alpha)}\frac{\phi_t}{\phi} + \frac{1}{2}\left(-2r\frac{\phi_x}{\phi}\right)^2 + \frac{1}{2}\left(-2r\frac{\phi_y}{\phi}\right)^2 = \\ r\left(-2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2} - 2r\frac{\phi_{yy}}{\phi} + 2r\frac{\phi_y^2}{\phi^2}\right) + \eta(t). \end{gathered} \tag{2.13}$$

The equation (2.13) can be reduced to:

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}) + \zeta(t)\phi, \tag{2.14}$$

where, $\zeta(t) = \dfrac{-\eta(t)}{2r}$.

Now, we state the following theorem to show that the calculation of the solutions $u(x,y,t)$ and $v(x,y,t)$ is independent of the function $\zeta(t)$.

Theorem 2.1. [33] *Let $\phi(x,y,t)$ be the solution* (2.14)*, $u(x,y,t)$ and $v(x,y,t)$ are defined in* (2.10)*, then the solution u and v are independent of $\zeta(t)$.*

Proof. Let

$$\beta(t) = \int \frac{1}{t^{1-\alpha}}\zeta(t)dt,$$

then,

$$\beta'(t) = \frac{1}{t^{1-\alpha}}\zeta(t).$$

Multiply by $e^{-\beta(t)}$ the two sides of (2.14), we find:

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} e^{-\beta(t)} = r(\phi_{xx} + \phi_{yy})e^{-\beta(t)} + \zeta(t)\phi e^{-\beta(t)}. \tag{2.15}$$

Using property 6 of theorem 1.1, the equation(2.15) becomes:

$$t^{1-\alpha}\frac{\partial \phi}{\partial t} e^{-\beta(t)} - \zeta(t)\phi e^{-\beta(t)} = r(\phi_{xx} + \phi_{yy})e^{-\beta(t)}. \tag{2.16}$$

It can also be written as,

$$t^{1-\alpha}\frac{\partial}{\partial t}\left(e^{-\beta(t)}\phi\right) = r((e^{-\beta(t)}\phi)_{xx} + (e^{-\beta(t)}\phi)_{yy}). \tag{2.17}$$

Let us now set,

$$\psi(x, y, t) = e^{-\beta(t)}\phi(x, y, t),$$

then $\psi(x, y, t)$ verify the following heat equation:

$$t^{1-\alpha}\frac{\partial \psi}{\partial t} = r(\psi_{xx} + \psi_{yy}), \tag{2.18}$$

which can be rewritten in the following form:

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} = r(\psi_{xx} + \psi_{yy}). \tag{2.19}$$

Note that the difference between the solution of equation (2.14) and that of equation (2.19) lies only in the presence of the factor $e^{-\beta(t)}$, so we have:

$$u(x, y, t) = \frac{\phi_x}{\phi} = \frac{e^{-\beta(t)}\phi_x}{e^{-\beta(t)}\phi} = \frac{\psi_x}{\psi}, \tag{2.20}$$

$$v(x, y, t) = \frac{\phi_y}{\phi} = \frac{e^{-\beta(t)}\phi_y}{e^{-\beta(t)}\phi} = \frac{\psi_y}{\psi}. \tag{2.21}$$

It is clear that the solution u and v are independent of the function $\zeta(t)$. □

To simplify the study, let us consider the particular case $\zeta(t) \equiv 0$, then the equation (2.14) becomes:

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}). \tag{2.22}$$

In the following, we determine the intial and boundary counditions corresponding to Eq.(2.22). For clarity, we consider:

$$\Omega = [0,b] \times [0,b], \;\; \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

with,

$$\Gamma_1 = \{0 \le x \le b,\, y = 0\}, \quad \Gamma_2 = \{0 \le x \le b,\, y = b\},$$
$$\Gamma_3 = \{x = 0,\;\, 0 \le y \le b\} \;\text{ and }\; \Gamma_4 = \{x = b,\, 0 \le y \le b\}.$$

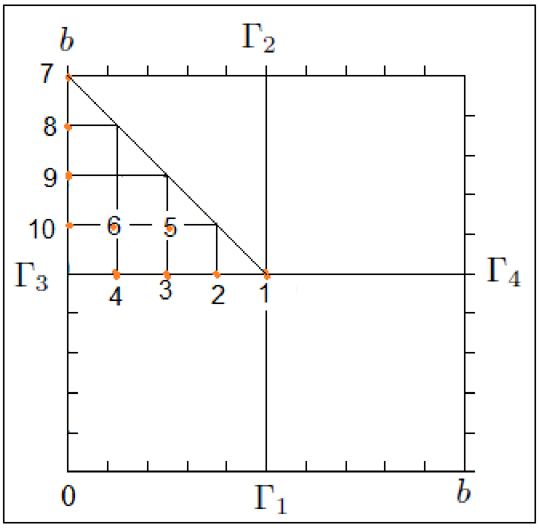

Figure 2.2: Solution domain

2.2.2 Determination of initial condition (IC)

Form Eq.(2.10), we have:

$$\frac{\phi_x}{\phi} = \frac{u(x,y,t)}{-2r}. \tag{2.23}$$

By integrating both sides of Eq.(2.23) with respect to x, we obtain:

$$\ln(\phi) = \frac{-1}{2r}\int_0^x u(s,y,t)ds + \ln(\phi(0,y,t)).$$

This implies,

$$\phi(x,y,t) = \phi(0,y,t)\exp\left(\frac{-1}{2r}\int_0^x u(s,y,t)ds\right), \tag{2.24}$$

where $\phi(0,y,t)$ is an unknown function. To determine it, we need to use the second form of Eq.(2.10) which can also be rewritten as:

$$\frac{\phi_y}{\phi} = \frac{v(x,y,t)}{-2r}. \tag{2.25}$$

Integrating the above equation with respect to y gives:

$$\ln(\phi) = \frac{-1}{2r}\int_0^y u(x,s,t)ds + \ln(\phi(x,0,t)).$$

This implies,

$$\phi(x,y,t) = \phi(x,0,t)\exp\left(\frac{-1}{2r}\int_a^y v(x,s,t)ds\right). \tag{2.26}$$

For $x = 0$, the Eq.(2.26) becomes:

$$\phi(0,y,t) = \phi(0,0,t)\exp\left(\frac{-1}{2r}\int_a^y v(0,s,t)ds\right). \tag{2.27}$$

By inserting Eq.(2.27) into Eq.(2.24), we obtain:

$$\phi(x,y,t) = \phi(0,0,t)\exp\left(-\frac{1}{2r}\int_0^y v(0,s,t)ds - \frac{1}{2r}\int_0^x u(s,y,t)ds\right). \tag{2.28}$$

For $t = 0$, we obtain the initial condition above:

$$\phi(x,y,0) = \phi(0,0,0)\exp\left(-\frac{1}{2r}\int_0^y v(0,s,0)ds - \frac{1}{2r}\int_0^x u(s,y,0)ds\right). \tag{2.29}$$

To show that $\phi(0,0,0)$ has no effect on the solution of the initial system of Burgers equations, we set $\tilde{\phi} = c\phi$, where c is a contant and we state the following proposition.

Proposition 2.1. [33] *Let $\tilde{\phi}$ be the solution of the heat equation* (2.22)*, let $\tilde{u}$ and $\tilde{v}$ the solution defined in relation* (2.10) *then, $\tilde{u}(x,y,t)$ and $\tilde{v}(x,y,t)$ are independent of the constant c.*

Proof. According to Eq.(2.10), we have:

$$\tilde{u}(x,y,t) = -2r\frac{(\tilde{\phi})_x}{\tilde{\phi}} = -2r\frac{c(\phi)_x}{c\phi} = -2r\frac{(\phi)_x}{\phi} = u(x,y,t),$$

and

$$\tilde{v}(x,y,t) = -2r\frac{(\tilde{\phi})_y}{\phi_c} = -2r\frac{c(\phi)_y}{c\phi} = -2r\frac{(\phi)_y}{\phi} = v(x,y,t).$$

Thus, the proof is complete. □

To simplify the study, we can consider $\phi(0,0,0) = 1$, which gives:

$$\phi_0(x,y) = \exp\left(-\frac{1}{2r}\int_0^y v(0,s,0)ds - \frac{1}{2r}\int_0^x u(s,y,0)ds\right). \tag{2.30}$$

2.2.3 Determination of boundary conditions (BC)

Using Eq.(2.10), the boundary conditions are reduced to:

$$\begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)), \\ \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \tag{2.31}$$

Therefore, the time conformable diffusion equation with the initial and Neumann boundary conditions is given by:

$$\begin{cases} Eq.: \quad \dfrac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}). \\ \\ CI: \quad \phi_0(x,y) = \exp\left(-\dfrac{1}{2r}\displaystyle\int_0^y v(0,s,0)ds - \dfrac{1}{2r}\int_0^x u(s,y,0)ds\right). \\ \\ CB: \quad \begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)), \\ \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \end{cases} \tag{2.32}$$

Reformulating problem (2.32) by using the property 6 of Theoreme 1.1 it yields,

$$\begin{cases} Eq.: & t^{(1-\alpha)}\dfrac{\partial\phi}{\partial t} = r(\phi_{xx} + \phi_{yy}). \\ CI\ : & \phi_0(x,y) = \exp\left(-\dfrac{1}{2r}\displaystyle\int_0^y v(0,s,0)ds - \dfrac{1}{2r}\int_0^x u(s,y,0)ds\right). \\ CB\ : & \begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t),\ (x,y,t) \in (\partial\Omega \times (0,T)), \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t),\ (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \end{cases} \tag{2.33}$$

2.3 Analytical solution

We determine the analytical solution of system (2.1) using the heat equation (2.33) and the inverse Cole-Hopf transformation.

2.3.1 Analytical solution of the heat equation

To solve l′ Eq.(2.33), we apply the Fourier transform (F.T),

$$F.T(\phi(x,y,t)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2i\pi(k_x.x+k_y.y)}\ \phi(x,y,t)dxdy = \widehat{\phi}(k_x,k_y,t), \tag{2.34}$$

and the inverse Fourier transform $(F.T^{-1})$,

$$F.T^{-1}(\phi(x,y,t)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2i\pi(k_x.x+k_y.y)}\ \widehat{\phi}(k_x,k_y,t)dk_xdk_y. \tag{2.35}$$

First, we apply the Fourier transform F.T to the term ϕ_t,

$$
\begin{aligned}
F.T\left(\frac{\partial \phi}{\partial t}\right) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2i\pi(k_x.x+k_y.y)}\,\frac{\partial \phi}{\partial t}\;dxdy,\\
&=\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2i\pi(k_x.x+k_y.y)}\,\phi\;dxdy,\\
&=\frac{\partial\widehat{\phi}(k_x,k_y,t)}{\partial t}.
\end{aligned}
\tag{2.36}
$$

Then, we need to examine the Fourier transform F.T of the term ϕ_x, i.e.:

$$
F.T(\phi_x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2i\pi(k_x.x+k_y.y)}\,\phi_x\;dxdy.
\tag{2.37}
$$

The integration by parts of Eq.(2.37), we define f and g' as follows:

$$
\begin{cases}
f = e^{-2i\pi(k_x.x+k_y.y)} \Longrightarrow f' = -2i\pi k_x\, e^{-2i\pi(k_x.x+k_y.y)},\\
g' = \phi_x \Longrightarrow g = \phi.
\end{cases}
\tag{2.38}
$$

This gives,

$$
F.T(\phi_x) = e^{-2i\pi(k_x.x+k_y.y)}\,\phi\Big|_{-\infty_{(x,y)}}^{+\infty} - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} -2i\pi k_x\; e^{-2i\pi(k_x.x+k_y.y)}\,\phi.
\tag{2.39}
$$

According to [26], the boundary conditions of the heat equation on the infinite interval: $\phi(x,y,t) = 0$ if $|x| = |y| = \infty$, thus we have:

$$
\begin{aligned}
F.T(\phi_x) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2i\pi k_x\; e^{-2i\pi(k_x.x+k_y.y)}\,\phi\\
&=2i\pi k_x \times F.T(\phi(x,y,t))\\
&=2i\pi k_x\,\widehat{\phi}(k_x,k_y,t).
\end{aligned}
\tag{2.40}
$$

We can iterate this result to obtain the F.T of the term ϕ_{xx}:

$$\begin{aligned} F.T(\phi_{xx}) =& F.T\left((\phi_x)_x\right) \\ =& 2i\pi k_x \times F.T\left(\phi_x\right) \\ =& (2i\pi k_x)^2\, \widehat{\phi}(k_x, k_y, t). \end{aligned} \tag{2.41}$$

In the same way, we proceed for the variable y, thus we have:

$$F.T(\phi_{yy}) = (2i\pi k_y)^2 \widehat{\phi}(k_x, k_y, t).$$

By substituting the above results into Eq.(2.33), we obtain:

$$t^{(1-\alpha)} \frac{\partial \widehat{\phi}}{\partial t} + r(2\pi)^2 (k_x^2 + k_y^2) \widehat{\phi} = 0. \tag{2.42}$$

Thus, the solution to Eq.(2.42) is given by:

$$\widehat{\phi} = A(k_x, k_y)\, e^{-4r\pi^2 (k_x^2 + k_y^2) t^\alpha / \alpha}, \tag{2.43}$$

where, $A(k_x, k_y)$ is the constant of integration,which is given by:

$$A(k_x, k_y) = \widehat{\phi}_0(k_x, k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi_0(x', y')\, e^{-2i\pi(k_x.x' + k_y.y')}\, dx' dy'.$$

By applying F.T^{-1} to Eq.(2.43), we obtain:

$$\begin{aligned} F.T^{-1}(\phi(x, y, t)) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2i\pi(k_x.x + k_y.y)}\, \widehat{\phi}(k_x, k_y, t)\, dk_x\, dk_y \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi_0(x', y') \times I \times dx' dy', \end{aligned}$$

where,

$$\begin{aligned} I &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2i\pi(k_x.x + k_y.y) - 2i\pi(k_x.x' + k_y.y') - 4r\pi^2(k_x^2 + k_y^2) t^\alpha / \alpha}\, dk_x\, dk_y \\ &= \left[\int_{-\infty}^{+\infty} e^{2i\pi(x - x')k_x - 4r\pi^2 k_x^2 t^\alpha / \alpha}\, dk_x \right] \times \left[\int_{-\infty}^{+\infty} e^{2i\pi(y - y')k_y - 4r\pi^2 k_y^2 t^\alpha / \alpha}\, dk_y \right]. \end{aligned}$$

Using the Maple program (See Appendix I), the solution of Eq.(2.32) is given by:

$$\phi(x,y,t) = \frac{\alpha t^{-\alpha}}{4r\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[\frac{-\alpha(x-x')^2 - \alpha(y-y')^2}{4rt^\alpha}\right] \phi_0(x',y')dx'dy'. \tag{2.44}$$

2.3.2 Analytical Solution of System

Let's calculate:

$$\phi_x(x,y,t) = \frac{-\alpha t^{-\alpha}}{8r\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\alpha(x-x') \times J}{rt^\alpha}\, \phi_0(x',y')dx'dy', \tag{2.45}$$

$$\phi_y(x,y,t) = \frac{-\alpha t^{-\alpha}}{8r\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\alpha(y-y') \times J}{rt^\alpha}\, \phi_0(x',y')dx'dy'. \tag{2.46}$$

where,

$$J = \exp\left(\frac{-\alpha(x-x')^2 - \alpha(y-y')^2}{4rt^\alpha}\right).$$

Once the functions $\phi(x,y,t)$, $\phi_x(x,y,t)$ and $\phi_y(x,y,t)$ are calculated and using (2.10), the solution is:

$$u(x,y,t) = \frac{\alpha \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-x')\, \exp\left[\frac{-\alpha(x-x')^2 - \alpha(y-y')^2}{4rt^\alpha}\right] \phi_0(x',y')dx'dy'}{t^\alpha \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[\frac{-\alpha(x-x')^2 - \alpha(y-y')^2}{4rt^\alpha}\right] \phi_0(x',y')dx'dy'}, \tag{2.47}$$

$$v(x,y,t) = \frac{\alpha \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (y-y')\, \exp\left[\frac{-\alpha(y-y')^2 - \alpha(y-y')^2}{4rt^\alpha}\right] \phi_0(x',y')dx'dy'}{t^\alpha \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[\frac{-\alpha(x-x')^2 - \alpha(y-y')^2}{4rt^\alpha}\right] \phi_0(x',y')dx'dy'}. \tag{2.48}$$

CHAPITRE 3

AN NUMERICAL STUDY OF COUPLED TWO DIMENSIONAL FRACTIONAL BURGERS EQUATION

CHAPTER 3

NUMERICAL STUDY OF COUPLED TWO DIMENSIONAL FRACTIONAL BURGERS EQUATION

In this chapter, we focus on the numerical solution of our problem. For illustration, we complete the study with numerical simulations to evaluate the effectiveness of the Cole-Hopf transformation.

3.1 Discrete Analogues

3.1.1 Discretization of the linearized problem using the FDM

We discretize the domain Ω using the FDM at nx points (resp. ny), each with an interval length $\Delta x = (b-a)/nx$ (resp.$\Delta y = (b-a)/ny$) along the x (resp.y) and define the points of the discrete mesh (x_i, y_j, t_n), for $(a + i\Delta x, a + j\Delta y, n\Delta t)$, where $i = 0, ..., nx,\ j = 0, ..., ny$ and $n = 0, ..., T$.

i) Explicit Scheme

The explicit scheme corresponding to Eq.(2.33) is given by:

$$t_n^{(1-\alpha)} \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t} = r\left(\frac{\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n}}{\Delta x^2} + \frac{\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n}}{\Delta y^2}\right).$$

Thus, for each interior point (x_i, y_j), where $i = 1, ..., nx-1, j = 1, ..., ny-1$, we have:

$$\phi_{i,j}^{n+1} = \vartheta_1 \phi_{i,j}^{n} + \vartheta_2(\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n}) + \vartheta_3(\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n}), \tag{3.1}$$

where,

$$\vartheta_1 = 1 - \frac{2r\Delta t}{(\Delta x)^2\, t_n^{(1-\alpha)}} - \frac{2r\Delta t}{(\Delta y)^2 t_n^{(1-\alpha)}},$$

$$\vartheta_2 = \frac{r\Delta t}{(\Delta x)^2\, t_n^{(1-\alpha)}} \quad \text{and} \quad \vartheta_3 = \frac{r\Delta t}{(\Delta y)^2\, t_n^{(1-\alpha)}}.$$

Calculation of Boundary Elements

We descretize the boundary conditions as follows:

$$\begin{cases} \phi_x(x_i, y_j, t_n) \simeq \dfrac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x} = -\dfrac{1}{2r} u^n_{i,j}\phi^n_{i,j}, \\ \\ \phi_y(x_i, y_j, t_n) \simeq \dfrac{\phi^n_{i,j+1} - \phi^n_{i,j-1}}{2\Delta y} = -\dfrac{1}{2r} v^n_{i,j}\phi^n_{i,j}. \end{cases} \tag{3.2}$$

Which are written as follows:

$$\begin{cases} \phi^n_{i+1,j} = \phi^n_{i-1,j} - \dfrac{\Delta x}{r} u^n_{i,j}\phi^n_{i,j}, \\ \\ \phi^n_{i,j+1} = \phi^n_{i,j-1} - \dfrac{\Delta y}{r} v^n_{i,j}\phi^n_{i,j}. \end{cases} \tag{3.3}$$

Boundary Conditions Calculation on the Borders $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

- On Γ_1, for $j = 0$, equation (3.3) simplifies to:

$$\begin{cases} \phi^n_{i+1,0} = \phi^n_{i-1,0} - \dfrac{\Delta x}{r} u^n_{i,0}\phi^n_{i,0}, \\ \\ \phi^n_{i,1} = \phi^n_{i,-1} - \dfrac{\Delta y}{r} v^n_{i,0}\phi^n_{i,0}. \end{cases} \tag{3.4}$$

By substituting equation (3.4) into equation.(3.1), for $i = 1, ..., nx$, we obtain:

$$\begin{aligned} \phi^{n+1}_{i,0} &= \vartheta_1\phi^n_{i,0} + \vartheta_2(\phi^n_{i+1,0} + \phi^n_{i-1,0}) + \vartheta_3(\phi^n_{i,1} + \phi^n_{i,-1}) \\ &= \vartheta_1\phi^n_{i,0} + \vartheta_2\left(2\phi^n_{i-1,0} - \frac{\Delta x}{r} u^n_{i,0}\phi^n_{i,0}\right) + \vartheta_3\left(2\phi^n_{i,1} - \frac{\Delta y}{r} v^n_{i,0}\phi^n_{i,0}\right). \end{aligned}$$

Similarly, following the same approach as for Γ_1, we can derive analogous expressions for Γ_2, $(j = ny)$ Γ_3, $(i = 0)$ and Γ_4, $(i = nx)$.

$$\phi_{i,ny}^{n+1} = \vartheta_1 \phi_{i,ny}^n + \vartheta_2 \left(2\phi_{i-1,ny}^n - \frac{\Delta x}{r} u_{i,ny}^n \phi_{i,ny}^n \right) + \vartheta_3 \left(2\phi_{i,ny-1}^n - \frac{\Delta y}{r} v_{i,ny}^n \phi_{i,ny}^n \right), \tag{3.5}$$

for $i = 1, ..., nx$.

$$\phi_{0,j}^{n+1} = \vartheta_1 \phi_{0,j}^n + \vartheta_2 \left(2\phi_{1,j}^n + \frac{\Delta x}{r} u_{0,j}^n \phi_{0,j}^n \right) + \vartheta_3 \left(2\phi_{0,j-1}^n - \frac{\Delta y}{r} v_{0,j}^n \phi_{0,j}^n \right), \tag{3.6}$$

$$\phi_{nx,j}^{n+1} = \vartheta_1 \phi_{nx,j}^n + \vartheta_2 \left(2\phi_{nx-1,j}^n - \frac{\Delta x}{r} u_{nx,j}^n \phi_{nx,j}^n \right) + \vartheta_3 \left(2\phi_{nx,j-1}^n - \frac{\Delta y}{r} v_{nx,j}^n \phi_{nx,j}^n \right), \tag{3.7}$$

for, $j = 1, ..., ny$.

For the lower-left vertex (x_0, y_0), we have:

$$\phi_{0,0}^{n+1} = \vartheta_1 \phi_{0,0}^n + \vartheta_2 \left(2\phi_{1,0}^n + \frac{\Delta x}{r} u_{0,0}^n \phi_{0,0}^n \right) + \vartheta_3 \left(2\phi_{0,1}^n + \frac{\Delta y}{r} v_{0,0}^n \phi_{0,0}^n \right). \tag{3.8}$$

ii) Implicit Scheme

By using a simple forward discretization in time t and a centered discretization in space x and y, around the point (x_i, y_j, t_n), the implicit scheme for Eq .(2.33) is obtained:

$$t_n^{1-\alpha} \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = r \left(\frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2} \right),$$

which can be rewritten as follows:

$$-\alpha(\phi_{i+1,j}^{n+1} + \phi_{i-1,j}^{n+1}) + \gamma \phi_{i,j}^{n+1} - \beta(\phi_{i,j+1}^{n+1} + \phi_{i,j-1}^{n+1}) = \phi_{i,j}^n, \tag{3.9}$$

where,

$$\alpha = \frac{r\Delta t}{(\Delta x)^2 t_n^{1-\alpha}}, \quad \beta = \frac{r\Delta t}{(\Delta y)^2 t_n^{1-\alpha}}, \quad \gamma = 1 + 2\alpha + 2\beta.$$

Or in matrix form:

$$\mathcal{A}.\mathcal{X} = \mathcal{B},$$

with,

$$\mathcal{A} = \begin{pmatrix} A & B & 0 & \cdots & 0 \\ C & D & K & 0 \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & C & D & K \\ 0 & \cdots & 0 & L & M \end{pmatrix}_{(nx\times ny, nx\times ny)},$$

$\mathcal{X}^t = (\phi_{0,0}^{n+1}, \phi_{0,1}^{n+1}, ..., \phi_{nx,ny}^{n+1})$ and $\mathcal{B} = (\phi_{0,0}^{n}, \phi_{0,1}^{n}, ..., \phi_{nx,ny}^{n})$.

A, B, C, D, K, L and M are submatrices of dimension (nx, ny) defined as follows:

$$A = \begin{pmatrix} a & -2\beta & 0 & \cdots & 0 \\ -2\beta & \gamma & 0 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2\beta & \gamma \end{pmatrix}, B = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & b_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & b_n \end{pmatrix},$$

$$C = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & 0 \\ 0 & \cdots & \cdots & 0 & -2\alpha \end{pmatrix}, D = \begin{pmatrix} d_1 & -2\beta & 0 & \cdots & 0 \\ -\beta & \gamma & -\beta & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma & -\beta \\ 0 & \cdots & 0 & -2\beta & d_{ny} \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, L = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -2\alpha \end{pmatrix},$$

$$M = \begin{pmatrix} m' & -2\beta & 0 & \cdots & 0 \\ -2\beta & m_1 & 0 & \cdots & \vdots \\ 0 & \ddots & m_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2\beta & m_{nx} \end{pmatrix}.$$

With,

$$a = -\frac{\alpha\Delta x}{r}u^n_{0,0} + \gamma - \frac{\beta\Delta y}{r}v^n_{0,0}.$$

$$b_j = -2\alpha - \frac{\alpha\Delta x}{r}u^n_{0,j} + \frac{\beta\Delta y}{r}v^n_{0,j} \text{ for } j = 1, ..., ny.$$

$$d_1 = -\frac{\alpha\Delta x}{r}u^n_{1,0} + \gamma - \frac{\beta\Delta y}{r}v^n_{1,0}.$$

$$d_{ny} = -\frac{\alpha\Delta x}{r}u^n_{nx,ny} + \gamma + \frac{\beta\Delta y}{r}v^n_{nx,ny}.$$

$$m' = d_1, m_i = -\frac{\alpha\Delta x}{r}u^n_{i,ny} + \gamma + \frac{\beta\Delta y}{r}v^n_{i,ny}, \text{ for } i = 1, ..., nx.$$

3.1.2 Discrete solution of the system

The solution of the system (2.1) can be obtained using the inverse Cole-Hopf transformation.
Let $D_x\phi^n_{i,j}$ and $D_y\phi^n_{i,j}$ be the derivatives of ϕ at the point (x_i, y_j, t_n) with respect to x and y. Then, $D_x\phi^n_{i,j}$ and $D_y\phi^n_{i,j}$ can be computed using the first-order centered diffrence formula, for $i = 1, .., nx - 1, j = 1, .., ny - 1$.

$$\begin{aligned} D_x\phi^n_{i,j} &= \frac{\partial\phi}{\partial x} \simeq \frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x}, \\ D_y\phi^n_{i,j} &= \frac{\partial\phi}{\partial y} \simeq \frac{\phi^n_{i,j+1} - \phi^n_{i,j-1}}{2\Delta y}. \end{aligned} \tag{3.10}$$

Knowing that the derivatives $D_y\phi^n_{0,j}, D_y\phi^n_{nx,j}, D_x\phi^n_{i,0}$ and $D_x\phi^n_{i,ny}$ at the boundries are already known. Once the approximate values of ϕ, ϕ_x and ϕ_y are computed at all discrete points (x_i, y_j, t_n), then the approximate values of u and v at discrete points, can be calculated using the following formulas:

$$\begin{cases} u^n_{i,j} = -2r\dfrac{D_x\phi^n_{i,j}}{\phi^n_{i,j}}, \\ \\ v^n_{i,j} = -2r\dfrac{D_y\phi^n_{i,j}}{\phi^n_{i,j}}, \text{ for } i = 1, .., nx, \quad j = 1, .., ny. \end{cases} \tag{3.11}$$

3.2 Numerical experiments

For illustration of the proposed method, we will report the accuracy of the method based on relative error L_1-norm and L_∞-norm defined by:

$$||Erreuru||_1 = \frac{||u_a - u_n||_1}{||u_a||_1}, \; ||Erreurv||_1 = \frac{||v_a - v_n||_1}{||v_a||_1}, \tag{3.12}$$

and

$$||Erreuru||_\infty = \frac{||u_a - u_n||_\infty}{||u_a||_\infty}, \; ||Erreurv||_\infty = \frac{||v_a - v_n||_\infty}{||v_a||_\infty}, \tag{3.13}$$

where the pair (u_a,v_a) represents the below analytical solution (3.14) (see, [17], p. 581, for the particular case $\alpha = 1$) for the system (2.1) and the pair (u_n,v_n) represents the above computed solution (3.11) for the system (2.1).

To simulate, we take the following exact solution for system (2.1) in over square domain Ω.

$$\begin{cases} u_a(x,y,t) = \dfrac{3}{4} - \dfrac{1}{4[1 + exp((-4x\alpha + 4y\alpha - t^\alpha)/32r\alpha)]}, \\ v_a(x,y,t) = \dfrac{3}{4} + \dfrac{1}{4[1 + exp((-4x\alpha + 4y\alpha - t^\alpha)/32r\alpha)]}. \end{cases} \tag{3.14}$$

Note that the initial and boundary conditions can be taken from the exact solutions. After computing, we evaluate respectively the relative errors Eq.(3.12) and Eq.(3.13). We use the explicit and implicit schemes for the conformable time-derivative 2D heat equation and see the convergence of each scheme in the following table 3.1 and table 3.2.

Erreur relative	$\|\|Erreur\ u\|\|_{L_1}$		$\|\|Erreur\ v\|\|_{L_1}$	
Scheme	Explicit	Implicit	Explicit	Implicit
	T=0.1			
$\Delta x = \Delta y = 0.2$	$3.30e-03$	$3.34e-03$	$3.20e-03$	$3.22e-03$
$\Delta x = \Delta y = 0.1$	$2.17e-03$	$2.17e-03$	$1.55e-03$	$1.55e-03$
$\Delta x = \Delta y = 0.05$	$1.46e-03$	$1.53e-03$	$8.19e-04$	$8.02e-04$
	T=0.5			
$\Delta x = \Delta y = 0.2$	$5.60e-03$	$5.64e-03$	$1.63e-03$	$1.58e-03$
$\Delta x = \Delta y = 0.1$	$4.69e-03$	$4.56e-03$	$1.58e-03$	$1.41e-03$
$\Delta x = \Delta y = 0.05$	$4.48e-03$	$4.52e-03$	$1.46e-03$	$1.37e-03$
	T=1			
$\Delta x = \Delta y = 0.2$	$7.85e-03$	$7.90e-03$	$1.43e-03$	$1.43e-03$
$\Delta x = \Delta y = 0.1$	$7.37e-03$	$7.47e-03$	$1.37e-03$	$1.31e-03$
$\Delta x = \Delta y = 0.05$	$7.26e-03$	$7.35e-03$	$1.06e-03$	$1.29e-03$

Table 3.1: Erreurs relatives L_1.

Erreur relative	$\|\|Erreur\ u\|\|_{L_\infty}$		$\|\|Erreur\ v\|\|_{L_\infty}$	
Scheme	Explicit	Implicit	Explicit	Implicit
	T=0.1			
$\Delta x = \Delta y = 0.2$	$3.35e-03$	$3.34e-03$	$3.20e-03$	$3.22e-03$
$\Delta x = \Delta y = 0.1$	$2.39e-03$	$2.39e-03$	$1.70e-03$	$1.70e-03$
$\Delta x = \Delta y = 0.05$	$1.52e-03$	$1.53e-03$	$8.29e-04$	$8.29e-04$
	T=0.5			
$\Delta x = \Delta y = 0.2$	$5.62e-03$	$5.64e-03$	$1.69e-03$	$1.69e-03$
$\Delta x = \Delta y = 0.1$	$4.76e-03$	$4.76e-03$	$1.57e-03$	$1.91e-03$
$\Delta x = \Delta y = 0.05$	$4.62e-03$	$4.62e-03$	$1.48e-03$	$1.87e-03$
	T=1			
$\Delta x = \Delta y = 0.2$	$7.88e-03$	$7.90e-03$	$1.68e-03$	$1.68e-03$
$\Delta x = \Delta y = 0.1$	$7.47e-03$	$7.47e-03$	$1.50e-03$	$1.51e-03$
$\Delta x = \Delta y = 0.05$	$7.34e-03$	$7.35e-03$	$1.47e-03$	$1.49e-03$

Table 3.2: Erreurs relatives L_∞.

We show through the Figure 3.1 the graphs of the numerical solution for 2D time-fractional heat equation (2.33), for $r = 0.5, \Delta x = \Delta y = 0.08$ and

$\alpha = 0.5$.

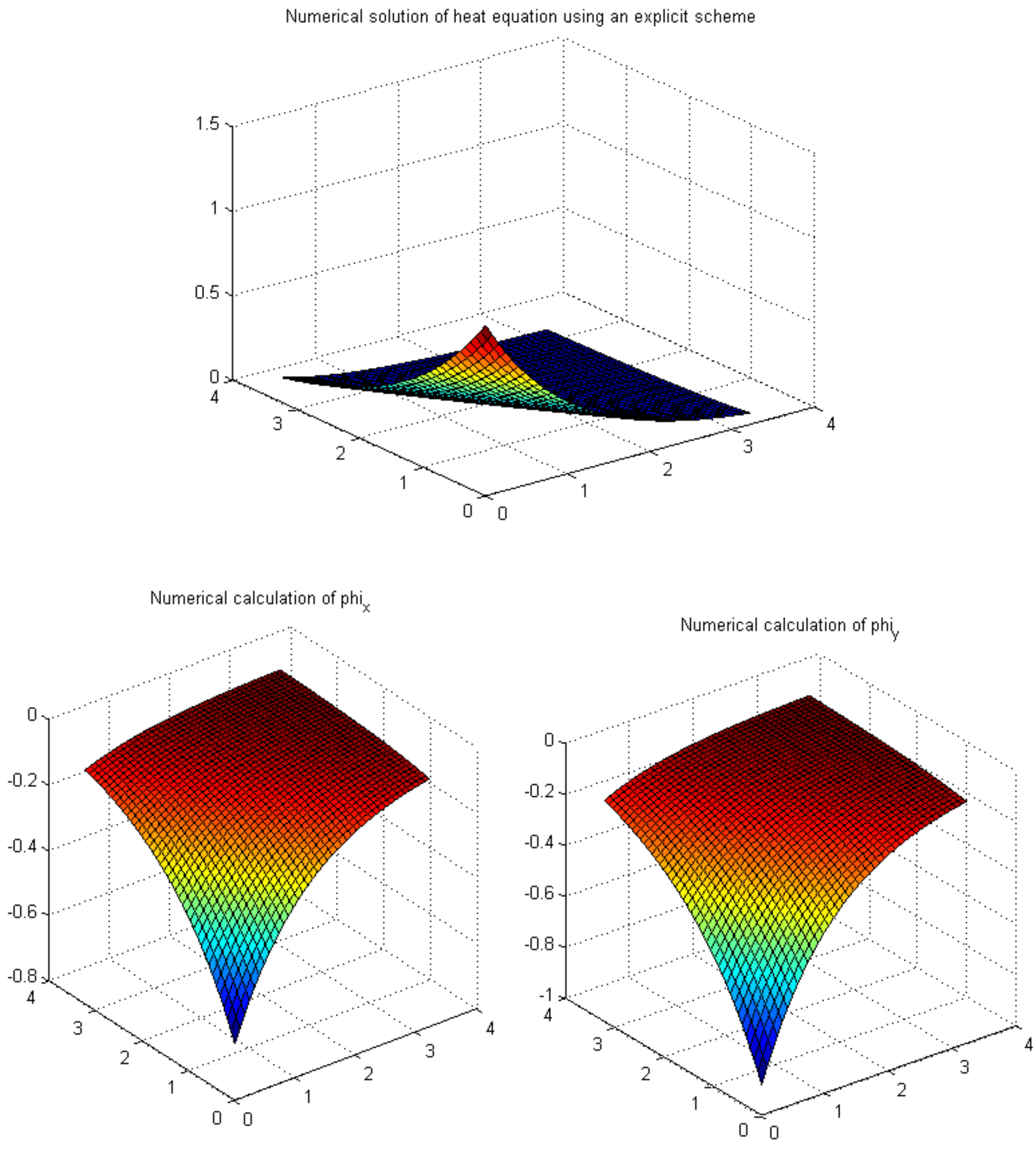

Figure 3.1: Graphs of numerical solution of heat equation and numerical calculation of ϕ_x and ϕ_y.

Now, we give the graphs of the exact and numerical solutions in Figure

3.2 for the system.(2.1), for $r = 0.5, \Delta x = \Delta y = 0.08$ and $\alpha = 0.5$.

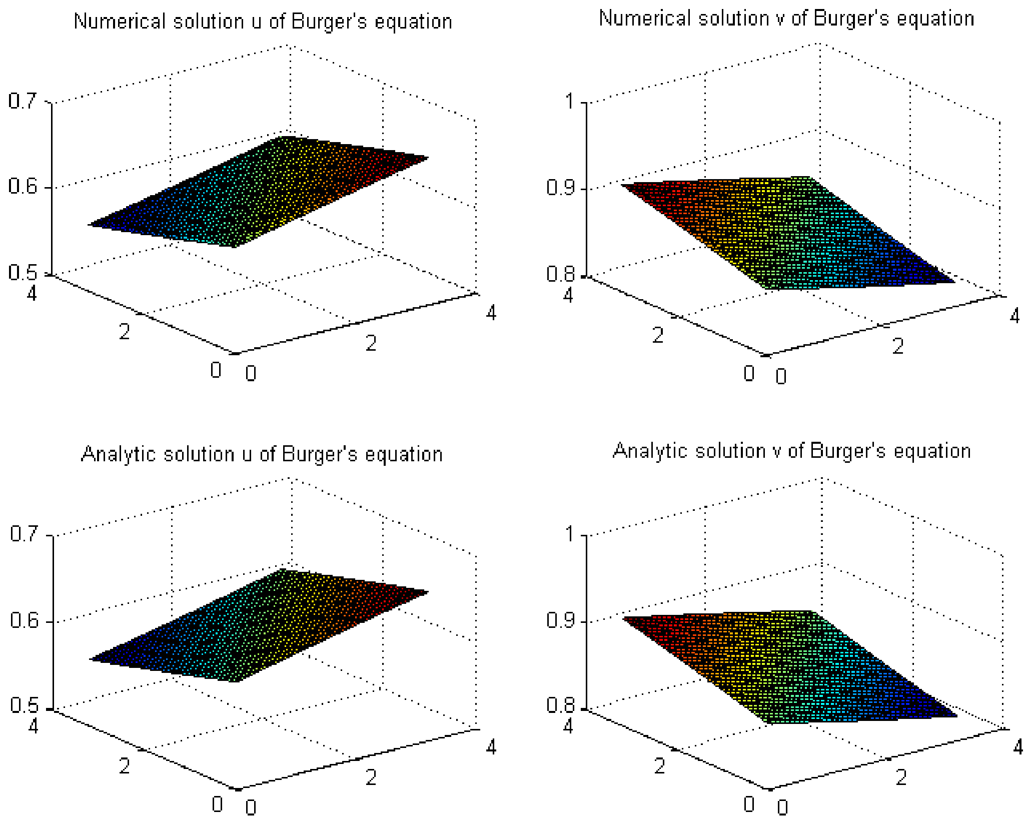

Figure 3.2: Graphs of exact and numerical solution for 2D time-fractional Burgers equations.

In same way, we give the graphs of the exact and numerical solutions in Figure 3.3 for the system.(2.1), for $r = 0.5, \Delta x = \Delta y = 0.08$ and $\alpha = 0.25, 0.75$ and 0.92.

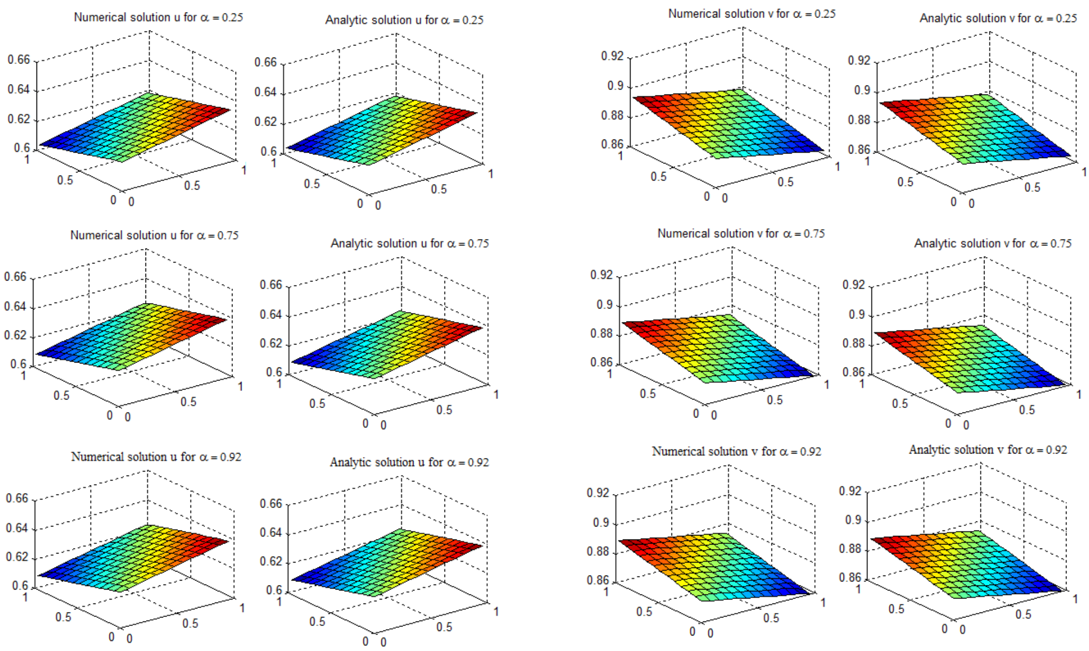

Figure 3.3: Graphs of exact and numerical solution for 2D time-fractional Burgers equations, for $r = 0.5, \Delta x = \Delta y = 0.08$ and $\alpha = 0.25;$ 0.75 and 0.92.

Discussion:

The results show a strong agreement between the numerical and analytical solutions for different values of the fractional order α, demonstrating the accuracy and efficiency of the adopted numerical method.

The behavior of the solution clearly changes with the value of α:

- For $\alpha = 0.25$, the solution remains low with slow diffusion.
- For $\alpha = 0.75$ and $\alpha = 0.92$, the solution increases gradually or sharply, with faster diffusion.

Smaller values of α imply a stronger memory effect in the system, leading to a more conservative response and slower evolution. As α increases, this memory influence weakens, and the behavior approaches that of classical models with more active and faster diffusion.

These results highlight the flexibility of α as a control parameter that can modulate the physical behavior of the system from slow and constrained dynamics to faster and more pronounced development. As a future direction, it is recommended to extend the simulation time and use more complex initial profiles to better capture the long-term influence of the fractional order on the system dynamics.

Conclusion.

In this study, the approach of the Cole-Hopf transformation shows its efficiency to deal with this class of fractional nonlinear problems and the proposed numerical algorithm is simple and effective.

CONCLUSION

Conclusion

In conclusion, we applied the Cole–Hopf transformation to a coupled 2D system of Burgers equations with conformable fractional derivatives in order to transform it into a linear fractional heat-type equation. By combining the solution obtained from the diffusion equation with the inverse Cole–Hopf transformation, we derived the solution of the original system of Burgers equations with conformable derivatives. This shows that the use of the Cole–Hopf transformation and its inverse is highly effective in solving this class of nonlinear fractional equations. The results obtained are very encouraging and it would be interesting to apply this approach to more complex higher-dimensional problems in order to draw more general conclusions.

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BIBLIOGRAPHY

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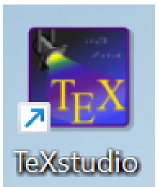
APPENDIX

APPENDIX I

☞ LaTeX program



LaTeX is a free tool for writing and formatting academic theses, widely used in scientific and engineering fields for its strong support of equations and scientific symbols. Developed by Leslie Lamport in 1985 as an extension of TeX by Donald Knuth, it allows users to insert tables, images, and generate contents automatically. While powerful, it can be challenging for beginners due to its markup-based syntax.

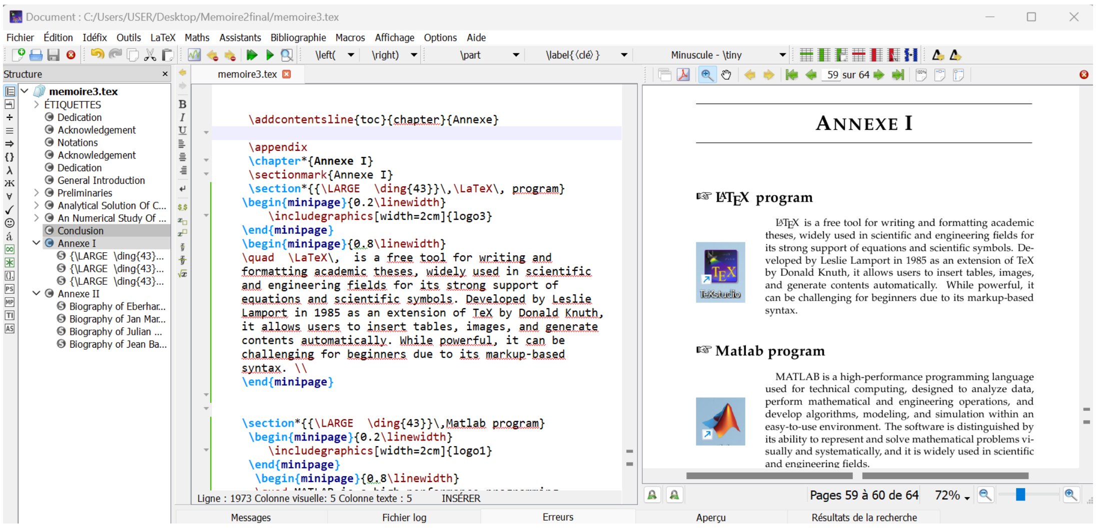


Figure 4: Interface view of TeXstudio program.

☞ Matlab program

Matlab is a high-performance programming language used for technical computing, designed to analyze data, perform mathematical and engineering operations, and develop algorithms, modeling, and simulation within an easy-to-use environment. The software is distinguished by its ability to represent and solve mathematical problems visually and systematically, and it is widely used in scientific and engineering fields.

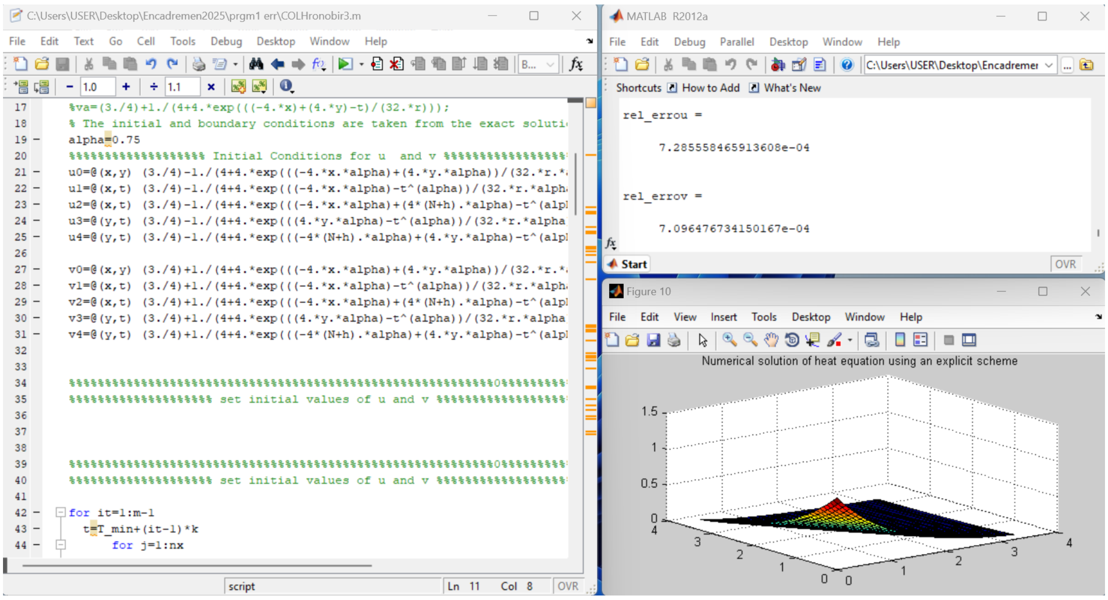


Figure 5: Interface view of Matlab program.

☞ Maple program

Maple is an advanced analytical computing system that uses mathematical and computational concepts to solve many simple and complex problems across various scientific, engineering, and economic fields. Maple is capable of providing both numerical and symbolic solutions for functions and mathematical operations. This means Maple can handle functions and mathematical expressions containing symbols like π or x without needing to know their exact or approximate numerical values. For example, Maple can determine and calculate the derivative of $\sin(x)$, which is $\cos(x)$, even if the value of the variable x is unknown. This enables Maple to provide precise calculations for a wide range of scientific, engineering, and analytical problems. Additionally, Maple has powerful capabilities for generating highly clear and accurate graphs and diagrams, which help visualize problems and the derived results.

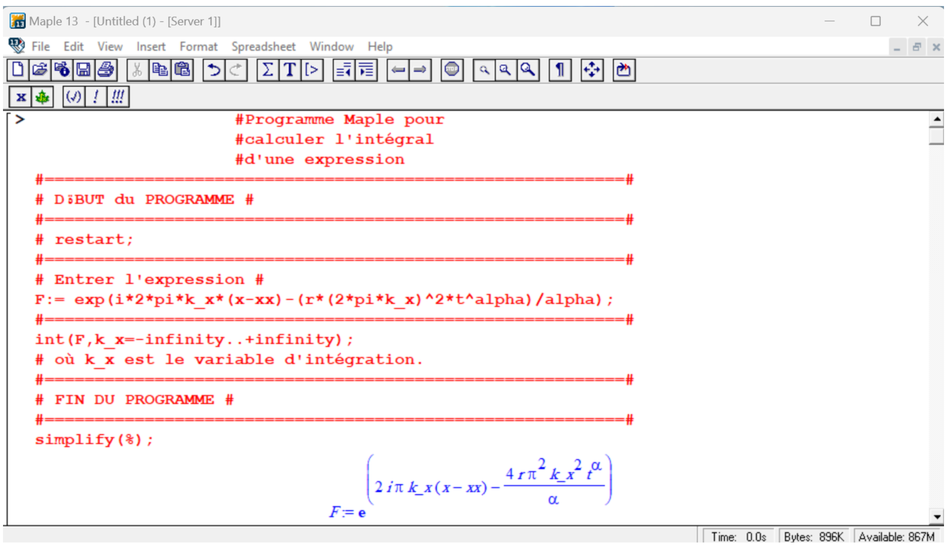


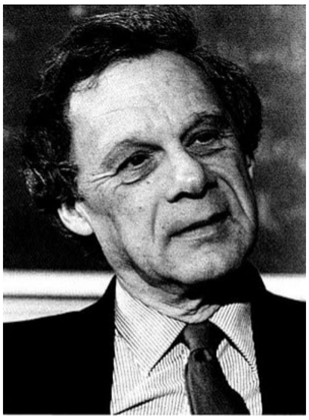
Figure 6: Interface view of Maple program.

APPENDIX II

Biography of Eberhard Hopf

EBERHARD HOPF (1902–1983) was a distinguished Austrian mathematician known for his influential work in pure and applied mathematics. He held academic roles in Europe and the U.S., notably at Indiana University (1949–1972), where he was a Research Professor and long-time journal editor. His major contributions include the Hopf Maximum Principle and foundational work in partial differential equations, ergodic theory, and the Navier–Stokes equations, greatly advancing mathematical physics.

Biography of Julian D. Cole



JULIAN D. COLE (1925–1999) was a renowned mathematician in engineering and aeronautics, recognized for his work on high-speed aerodynamic flows and modern perturbation methods for nonlinear problems. He held degrees from Cornell and Caltech and worked at institutions such as Caltech, UCLA, and RPI. Cole played a key role in developing shock-free transonic airfoils with Hans Liepmann and later helped revolutionize transonic flow calculations with Earll Murman at Boeing. His influence extended beyond aeronautics to fields like bioengineering and semiconductor physics.

Biography of Johannes Martinus Burgers

JOHANNES MARTINUS BURGERS was born in Arnhem in The Netherlands on January 13, 1895. He received his primary and secondary education in Arnhem from 1901 to 1912. To qualify for a university education he took supplementary courses in Latin and Greek from 1912 to 1914 and started studying advanced mathematics and subjects related to theoretical physics. From 1914 to 1918 Jan Burgers studied at the University of Leiden.
An account of his environment at home and about his education can be found in a set of autobiographical notes. Additional information about the interactions of Jan Burgers with his PhD advisor, Paul Ehrenfest, at the University of Leiden can be found in he potential of Jan Burgers as an outstanding scholar was recognized early and he started to work as a Professor of Aerodynamics and Hydrodynamics at the Technical University in Delft in 1918, two months before he received his PhD in the Physical and Mathematical Sciences from the University of Leiden under the supervision of Paul Ehrenfest. After an impressive career in fluid mechanics in The Netherlands, Jan Burgers became a Research Professor at the Institute for Fluid Dynamics and Applied Mathematics (IFDAM) of the University of Maryland in College Park, MD in 1955.

Biography of Jean Baptiste Joseph Fourier

JEAN-BAPTISTE JOSEPH FOURIER (1768–1830) was a French mathematician and physicist known for developing Fourier series and the Fourier transform, which break down complex functions into sine and cosine waves—key tools in science and engineering. He also formulated Fourier's Law of Heat Conduction, laying the groundwork for thermodynamics.

Fourier was the first to describe the greenhouse effect, suggesting the atmosphere traps heat. He joined Napoleon's expedition to Egypt, served as a governor, and conducted major research on heat propagation in Grenoble. His influential 1822 work, The Analytic Theory of Heat, introduced the heat equation and dimensional analysis.

Fourier's legacy lives on in mathematical physics, with his name engraved on the Eiffel Tower and an Egyptian-style tomb in Père Lachaise Cemetery honoring his contributions.