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Existence, uniqueness and Ulam-Hyers stability results for certain generalized fractional differential equations

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Dedication

We dedicate this work:

To our parents.

To our sisters and brothers.

To all our families, and our friends.

To our proficient supervisor Dr Kenef Asma.

Contents

ملخص

نركز اهتمامنا في هذا العمل على دراسة بعض المسائل الحدية ذات المعادلات التفاضلية من الرتب الكسرية حيث كان البعض منها هجينة من النوع الاول. تضمنت الدراسة وجود ووحدانية الحل باستخدام بعض نظريات النقطة الصامدة بالاضافة لدراسة الاستقرار وفقا لمفهوم اولام هايرز.

الكلمات المفتاحية: التكامل الكسري المعمم لريمان ليوفيل، المشتقات الكسرية المعممة، وجود ووحدانية الحل، نظرية النقطة الصامدة لبناخ، استقرار اولام هايرز.

Résumé

Dans ce travail, notre attention est concentrée sur l'étude de certains problèmes aux limites impliquant des équations différentielles fractionnaires, dont certaines sont de type hybride de premier ordre. L'étude comprend l'existence et l'unicité des solutions en utilisant certains théorèmes du point fixe, ainsi que l'investigation de la stabilité selon Ulam-Hyers.

Mots-clé: *Intégrale fractionnaire généralisée de Riemann-Liouville, dérivées fractionnaires généralisées, existence et unicité de la solution, théorème du point fixe de Banach, stabilité d'Ulam-Hyers.*

Abstract

In this work, our focus is on studying certain boundary value problems involving fractional differential equations, some of which are of the first hybrid type. The study includes the existence and uniqueness of solutions using some fixed point theorems, in addition to investigating stability according to the Ulam-Hyers.

***Keywords**: Generalized Riemann–Liouville fractional integral, generalized fractional derivatives, existence and uniquness of solution, Banach fixed point theorem, Ulam-Hyers stability.*

Introduction

Fractional calculus is a branch of mathematics that focuses on the properties of derivatives and integrals of fractional orders in $\mathbb{R}$ or $\mathbb{C}$. This field includes methods and approaches for studying the existence and properties of solutions to fractional differential equations, which involve fractional derivatives.

Fractional differential equations appear in various research fields and they are used to model many phenomena in physics, chemistry, mechanics, electricity, biology, as well as in control, signal processing, and image processing[15].

The theory of fractional calculus begin almost at the same time as classical calculus theory. On September 30, 1695, Marquis de L'Hôpital sent a letter to Gottfried Wilhelm Leibniz inquiring about a remark related to his research on the derivative of order n for the function f definded $f(x) = x$ if $n = \frac{1}{2}$. The response was, "The question seems paradoxical. Nevertheless, it is possible to find a result if $n = \frac{1}{2}$". After that, researches in the subject continued to develop. The following figure illustrates the evolution of fractional calculus theory from the year 1650 to 1971 and highlights the contributing scientists.

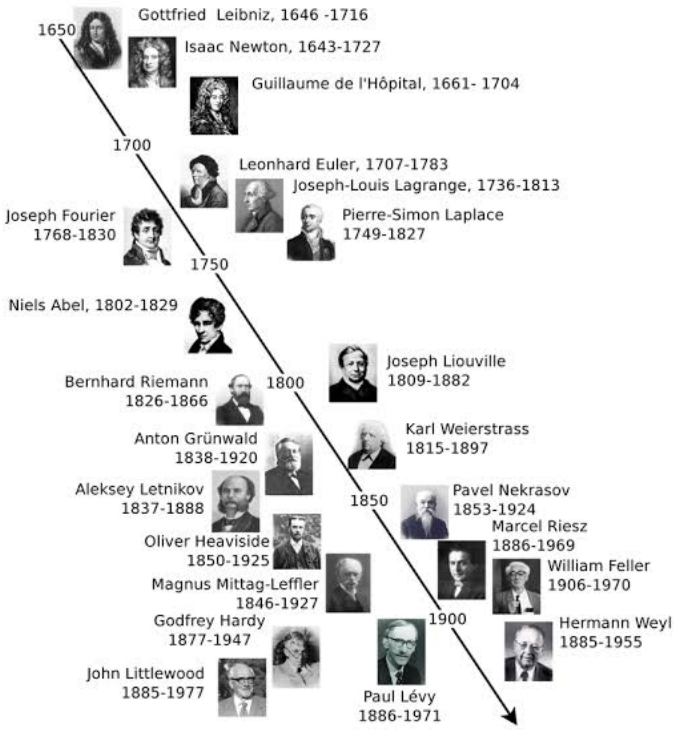

Figure 1: Timeline of main scientists in the area of fractional calculus.

Due to the importance of fractional derivatives and integrals, as well as their applications, the topic of fractional calculus has attracted the interest of scientists and researchers. Consequently, in the recent years, numerous articles and books have been written on this subject. The study of the existence and uniqueness of solutions to differential equations has been a very active research field in mathematics. Many methods are used to demonstrate the existence and uniqueness of the solution, such as the method of upper and lower solutions, Mawhin's theory, fixed point theorems..., and to calculate the solutions of these equations, whether through analytical methods or numerical methods, it is necessary to specify certain conditions. These methods are not free from errors resulting from the measurement of experimental results or performing calculations on them. Any disturbance, no matter how small, in the differential equation can lead to a change in the behavior of the solution of the equation. This change may be significant to the extent that the solution no longer describes the phenomenon accurately or even approximately.

There are many types of fractional derivatives, one of the most important is the ψ*-Riemann-Liouville and* ψ*-Caputo fractional derivatives* $[3, 4, 6, 10, 14]$*, which have gained the attention of many researchers. While both types have their uses and benefits, the Caputo derivative is preferred in fields that require standard initial conditions, such as physical and* *engineering models. On the other hand, the Riemann-Liouville derivative can be useful in pure mathematical and theoretical applications.*

Recently, a new class of mathematical modelings based on hybrid fractional differential equations, which can be employed in modeling and describing non-homogeneous physical phenomena that take place in their form [1].

Therefore, studying the conditions that prevent solutions from deviating from the desired behavior after disturbances to the equation is of paramount importance when studying the practical applications of differential equations. This is known as stability according to Ulam-Hyers. In our memorandum, we will focus on addressing the existence and uniqueness of the solution using the Banach fixed point theorem . Additionally, we will study the stability according to Ulam-Hyers of the solution for some problems with fractional differential equations, some of them are hybrid. This work has been divided into three chapters as follows:

In the first Chapter(1) *we presented some general concepts, definitions, and some properties of fractional integration and differentiation, in addition to Banach fixed point theorem that we will rely on in the upcoming chapters.*

In the second Chapter(2) *we studied the existence, uniqueness, and stability of solutions for the following problem with the first type hybrid fractional differential equations.*

$$\begin{cases} D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right)+g\left(t,x(t),D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right)\right)=0, t\in J,\\ \lim\limits_{t\to 0}\left[(\varphi(t)-\varphi(0))^{2-\alpha}\dfrac{x(t)}{f(t,x(t))}\right]=0,\\ x(1)=\eta f(1,x(1)). \end{cases}$$

Where

- $J=[0,1], \eta\in\mathbb{R}$.
- $D_{0^+}^{\alpha,\varphi}$ *is the* φ*-Riemann-Liouville fractional derivative, with* $1<\alpha<2$.
- *The function* $\varphi:[0,1]\to[0,1]$ *is a strictly increasing function, such that* $\varphi\in C^2[0,1], \varphi'(x)\neq 0$ *for all* $x\in[0,1]$.
- f *and* g *are two functions satisfying certain conditions, which we will mention later.*

The third Chapter(3) *concerns the existence, uniqueness, and stability of solutions for the* *boundry value problem with fractional differential equations.*

$$\begin{cases} {}^{c}D_{0^+}^{\alpha,\varphi} z(x) - f(x, z(x)) = 0, \quad x \in [0,1], \\ z(0) + z'(0) = 0, \\ z(1) + z'(1) = 0. \end{cases}$$

Where

- ${}^{c}D_{0^+}^{\alpha,\varphi}$ *is the* φ*-caputo fractional derivative, with* $1 < \alpha < 2$.
- *The function* $\varphi : [0,1] \to [0,1]$ *is a strictly increasing function, such that* $\varphi \in C^2[0,1]$, $\varphi'(x) \neq 0$, *for all* $x \in [0,1]$.
- f *is a function satisfying certain conditions, which we will mention later.*

CHAPTER 1

Preliminaries

In this chapter, we will introduce the necessary concepts for a proper understanding of this work. It is divided into four parts. The first part contains a brief overview of function spaces. The second part is dedicated to introduce definitions and properties of the Riemann-Liouville and Caputo fractional derivatives that are most often used in applications. The third part is devoted to introduce the Ulam-Hyers stability theorem. End the chapter with Banach fixed point theorem used in this work.

1.1 Generalities about Function Spaces

Let $\Omega = [a, b], (-\infty < a < b < +\infty)$ *be a finite interval of the real axis* $\mathbb{R} = (-\infty, +\infty)$.

- *Let* $C(\Omega, \mathbb{R})$ *be the Banach space of all continuous functions from* Ω *to* $\mathbb{R}$ *with the norm*

$$\|u\|_{C(\Omega,\mathbb{R})} = \max_{t\in\Omega} |u(t)|.$$

- *We donate by* $C^m(\Omega, \mathbb{R})$ *the space of functions* f *that are* m *times continuosly differentiable on* Ω *with the norm*

$$\|u\|_{C^m(\Omega,\mathbb{R})} = \sum_{k=0}^{m} \|u^{(k)}\|_{C(\Omega,\mathbb{R})}, m \in \mathbb{N}.$$

For more details we refer[10]

1.2 Generalities about Functional Analysis

Let Ω=$[a, b]$ be a finite interval of the real axis $\mathbb{R}$. Let $(E, \|.\|)$ a Banach space.

Definition 1.1. We say that an operator $A : E \to E$ is a contraction on E if there existes a real number k such that $0 < k < 1$ satisfies

$$\forall x, y \in E, \|Ax - Ay\| \leq k\|x - y\|.$$

1.3 Fractional calculus

Let $\Omega = [a, b], (-\infty < a < b < +\infty)$, be a finite interval on $\mathbb{R}$.

1.3.1 Fractional integral

Definition 1.2. [11,14] Let $z : \Omega \longrightarrow \mathbb{R}$ be an integrable function and $\varphi : \Omega \longrightarrow \mathbb{R}$ be an increasing function such that $\varphi'(x) \neq 0$, for all $x \in \Omega$. The φ-Riemann-Liouville fractional integral of z of order α is defined as follows:

$$I_{a^+}^{\alpha,\varphi} z(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \varphi'(v) \left(\varphi(x) - \varphi(v)\right)^{\alpha-1} z(v) dv,$$

such that $\Gamma(.)$ is defined as follows:

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \exp(-t) dt, \alpha > 0,$$

which achieves some properties, including

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

In particular when we choose $\varphi(x) = x$, for all $x \in \Omega$, we obtain a well-known fractional integral called Riemann-Liouville fractional integral defined by

$$I_{a^+}^{\alpha} z(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - v)^{\alpha-1} z(v) dv.$$

Example 1.3. We defined the function z as follows

$$z(x) = c, \ x \in \Omega.$$

Where $c \in \mathbb{R}$. The φ-Riemann-Liouville fractional intrgral of the function z is

$$(I_{a^+}^{\alpha,\varphi} z)(x) = \frac{c}{\Gamma(\alpha+1)}(\varphi(x) - \varphi(v))^{\alpha},\ x \in \Omega.$$

Lemma 1.4. *[3, 4] The following properties are valid for fractional integrals*

(1) *If $\alpha, \beta > 0$, then the equation*

$$I_{a^+}^{\alpha,\varphi} I_{a^+}^{\beta,\varphi} z(x) = I_{a^+}^{\alpha+\beta,\varphi} z(x).$$

is satisfied at almost every point $x \in \Omega$. If $\alpha + \beta > 1$ then the equation above holds at any point of Ω.

(2) *The φ-Riemann-Liouville fractional integral of order $\alpha > 0$ is a linear operator.*

1.3.2 Fractional derivatives

Definition 1.5. [13] The φ-Riemann-Liouville fractional derivative of order α of a function z corresponding to φ-Riemann-Liouville fractional integral is defined as follows:

$$D_{a^+}^{\alpha,\varphi} z(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n \int_a^x \varphi'(v) \left(\varphi(x) - \varphi(v) \right)^{n-\alpha-1} z(v) dv,\ x \in \Omega.$$

with $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of α, $n = \alpha$ for $\alpha \in \mathbb{N}$.
In particular when we choose $\varphi(x) = x$, for all $x \in \Omega$, we obtain a well-known fractional derivative called Riemann-Liouville fractional derivative defined by

$$D_{a^+}^{\alpha} z(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-v)^{n-\alpha-1} z(v) dv.$$

Definition 1.6. [3, 4]The φ-Caputo fractional derivative of order α is defined as follows:

$${}^c D_{a^+}^{\alpha,\varphi} z(x) = D_{a^+}^{\alpha,\varphi} \left[z(x) - \sum_{k=0}^{n-1} \frac{z_{\varphi}^{[k]}(a)}{k!} \left(\varphi(x) - \varphi(a) \right)^k \right], x \in \Omega.$$

where

$$n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}, n = \alpha \text{ for } \alpha \in \mathbb{N},$$

and

$$z_{\varphi}^{[k]}(x) = \left(\frac{1}{\varphi'(x)} \frac{d}{dt} \right)^k z(x), x \in \Omega.$$

In particular when we choose $\varphi(x) = x$, for all $x \in \Omega$, we obtain a well-known fractional derivative called Caputo fractional derivative defined by

$$^{c}D^{\alpha}_{a+}z(x) = D^{\alpha}_{a+}\left[z(x) - \sum_{k=0}^{n-1}\frac{z^{(k)}(a)}{k!}(x-a)^{k}\right].$$

Theorem 1.7. *[3, 4, 13] Given a function $f \in C(\Omega, \mathbb{R})$ and $1 < \alpha < 2$, we have*

$$I^{\alpha,\varphi}_{a^+}D^{\alpha,\varphi}_{a^+}f(x) = f(x) + c_1(\varphi(x)-\varphi(a))^{\alpha-1} + c_2(\varphi(x)-\varphi(a))^{\alpha-2},\ x \in \Omega,\ c_1, c_2 \in \mathbb{R}.$$

$$I^{\alpha,\varphi}_{a^+}{}^{c}D^{\alpha,\varphi}_{a^+}f(x) = f(x) + k_0 + k_1(\varphi(x)-\varphi(a)),\ x \in \Omega,\ k_0, k_1 \in \mathbb{R}.$$

Theorem 1.8. *[3, 4, 13] Given a function $f \in C(\Omega, \mathbb{R})$ and $\alpha > 0$, we have*

$$D^{\alpha,\varphi}_{a^+}I^{\alpha,\varphi}_{a^+}f(x) = f(x),\ x \in \Omega.$$
$$^{c}D^{\alpha,\varphi}_{a^+}I^{\alpha,\varphi}_{a^+}f(x) = f(x),\ x \in \Omega.$$

1.4 Ulam-Hyers stability

To define the concept of stability according to Ulam-Hyers, we rely on the definitions found in the reference [7].
Let the following fractional differential equation be:

$$^{c}D^{\alpha,\varphi}_{a^+}u(t) = f(t, u(t)),\ t \in J. \tag{1.1}$$

Definition 1.9. We say that equation (1.1) is Ulam-Hyers stable if there exists $C_f > 0$ for all $\varepsilon > 0$ and for all $y \in C(J, \mathbb{R})$ solution of the inequality:

$$\left|^{c}D^{\alpha,\varphi}_{a^+}y(t) - f(t, y(t))\right| \leq \varepsilon,\ t \in J. \tag{1.2}$$

Then there exists $u \in C(J, \mathbb{R})$ a solution of equation (1.1) , that sutisfies:

$$|y(t) - u(t)| \leq C_f\varepsilon.$$

Remark 1.10. The function $y \in C(J, \mathbb{R})$ is the solution of the inequality (1.2) if and only if there exists a function $\psi \in C(J, \mathbb{R})$ (related to y) such that

(1) $|\psi(t)| < \varepsilon$, for all $t \in J$.

(2) $^{c}D^{\alpha,\varphi}_{a^+}y(t) = f(t, y(t)) + \psi(t),\ t \in J.$

1.5 Banach fixed point theorem

The Banach fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points.

Theorem 1.11. *[5] Let Ω be a non-empty closed subset of a Banach space E, then any contraction A of Ω into itself has a unique fixed point. i.e.,*

$$\exists!\, x \in E;\ Ax = x.$$

CHAPTER 2

Existence, uniqueness and stability of some first-type hybrid generalized problems.

2.1 Introduction

Due to the significant importance that fractional calculus and integration have gained in recent years, many scientists and researchers in various fields have turned to them, as they possess numerous applications in various domains. For instance, studies in physics, mechanics, optics..., for more information we refer [9]
Another intresting class of problems involves hybrid differential equations, where the problem of the following form was stadied in the refrence [16].

$$\begin{cases} D^{\alpha}_{0^+}\left(\frac{x(t)}{g(t,x(t))}\right) = f(t,x(t)),\ t \in [0,T], \\ x(0) = 0. \end{cases}$$

where:

- $T > 0$.
- $D^{\alpha}_{0^+}$ *is the Riemann-Liouville fractional derivative of order* $0 < \alpha < 1$.
- $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $g : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}^*$ *are two continous functions.*

By using the Banach fixed-point theorem, the researchers proved the existence and uniqueness of the solution of the preceding problem.

In this chapter, we will present the study of the existence, uniqueness and stability of the solution of the problem involving the following first type hybrid fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right)+g\left(t,x(t),D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right)\right)=0,\ t\in J, \\ \lim\limits_{t\to 0}\left[(\varphi(t)-\varphi(0))^{2-\alpha}\dfrac{x(t)}{f(t,x(t))}\right]=0, \\ x(1)=\eta f(1,x(1)). \end{cases} \tag{P}$$

where:

- $J=[0,1]$ *and* $\eta\in\mathbb{R}$.
- $D_{0^+}^{\alpha,\varphi}$ *is the* φ*-Riemann-Liouville fractional derivative of order* $1<\alpha<2$.
- $f\in C(J\times\mathbb{R},\mathbb{R})$ *and* $g\in C(J\times\mathbb{R}^2,\mathbb{R})$ *are two nonlinear functions.*
- $\varphi\colon J\to J$ *is a strictly increasing function such that* $\varphi\in C^2(J,\mathbb{R}), \varphi'(t)\neq 0$ *for all* $t\in J$.

For that, we transform the deffirential equation to an equivalent integral equation, then we prove the existence of a unique solution by the help of Banach fixed point theorem, at the end we prove that the solution is Ulam-Hyers stable.

2.2 Existence, uniqueness and stability results

At first, we need to define the solution of the problem (P)

Definition 2.1. We say that the function x from $C(J, \mathbb{R})$ is a solution of the problem (P) if it satisfies the equation:

$$D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right) + g\left(t, x(t), D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right)\right) = 0,\ t \in J, \tag{2.1}$$

and the conditions :

$$\lim_{t \to 0}\left[(\varphi(t) - \varphi(0))^{2-\alpha}\frac{x(t)}{f(t,x(t))}\right] = 0, \tag{2.2}$$

$$x(1) = \eta f(1, x(1)). \tag{2.3}$$

Lemma 2.2. *Let $h\colon J \to \mathbb{R}$ be a continious function, then the function x is a solution of the fractional order differential equation :*

$$D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right) + h(t) = 0, t \in J, \tag{2.4}$$

and the initial conditions (2.2) *and* (2.3) *are satisfied if and only if x satisfied the fractional order integral equation:*

$$x(t) = \eta\gamma(t)f(t,x(t)) + \int_0^1 \varphi'(s)G(t,s)f(t,x(t))h(s)ds, t \in J.$$

where:

$$G(t,s) = \frac{\gamma(t)}{\Gamma(\alpha)}\begin{cases}(\varphi(1) - \varphi(s))^{\alpha-1} - \dfrac{1}{\gamma(t)}(\varphi(t) - \varphi(s))^{\alpha-1}, & 0 \le s < t \le 1,\\ (\varphi(1) - \varphi(s))^{\alpha-1}, & 0 \le t < s \le 1.\end{cases} \tag{2.5}$$

with:

$$\gamma(t) = \frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}},\ t \in J.$$

Proof. Aplying the fractional integral $I_{0^+}^{\alpha,\varphi}$ to the equation (2.1) we find:

$$\frac{x(t)}{f(t,x(t))} = -I_{0^+}^{\alpha,\varphi}h(t) + c_0(\varphi(t) - \varphi(0))^{\alpha-1} + c_1(\varphi(t) - \varphi(0))^{\alpha-2},$$

by using the condition (2.2) we obtain $c_1 = 0$, then:

$$\frac{x(t)}{f(t,x(t))} = -I_{0^+}^{\alpha,\varphi}h(t) + c_0(\varphi(t) - \varphi(0))^{\alpha-1}.$$

On the other hand, by using the condition (2.3) we get:

$$-I_{0^+}^{\alpha,\varphi}h(1) + c_0(\varphi(1)-\varphi(0))^{\alpha-1} = \frac{x(1)}{f(1,x(1))} = \eta,$$

so:

$$c_0 = (I_{0^+}^{\alpha,\varphi}h(1)+\eta)\frac{1}{(\varphi(1)-\varphi(0))^{\alpha-1}}.$$

By substituation we find

$$\begin{aligned} x(t) &= f(t,x(t))\left[-I_{0^+}^{\alpha,\varphi}h(t) + (I_{0^+}^{\alpha,\varphi}h(1)+\eta)\frac{1}{(\varphi(1)-\varphi(0))^{\alpha-1}}(\varphi(t)-\varphi(0))^{\alpha-1}\right] \\ &= -\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t \varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}h(s)ds \\ &+ f(t,x(t))\left(\frac{1}{\Gamma(\alpha)}\int_0^1 \varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}h(1)ds+\eta\right)\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{(\varphi(1)-\varphi(0))^{\alpha-1}}. \end{aligned}$$

Finally the solution satisfied the equation:

$$x(t) = \eta\gamma(t)f(t,x(t)) + \int_0^1 \varphi'(s)G(t,s)f(t,x(t))h(s)ds.$$

The converse implication is clear from Theorem (1.8).

□

Lemma 2.3. *The following properties are satisfied by the Green function G defined by equation* (2.5)*:*

(i) $G(t,s) \geq 0$, *for all* $t,s \in J$.

(ii) $G(t,s) \leq \dfrac{(\varphi(1)-\varphi(0))^{\alpha-1}}{\Gamma(\alpha)}$, *for all* $t,s \in J$.

Proof. **We prove (i):** Since φ is a strictly increasing function, we have $\varphi(1) > \varphi(s)$ whenever $s \leq 1$, so one can easly conclude from equation (2.8) that $G(t,s) \geq 0$ for $0 \leq t < s \leq 1$, and for $0 \leq s < t \leq 1$, we consider $\varphi(1)-\varphi(t) \geq 0$, multiplying both sides by $\varphi(s)-\varphi(0) \geq 0$, we have $(\varphi(1)-\varphi(t))(\varphi(s)-\varphi(0)) \geq 0$, which implies

$$(\varphi(1)\varphi(s) + \varphi(0)\varphi(t) \geq \varphi(1)\varphi(0) + \varphi(t)\varphi(s),$$

multiplying both sides of the above inequality by (-1), we get

$$-\varphi(1)\varphi(0) - \varphi(t)\varphi(s) \leq -\varphi(1)\varphi(s) - \varphi(0)\varphi(t),$$

adding $\varphi(1)\varphi(t) + \varphi(0)\varphi(s)$ to both sides, we obtain

$$(\varphi(t) - \varphi(0))(\varphi(1) - \varphi(s)) \leq (\varphi(t) - \varphi(s))(\varphi(1) - \varphi(0)),$$

raising both sides to the power $(\alpha - 1)$ and then dividing by $(\varphi(t) - \varphi(0))^{\alpha-1}$, we get

$$(\varphi(t) - \varphi(s))^{\alpha-1} - \frac{(\varphi(1) - \varphi(0))^{\alpha-1}}{(\varphi(t) - \varphi(0))^{\alpha-1}}(\varphi(t) - \varphi(s))^{\alpha-1} \geq 0,$$

hence $G(t, s) \geq 0$, for all $t, s \in J$.

We prove (ii): Since φ is a strictly increasing function, we have $\varphi(t) - \varphi(0) \leq \varphi(1) - \varphi(0)$, for all $t \in J$, which implies $\gamma(t) \leq 1$, for $0 \leq t < s \leq 1$ we can easily conclude that:

$$\frac{\gamma(t)}{\Gamma(\alpha)}(\varphi(1) - \varphi(s))^{\alpha-1} \leq \frac{(\varphi(1) - \varphi(0))^{\alpha-1}}{\Gamma(\alpha)},$$

and for $0 \leq s < t \leq 1$:

$$\begin{aligned}\frac{\gamma(t)}{\Gamma(\alpha)}(\varphi(1) - \varphi(s))^{\alpha-1} - \frac{1}{\gamma(t)}(\varphi(t) - \varphi(s))^{\alpha-1} &\leq \frac{1}{\Gamma(\alpha)}\big((\varphi(1) - \varphi(0))^{\alpha-1} \\ &\quad - \frac{(\varphi(1) - \varphi(0))^{\alpha-1}(\varphi(t) - \varphi(s))^{\alpha-1}}{(\varphi(t) - \varphi(0))^{\alpha-1}}\big) \\ &\leq \frac{1}{\Gamma(\alpha)}(\varphi(1) - \varphi(0))^{\alpha-1}\left(1 - \frac{(\varphi(t) - \varphi(s))^{\alpha-1}}{(\varphi(t) - \varphi(0))^{\alpha-1}}\right) \\ &\leq \frac{1}{\Gamma(\alpha)}(\varphi(1) - \varphi(0))^{\alpha-1},\end{aligned}$$

hence

$$G(t, s) \leq \frac{(\varphi(1) - \varphi(0))^{\alpha-1}}{\Gamma(\alpha)},\ t, s \in J.$$

□

Let us define the operator $T\colon C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by:

$$T(x(t) = \eta\gamma(t)f(t, x(t)) + \int_0^1 \varphi'(s)G(t, s)\sigma_x(s)f(t, x(t))ds,\ t \in J. \tag{2.6}$$

such that :

$$\sigma_x(t) = g\left(t, x(t), D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t, x(t))}\right)\right), t \in J.$$

Where $\mathrm{C}(\mathrm{J}, \mathbb{R})$ *is equipped with the norm:*

$$\|x\|_\infty = \max_{t\in J}|x(t)|.$$

To prove the existence of the solution of the problem (P) it is sifficient to prove that the operator T has a unique fixed piont.

2.2.1 Existence and uniqueness results

In this subsection we will study the existence and the uniqueness of the solution of the problem (P) using the Banach fixed point theorem under certain conditions imposed on the functions g and f. Therefor, we impose the following conditions:

(H_1) *There exists a constant $M_g \in \mathbb{R}_+^*$ such that:*

$$|g(t,u,v)| \leq M_g,$$

for all $t \in J$ and $u, v \in \mathbb{R}$.

(H_2) *There exists a constant $M_f \in \mathbb{R}_+^*$ such that:*

$$|f(t,u)| \leq M_f,$$

for all $t \in J$ and $u \in \mathbb{R}$.

(H_3) *There exists constants $k_2 \in (0,1)$ and $k_1, k_3 \in \mathbb{R}_+^*$ such that :*

$$|g(t,u,v) - g(t,\bar{u},\bar{v})| \leq k_1|u-\bar{u}| + k_2|v-\bar{v}|,$$

and

$$|f(t,u) - f(t,\bar{u})| \leq k_3|u-\bar{u}|$$

for all $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

Theorem 2.4. *Assuming that conditions $(H_1) - (H_3)$ are hold and if*

$$0 < \rho < 1, \tag{2.7}$$

such that:

$$\rho = |\eta|k_3 + \frac{M_f k_1}{(1-k_2)\Gamma(\alpha)}(\varphi(1)-\varphi(0))^{\alpha} + \frac{M_g k_3}{\Gamma(\alpha)}(\varphi(1)-\varphi(0))^{\alpha}.$$

Then the problem (P) admits a unique solution in $\mathrm{C}(J,\mathbb{R})$.

Proof. It is clear that the fixed point of T is a solution of the problem (P). We will now demonstrate the existence of a fixed point for T through the proof that T is a contraction.

Let $x,\ y \in C(J, \mathbb{R})$, then, for all $t \in J$, we have:

$$\begin{aligned}
|T(x(t)) - T(y(t))| &= |\eta\gamma(t)f(t,x(t)) + \int_0^1 \varphi'(s)G(t,s)\sigma_x(s)f(t,x(t))ds \\
&- \eta\gamma(t)f(t,y(t)) - \int_0^1 \varphi'(s)G(t,s)\sigma_y(s)f(t,y(t))ds| \\
&\leq |\eta|\gamma(t)|f(t,x(t)) - f(t,y(t))| + |f(t,x(t) \int_0^1 \varphi'(s)G(t,s)\sigma_x(s)ds \\
&- f(t,y(t)) \int_0^1 \varphi'(s)G(t,s)\sigma_y(s)ds| \\
&\leq |\eta|\gamma(t)|f(t,x(t)) - f(t,y(t))| + |f(t,x(t)) \int_0^1 \varphi'(s)G(t,s)\sigma_x(s)ds \\
&- f(t,y(t)) \int_0^1 \varphi'(s)G(t,s)\sigma_x(s)ds + f(t,y(t)) \int_0^1 \varphi'(s)G(t,s)\sigma_x(s)ds \\
&- f(t,y(t)) \int_0^1 \varphi'(s)G(t,s)\sigma_y(s)ds| \\
&\leq |\eta|\gamma(t)|f(t,x(t)) - f(t,y(t))| + |f(t,x(t)) - f(t,y(t))| \int_0^1 \varphi'(s)G(t,s)|\sigma_x(s)|ds \\
&+ |f(t,y(t))| \int_0^1 \varphi'(s)G(t,s)|\sigma_x(s) - \sigma_y(s)|ds \\
&\leq |\eta|\gamma(t)k_3|x(t) - y(t)| + \frac{k_3 M_g(\varphi(1) - \varphi(0))^\alpha}{\Gamma(\alpha)}|x(t) - y(t)| \\
&+ \frac{M_f}{\Gamma(\alpha)} \int_0^1 \varphi'(s)G(t,s)|\sigma_x(s) - \sigma_y(s)|ds.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
|\sigma_x(t) - \sigma_y(t)| &\leq |g(t,x(t),\sigma_x(t)) - g(t,y(t),\sigma_y(t))| \\
&\leq k_1|x(t) - y(t)| + k_2|\sigma_x(t) - \sigma_y(t)| \\
&\leq \frac{k_1}{1-k_2}|x(t) - y(t)|.
\end{aligned}$$

Through substitution we obtain:

$$\begin{aligned}
|T(x(t)) - T(y(t))| &\leq |\eta|\gamma(t)k_3|x(t) - y(t)| + \frac{k_3 M_g(\varphi(1) - \varphi(0))^\alpha}{\Gamma(\alpha)}|x(t) - y(t)| \\
&+ \frac{k_1 M_f}{(1-k_2)\Gamma(\alpha)} \int_0^1 \varphi'(s)G(t,s)|x(t) - y(t)|ds \\
&\leq k_3\left(|\eta| + \frac{M_g}{\Gamma(\alpha)}\right)\|x-y\|_\infty + \frac{k_1 M_f}{(1-k_2)\Gamma(\alpha)}\|x-y\|_\infty \\
&\leq \left(k_3|\eta| + \frac{k_3 M_g}{\Gamma(\alpha)}(\varphi(1) - \varphi(0))^\alpha + \frac{k_1 M_f}{(1-k_2)\Gamma(\alpha)}(\varphi(1) - \varphi(0))^\alpha\right)\|x-y\|_\infty.
\end{aligned}$$

Therefore:

$$\|T(x(.)) - T(y(.))\|_\infty \leq \rho\|x-y\|_\infty.$$

The operator T is a contraction, and according to Banach fixed-point theorem, the problem (P) admits a unique solution in $C(J,\mathbb{R})$. □

Example 2.5. Corresponding to the proposed problem, we provide the following example that demonstrate consistency to the main theorems.

$$\begin{cases} D_{0^+}^{\frac{3}{2},\frac{e^t}{3}}\left(\frac{x(t)}{1+\frac{1}{10}e^{-t^2}\cos(x(t))}\right)+\frac{1}{5}t\cos(x(t))+\frac{1}{5+\left|D_{0^+}^{\frac{3}{2},\frac{e^t}{3}}\left(\frac{x(t)}{1+\frac{1}{10}e^{-t^2}\cos(x(t))}\right)\right|}=0, \\ \lim_{t\to 0}(\frac{e^t}{3}-\frac{1}{3})^{\frac{1}{2}}x(t)=1+\frac{1}{10}\cos(x(0)), \\ x(1)=1+\frac{1}{10}e^{-1}\cos(x(1)). \end{cases} \tag{P_1}$$

Where f and g two continuous functions defined for all $t\in J$ as follows:

$$f(t,x(t))=1+\frac{1}{10}e^{-t^2}\cos(x(t)),$$
$$g\left(t,x(t),D_{0^+}^{\alpha,\varphi}\left(\frac{x(t)}{f(t,x(t))}\right)\right)=\frac{1}{5}t\cos(x(t))+\frac{1}{5+\left|D_{0^+}^{\frac{3}{2},\frac{e^t}{3}}\left(\frac{x(t)}{1+\frac{1}{10}e^{-t^2}\cos(x(t))}\right)\right|}.$$

Let's put:

$$f(t,u)=1+\frac{1}{10}e^{-t^2}\cos(u),$$
$$g(t,u,v)=\frac{1}{5}t\cos(u)+\frac{1}{5+|v|}.$$

For all $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in J$ we have:

$$|f(t,u)-f(t,\bar{u})|\le\frac{1}{10}|u-\bar{u}|,$$
$$|g(t,u,v)-g(t,\bar{u},\bar{v})|\le\frac{1}{5}|u-\bar{u}|+\frac{1}{25}|v-\bar{v}|,$$
$$|f(t,u)|\le\frac{11}{10},$$
$$|g(t,u,v)|\le\frac{2}{5},$$

So, all conditions (H_1), (H_2), and (H_3) are satisfied, and $\rho=0,16374<1$ with:

$$k_1=\frac{1}{5},k_2=\frac{1}{5},k_3=\frac{1}{25},M_f=\frac{2}{5},M_g=\frac{11}{10}.$$

Consequently, from Theorem (2.4) the problem (P_1) admits a unique solution in $C(J,\mathbb{R})$.

2.2.2 Ulam-Hyers stability results

In this section, we will study a type of stability for problem (P), known as Ulam-Hyers stability.

Lemma 2.6. *Let's assume that condition (H_2) is satisfied, and if y is a solution of the fractional differential inequality given, for $\varepsilon > 0$:*

$$\left| D_{0^+}^{\alpha,\varphi}\left(\frac{y(t)}{f(t,y(t))}\right) + g(t, y(t), D_{0^+}^{\alpha,\varphi}\left(\frac{y(t)}{f(t,y(t))}\right)\right| < \varepsilon,\ t \in J. \tag{2.8}$$

Then y is a solution of the following inequality:

$$|y(t) - T(y(t))| < \frac{M_f(\varphi(1) - \varphi(0))^\alpha}{\Gamma(\alpha)}\varepsilon,\ t \in J.$$

Proof. Let $y \in C(J,\mathbb{R})$ be a solution of the inequality (2.8), for $\varepsilon > 0$, using the Lemma (2.2) and Remark (1.10) related to the continuous function ψ where $|\psi(t)| < \varepsilon$, for all $t \in J$.

Then we have:

$$y(t) = \eta\gamma(t)f(t,y(t)) + f(t,y(t))\int_0^1 \varphi'(s)G(t,s)\left[g\left(s, y(s), D_{0^+}^{\alpha,\varphi}\left(\frac{y(s)}{f(s,y(s))}\right)\right) + \psi(s)\right]ds.$$

According to Remark (1.10) and condition (H_2), we obtain:

$$\begin{aligned}
|y(t) - T(y(t))| &= \left|f(t,y(t))\int_0^1 \varphi'(s)G(t,s)\psi(s)ds\right| \\
&\leq |f(t,y(t))|\int_0^1 \varphi'(s)G(t,s)|\psi(s)|ds \\
&\leq \frac{M_f(\varphi(1) - \varphi(0))^\alpha}{\Gamma(\alpha)}\varepsilon.
\end{aligned}$$

And this satisfies the inequality (2.8).

□

Theorem 2.7. *Assuming that conditions $(H_1) - (H_3)$ are satisfied along with condition (2.7), then the problem (P) is stable according to Ulam-Hyers stability.*

Proof. Under conditions $(H_1)-(H_3)$ and (2.7), the problem (P) has a unique solution in $C(J,\mathbb{R})$.

Let $y \in C(J,\mathbb{R})$ be a solution of the inequality (2.8). Then for all $t \in J$, we have:

$$\begin{aligned}
|y(t)-x(t)| &= \left|y(t) - \left[\eta\gamma(t)f(t,x(t)) + \int_0^1 \varphi'(s)G(t,s)f(t,x(t))h(s)ds\right]\right| \\
&= |y(t) - T(y(t)) + T(y(t)) - T(x(t))| \\
&\leq |y(t) - T(y(t))| + |T(y(t)) - T(x(t))| \\
&\leq \frac{M_f(\varphi(1)-\varphi(0))^\alpha}{\Gamma(\alpha)}\varepsilon + \rho\|y-x\|_\infty.
\end{aligned}$$

Therefore:

$$\|y-x\|_\infty \leq \frac{M_f(\varphi(1)-\varphi(0))^\alpha}{(1-\rho)\Gamma(\alpha)}\varepsilon.$$

Setting:

$$C_f = \frac{M_f(\varphi(1)-\varphi(0))^\alpha}{(1-\rho)\Gamma(\alpha)},$$

we obtain:

$$\|y-x\|_\infty \leq C_f\varepsilon.$$

Consequently the problem (P) is stable according to Ulam-Hyers. □

Example 2.8. Considering the problem (P_1) in the Example (2.5).

Since the conditions $(H_1)-(H_3)$ are satisfied and we have $C_f = 0.23946 > 0$.

Hense, from Theorem (2.7) the problem (P_1) is stable according to Ulam-Hyers.

CHAPTER 3

Existence, uniqueness and stability for certain generalized fractional boundary value problems

3.1 Introduction

As we previously mentioned, fractional differential equations are a natural generalization of ordinary equations. Where the following problem was discussed in the refrence [8].

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) + h(t,u(t)) = 0,\ t \in [c,d], \\ u(c) = u'(c) = 0,\ u(d) = ku(\beta). \end{cases}$$

With

- $k \in \mathbb{R}$ and $\beta \in]c,d[$
- ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative of order $2 < \alpha < 3$.
- $h \in C([c,d] \times \mathbb{R}, \mathbb{R})$.

Where they prove the exictence and uniquness of the solution for the above problem using Banach fixed point theorem.
The purpose of this Chapter, is to establish existence, uniqueness and stability results to the following fractional bondary value problem:

$$\begin{cases} {}^{c}D_{0^+}^{\alpha,\varphi} z(x) - f(x, z(x)) = 0, \quad x \in [0,1], \\ z(0) + z'(0) = 0, \\ z(1) + z'(1) = 0. \end{cases} \tag{S}$$

where:

- ${}^{c}D_{0^+}^{\alpha,\varphi}$ is the φ-Caputo fractional derivative of order $1 < \alpha < 2$.
- $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.
- The function $\varphi : [0,1] \to [0,1]$ is a strictly increasing function, such that $\varphi \in C^2[0,1]$, $\varphi'(x) \neq 0$, for all $x \in [0,1]$.

For that, we transform the problem (S) to an equivalent integral equation, then we prove the existence of a unique solution by the help of Banach fixed point theorem, at the end we prove that the solution is Ulam-Hyers stable[4].

3.2 Existence, uniqueness and stability results

First of all, we will define the solution of the problem (S).

Definition 3.1. We say that the function z from $C([0,1],\mathbb{R})$ is a solution of the problem (S) if it satisfies the equation:

$$^{c}D_{0^+}^{\alpha,\varphi} z(x) - f(x, z(x)) = 0,\ x \in [0,1], \tag{3.1}$$

and the conditions

$$z(0) + z'(0) = 0, \tag{3.2}$$

and

$$z(1) + z'(1) = 0, \tag{3.3}$$

Lemma 3.2. *Let $g\colon [0,1] \to \mathbb{R}$ is a continious function then the function z is a solution of the fractional order differential equation:*

$$^{c}D_{0^+}^{\alpha,\varphi} z(x) - g(x) = 0,\ x \in [0.1], \tag{3.4}$$

and the initial conditions (3.2) *and* (3.3) *are satisfied if and only if z is a solution of the fractional order integral equation:*

$$z(x) = \int_0^1 W(x,v)\varphi'(v)g(v))dv, x \in [0,1],$$

where:

$$W(x,v) = \left(\Gamma(\alpha-1)[\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0)]\right)^{-1} P(x,v),\ x, v \in [0,1], \tag{3.5}$$

and

$$P(x,v) = \begin{cases} (\varphi'(0) + \varphi(0) - \varphi(x))[\varphi'(1)(\varphi(1) - \varphi(v))^{\alpha-2} + \frac{1}{\alpha-1}(\varphi(1) - \varphi(v))^{\alpha-1}] & \\ + \frac{(\varphi(1)-\varphi(0)+(\varphi'(1)-\varphi'(0))}{\alpha-1})(\varphi(x) - \varphi(v))^{\alpha-1}, & 0 \leq v < x \leq 1, \\ (\varphi'(0) + \varphi(0) - \varphi(x))[\varphi'(1)(\varphi(1) - \varphi(v))^{\alpha-2} + \frac{1}{\alpha-1}(\varphi(1) - \varphi(v))^{\alpha-1}], & 0 \leq x < v \leq 1. \end{cases}$$

Proof. We have:

$$z(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \varphi'(v)(\varphi(x) - \varphi(v))^{\alpha-1} g(v)dv + c_0 + c_1(\varphi(x) - \varphi(0)),\ x \in [0,1],$$

by differentiation we get

$$z'(x) = \frac{(\alpha-1)\varphi'(x)}{\Gamma(\alpha)} \int_0^x \varphi'(v)(\varphi(x) - \varphi(v))^{\alpha-2} g(v)dv + c_1\varphi'(x),\ x \in [0,1],$$

by using the boundary condition (3.2), we find $z(0) + z'(0) = c_0 + c_1\varphi'(0)$, then $c_0 = -c_1\varphi'(0)$.

On the other hand, using the condition (3.3), we obtain

$$\begin{aligned} z(1) + z'(1) = {} & \frac{1}{\Gamma(\alpha)} \int_0^1 \varphi'(v)(\varphi(1) - \varphi(v))^{\alpha-1} g(v)dv - c_1\varphi'(0) + c_1(\varphi(1) - \varphi(0)) \\ & + \frac{(\alpha-1)\varphi'(1)}{\Gamma(\alpha)} \int_0^1 \varphi'(v)(\varphi(1) - \varphi(v))^{\alpha-2} g(v)dv + c_1\varphi'(1), \end{aligned}$$

and from it, it follows that:

$$\begin{aligned} c_1 = {} & \frac{-1}{\Gamma(\alpha)(\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0))} \int_0^1 \varphi'(v)(\varphi(1) - \varphi(v)^{\alpha-1} g(v)dv \\ & + \frac{-(\alpha-1)\varphi'(1)}{\Gamma(\alpha)(\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0))} \int_0^1 \varphi'(v)(\varphi(1) - \varphi(v))^{\alpha-2} g(v)dv, \end{aligned}$$

by substituation:

$$\begin{aligned} z(x) = {} & \frac{1}{\Gamma(\alpha)} \int_0^x \varphi'(v)(\varphi(x) - \varphi(v))^{\alpha-1} g(v)dv \\ & + \frac{\varphi'(0)}{\Gamma(\alpha)(\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0))} \int_0^1 \varphi'(v)((\varphi(1) - \varphi(v))^{\alpha-1} g(v)dv \\ & + \frac{\varphi'(0)}{\Gamma(\alpha)(\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0))} \int_0^1 \varphi'(v)(\alpha-1)\varphi'(1)(\varphi(1) - \varphi(v))^{\alpha-2}) g(v)dv \\ & - \frac{\varphi(x) - \varphi(0)}{\Gamma(\alpha)(\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0))} \int_0^1 \varphi'(v)((\varphi(1) - \varphi(v))^{\alpha-1} g(v)dv \\ & - \frac{\varphi(x) - \varphi(0)}{\Gamma(\alpha)(\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0))} \int_0^1 \varphi'(v)(\alpha-1)\varphi'(1)(\varphi(1) - \varphi(v))^{\alpha-2} g(v)dv. \end{aligned}$$

Finaly the solution satisfied the equation

$$z(x) = \int_0^1 W(x,v)\varphi'(v)g(v)dv,\ x \in [0,1].$$

We can easily deduce the converse implication of the lemma from Theorem (1.8).

□

Lemma 3.3. *The following properties are satisfied by the function W defined by equation* (3.5)

(i) $W(x,v) \geq 0,$ *for all* $x, v \in [0,1]$.

(ii) $W(x,v) \leq \dfrac{(\varphi(1)-\varphi(0))^{\alpha-1}}{\Gamma(\alpha)},$ *for all* $x, v \in [0,1]$.

Proof. **We prove (i).**
Let $\varphi(x) \leq \varphi(0) + \varphi'(0),\ x \in [0,1]$, Since φ is a strictly increasing function, we have $\varphi(x) > \varphi(v)$, whenever $v \leq 1$, so one can easily conclude from equation (3.5) that for $0 \leq v < x \leq 1, W(x,v) \geq 0$.
And for $0 \leq x < v \leq 1$ we have $\varphi(1) > \varphi(v)$ then $W(x,v) \geq 0$.
We prove (ii)
since $\varphi(x) \leq \varphi(0) + \varphi'(0)$, we have

$$\begin{aligned}(\varphi'(0)+\varphi(0)-\varphi(x))&\left[(\varphi(1)-\varphi(v))^{\alpha-2}+\frac{1}{\alpha-1}(\varphi(1)-\varphi(v))^{\alpha-1}\right]\\&+\frac{\varphi(1)-\varphi(0)+\varphi'(1)-\varphi'(0)}{\alpha-1}(\varphi(x)-\varphi(v))^{\alpha-1}\\&\leq\frac{\varphi(1)-\varphi(0)+\varphi'(1)-\varphi'(0)}{\alpha-1}(\varphi(x)-\varphi(v))^{\alpha-1}\\&\leq\frac{\varphi(1)-\varphi(0)+\varphi'(1)-\varphi'(0)}{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}.\end{aligned}$$

Then

$$W(x,v) \leq \frac{1}{\Gamma(\alpha)}(\varphi(1)-\varphi(0))^{\alpha-1}, \quad \text{for all } x, v \in [0,1].$$

□

Let us define the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$F(z(x)) = \int_0^1 \varphi'(v)W(x,v)f(v,z(v))dv,\ x \in [0,1]. \tag{3.6}$$

where $\mathrm{C}([0,1]),\mathbb{R})$ is equipped with the norm:

$$\|z\|_\infty = \max_{x\in[0,1]}|z(x)|.$$

To prove the existence and uniquness of solution of the problem (S) it is sifficient to prove that the operator F has a unique fixed point.

3.2.1 Existence and uniqueness results

At first, we will establish the existence of a unique solution of the problem (S) using the Banach fixed point theorem under certain conditions imposed on the function f therefor, we impose the following condition

(H) : for all $u, \bar{u} \in \mathbb{R}$, $k \in \mathbb{R}_+^*$ and $x \in [0,1]$

$$|f(x,u) - f(x,\bar{u})| \leq k|u - \bar{u}|$$

Theorem 3.4. *Assuming that the function $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ satisfies the condition (H) and if*

$$\rho = \frac{k}{\Gamma(\alpha)}(\varphi(1) - \varphi(0))^{\alpha} < 1. \tag{3.7}$$

Then the problem (S) *admits a unique solution in* $C([0,1], \mathbb{R})$.

Proof. It is clear that the fixed point of F defined in (3.6) is a solution of the problem (S). We will now demonstrate the existence of a unique fixed point for F through the proof that F is a contraction.
Let $z_1, z_2 \in C([0,1], \mathbb{R})$, then for all $x \in [0,1]$, we have:

$$\begin{aligned} |F(z_1(x)) - F(z_2(x))| &= \left| \int_0^1 \varphi'(v) W(x,v) f(v, z_1(v)) dv - \int_0^1 \varphi'(v) W(x,v) f(v, z_2(v)) dv \right| \\ &\leq \left| \int_0^1 \varphi'(v) W(x,v) \left(f(v, z_1(v)) - f(v, z_2(v)\right) dv \right| \\ &\leq \frac{k(\varphi(1) - \varphi(0))^{\alpha}}{\Gamma(\alpha)} \|z_1 - z_2\|_{\infty}, \end{aligned}$$

therfore:

$$\|F(z_1(.)) - F(z_1(.))\|_{\infty} \leq \rho \|z_1 - z_2\|_{\infty}.$$

The operator F is a contraction, and according to Banach fixed point theorem, the problem (S) admits a unique solution. □

Example 3.5.

$$\begin{cases} {}^cD_{0+}^{\frac{3}{2}, \sin(x)} z(x) - \frac{1}{25} e^{-x} z(x) = 0, \ x \in [0,1], \\ z(0) + z'(0) = 0, \ \ z(1) + z'(1) = 0. \end{cases} \tag{S_1}$$

We have f a continuous function defined for all $x \in [0,1]$ and $z \in C([0,1], \mathbb{R})$ as follows:

$$f(x, z(x)) = \frac{1}{25} e^{-x} z(x).$$

Let's put: $x \in [0,1]$,

$$f(x,u) = \frac{1}{25}e^{-x}u,$$

for all $u, \bar{u} \in \mathbb{R}$ and $x \in [0,1]$, we have:

$$|f(x,u) - f(x,\bar{u})| \leq \frac{1}{25}|u - \bar{u}|,$$

so, the condition (H) is satisfied, and $\rho = 3.484 \times 10^{-2} < 1$ with $k = \frac{1}{25}$.
Consquantly, the problem (S_1) has a unique solution according to Banch fixed point theorem.

3.2.2 Ulam-Hyers stability results

In this section, we will study Ulam-Hyers stability for the problem (S).

Lemma 3.6. *Let assume that condition (H) is satisfied, and if y is a solution of the fractional differential inequality given, for $\mu > 0$:*

$$|{}^cD_0^{\alpha,\varphi}y(x) - f(x,y(x))| < \mu, \tag{3.8}$$

then y is a solution of the following inequality:

$$|y(x) - F(y(x))| < \frac{(\varphi(1)-\varphi(0))^\alpha}{\Gamma(\alpha)}\mu. \tag{3.9}$$

Proof. Let $y \in C([0,1],\mathbb{R})$ be a solution of the inequality (3.8). For $\mu > 0$, using the Lemma (3.2) and Remark (1.10) related to the continuous function ψ where $|\psi(x)| < \mu$ for all $x \in [0,1]$, then we have:

$$y(x) = \int_0^1 \varphi'(v)W(x,v)\left[f(v,y(v)) + \psi(v)\right]dv,$$

$$\begin{aligned} |y(x) - F(y(x))| &= \left|\int_0^1 \varphi'(v)W(x,v)\psi(v)dv\right| \\ &\leq \int_0^1 \varphi'(v)W(x,v)|\psi(v)|dv \\ &\leq \frac{(\varphi(1)-\varphi(0))^\alpha}{\Gamma(\alpha)}\mu. \end{aligned}$$

And this satisfies the inequality (3.9).

□

Theorem 3.7. *Assuming that conditions* (H) *and* (3.7) *are satisfied, then the problem* (S) *is Ulam-Hyers stable.*

Proof. Under conditions (H) and (3.7), the problem (S) has a unique solution in $C([0,1],\mathbb{R})$. Let $y \in C[0,1]$ be a solution of the inequality(3.8), then for $x \in [0,1]$, we have:

$$\begin{aligned}
|y(x)-z(x)| &= \left| y(x) - \left[\int_0^1 \varphi'(v) G(x,v) f(x,z(x)) dv \right] \right| \\
&= |y(x) - (F(y(x)) + (F(y(x)) - (F(z(x))| \\
&\le |y(x) - (F(y(x))| + |(F(y(x)) - (F(z(x))| \\
&\le \frac{(\varphi(1)-\varphi(0))^\alpha}{\Gamma(\alpha)} \mu + \rho \|y-z\|_\infty,
\end{aligned}$$

therefore:

$$\|y-z\|_\infty \le \frac{(\varphi(1)-\varphi(0))^\alpha}{(1-\rho)\Gamma(\alpha)} \mu,$$

setting:

$$C = \frac{(\varphi(1)-\varphi(0))^\alpha}{(1-\rho)\Gamma(\alpha)},$$

we obtain:

$$\|y-z\|_\infty \le \mu C.$$

Consequently, the problem (S) is Ulam-Hyers stable. □

Example 3.8. Considering the problem (S_1) in the example (3.5), since the condition (H) is satisfied and we have $C = 0.9983 > 0$. Hense from Theorem (3.7) the problem (S_1) Ulam-Hyers stable.

Conclusion

The subject of fractional order differential equations is vast, and researching it is lengthy and requires a lot of patience. This memorandum is just a drop in the ocean, in which we have diligently tried to achieve the desired goal, which is to study the existence, uniqueness and Ulam-Hyers stability of the solution for some problems involving generalized fractional derivatives equations which have several types, including Caputo type and hybrid Riemann-Liouville type. Hope that we have contributed, even a little, to explain how to study these problems, and we wish for continued research in this field through the following topics:

Studying these problems for another type of fractional derivatives and in infinite domains, studying these problems in more general Banach spaces and use numerical methods to solve some of those problems for example.

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