People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Higher Normal School of Technological Education

- Skikda -

Department of Mathematics and computer science

Dissertation

Presented to obtain a degree in Mathematics as a teacher of middle school

Entitled

An Introduction to Topological groups

Presented by:

KOUACHI Dhikra

MENAI Souha

Board of Examiners:

Chairman: MAIZI Mohamed	ENSET SKIKDA
Supervisor: AIECH Messab	ENSET SKIKDA
Examiner: FERRAG Azouz	ENSET SKIKDA
Examiner: KIBECHE Khoudir	ENSET SKIKDA

Junes's session 2025

Acknowledgment

First and foremost, all praise and gratitude are due to Allah, whose mercy and guidance have accompanied us throughout this journey. Without His will, none of this would have been possible.

We would also like to thank our supervisor, **Dr. AIECH Messab**, for accepting to supervise this work.

Our sincere thanks go to the members of the defense committee: **Dr. MAIZI Mohamed**, **Dr. FERRAG Azouz**, and **Dr. KIBECHE Khoudir**, for dedicating their time to evaluate our research and for their valuable comments.

Finally, we extend our gratitude to everyone who in one way or another contributed to the completion and success of this work.

Dedication

To the people who made this all possible , **My dear family** words can never do you justice .even if all the words in the world came together they would still fall short when it came to honour you .

I will like to say thank you to my mother ,**Mother** thank you for raising us believing in us loving us and making it look easy whilst doing it .Thank you for refusing to let me settle and for seeing my potential even when I'm too scared to see it myself. You were always the role model i turned to to look for the woman i inspire to be . My unshakable pillar,**My dear father** thank you father you were there for us in every step thank you for providing us with the love and support to get through life . your trust in my words gave me the confidence to keep going .

My sister ,Thank you **lamis**, your support is not just occasional it's been a reliable constant throughout my life .growing through life with you has been the greatest gift ever given to me. i'm so proud of the people we're becoming, the past versions of ourselves, and the people we're aspiring to be.

My Little brother ,Thank you **mouhaned** for standing by and for see my worth and i hope you know how much you mean to me and hope you know how proud i am of you ,watching you grow up to be the man you are .

Thank you **My family** because you loved me enough to start over and over because of your sacrifices and your support and because of your love i am here .

all of my accomplishments i dedicate to you .

To **every teacher** along my academic journey thank you for encouraging us to grow and learn from our mistakes.

To **my big family** each one of you has shaped a part of who I am. Thank you for walking this path with me.

This is have been extremely challenging time but we have reached our destination.this moment marks just not a milestone in our academic journeys . But a moment of reflection on the experience that have shaped us over the course of of my degree i've realised that success isn't just measured by our academic achievements but by how we use the skills we've acquired and the lessons to make a difference.

MENAI Souha

Dedication

"All praise is due to Allah, who eased our beginnings and, through His grace and mercy, allowed us to reach the end"

To my first support, my refuge, and my faithful companion throughout this journey. To the one who endured so much for me and never stopped giving, to you **my mother**.

To the dear one whose name I carry with pride, to the man who spent his life striving for us to be our best, to the constant source of strength **my father**.

To my dearest grandfather, **el-Hadj Ibrahim**, who never said no to me and stood by my side every step of the way. And to **my grandmother**, may Allah grant her healing and protect her always.

To my dear grandfather **Hamid**, and to **my beautiful grandmother Warda**, whose loving prayers were a light on every step of this challenging path.

To my dear sister, **Ghofrane**, to whom I wish all the success in her life.

And to the one who has been my support after my father to my dear brother, **Mouatez**, may Allah grant you success in your life.

To the first one who believed in me and in my potential, my earliest supporter since childhood, to the one I wish all the happiness in the world, to my dear aunt **Leila**.

To my big family, who filled my life with love, To my aunts and uncles from my mother's side, **Houssem**, **Meriem**, **Samira**, **Nasser Eddine**, **Hichem** , **A.erahim**, **Minou** and their children, thank you for your kindness, your presence, and your love.

To my paternal aunts and uncles, **Amel**, **Aicha**, **Oum Kalthoum**, **Ali**, **Bilal**, and their children.

To the friends of my heart, those who shared my journey from the very beginning, and those whom the years at the Higher Normal School of Technological Education-Skikda gifted me, **Abir**, **Souha**, **Rokaya**, **Israa**, **Zineb**, **Jihane**, **Assil**, your presence made every step brighter.

To all those whose kindness, support, or presence guided me to this stage.

And lastly, I thank myself for not giving up, for growing, and for getting here.

KOUACHI Dhikra

Abstract

This thesis introduces the theory of Topological groups highlighting their fundamental properties and mathematical significance. after a brief overview of key concepts from topology and algebra in chapter 1 . the core of the thesis lies in chapter 2 where we define Topological groups and study their main structural features. these include the continuity of group operations , neighborhood systems at the identity , subgroups , quotient topological groups and the uniform structure associated with topological groups. chapter 3 outlines selected applications in harmonic analysis (including Haar measure and Pontryagin duality) lie Theory and functional analysis with further examples from physics and computer science.

Contents

Introduction

Algebra and topology are two fundamental branches of mathematics, each with a distinct focus. Topology examines continuity, convergence, and the structure of spaces, often employing qualitative methods suited to infinite processes. Algebra, by contrast, centers on operations and structural rules, emphasizing symbolic reasoning and computation.

Due to their differing natures, algebra and topology have often developed along separate lines. Nonetheless, their interaction emerges naturally in advanced fields such as functional analysis, dynamical systems, and representation theory, where both algebraic structure and topological concepts are essential.

Topological groups provide a natural environment where algebra (more specifically, group theory) and topology interact in a very fruitful way. The two subjects — the theory of topological spaces, by now a fundamental tool in analysis and the subject of standard texts, and group theory — were combined to form the notion of a topological group. This is an entity which is both a group and a topological space, in which the group operations are continuous.

Although the theory of topological groups was developed mainly in order to study groups of Lie type and its impetus came from problems in analysis, it soon proved to be useful also in purely algebraic contexts. Certain algebraic constructions lead to groups having natural topological structures which are somewhat pathological from an analyst's point of view.

Abstract topological groups were first defined by Schreier in 1926, though the idea was implicit in much earlier work on continuous groups of transformations.

A.D. Alexandroff, N. Bourbaki, M.I. Graev, S. Kakutani, E. van Kampen, A.N. Kolmogorov, A.A. Markov, and L.S. Pontryagin were among the first contributors to the theory of topological groups. Among those who contributed greatly to this field are

W.W. Comfort, M.M. Choban, E. van Douwen, V.I. Malykhin, J. van Mill, and B.A. Pasynkov.

The fundamental topic of various types of continuity of algebraic operations was developed in the works of A. Bouziad, R. Ellis, D. Montgomery, I. Namioka, J. Troallic, and L. Zippin. The recent excellent book of N. Hindman and D. Strauss contains a wealth of material on algebraic operations on compacta satisfying weak continuity requirements.

In these papers we persue to achieve goals. First, we believe that it can be used as reasonably complete introduction to the theory of general topological groups . Second, we expect that it may lead advanced students to the very boundaries of modern topological algebra, providing them with goals and with powerful techniques (and maybe, with inspiration!). One can use these papers in a research seminar on topological algebra and also in advanced courses. Fourth, we expect that these papers will serve quite effectively as a reference, and will be helpful to mathematicians working in other domains of mathematics.

To reach that , in the chapter 1 we have collected the topological and algebraic terms and definitions that will be needed to read and understand these papers and that's to make it more self contained .

In chapter 2 chapter we focused on introducing the fundamental concepts of topological groups, which are based on the interaction between algebra and topology. We start by giving a precise definition of a topological group,showing the continuity conditions that connect the group operations with the underlying topology. We then study the neighborhood system at the identity element, which plays a key role in describing the local structure of the group. The chapter also covers essential constructions such as subgroups and quotient topological groups, and the topologies they naturally inherit. Finally, we present the uniform structure associated with topological groups, providing a broader framework for analyzing uniform continuity.

Chapter 3 explores various applications of topological groups in different areas of mathematics and science. In mathematics, topological groups play an important role in harmonic analysis, where tools like the Haar measure, the Fourier transform on locally compact abelian groups, and Pontryagin duality are essential. We also discuss their connections with Lie groups, which are used to study continuous symmetries, and their role in functional analysis, especially in the study of operator algebras and topological vector spaces. In physics, topological groups appear naturally in classical mechanics and quantum mechanics, where they describe symmetry and conservation laws. In addition, we briefly mention applications in computer science, where concepts from group theory and topology are used in areas such as cryptography, coding theory, and data analysis. This chapter shows how the abstract theory of topological groups has meaningful and powerful applications across many scientific fields.

In the development of this thesis, we have drawn extensively on the foundational work of many mathematicians who contributed significantly to the theory of topological groups and their applications. Several key references have guided our understanding and shaped the structure of this work.Among them Abstract Harmonic Analysis by E. Hewitt and K. A. Ross, Introduction to Topological Groups by Taqdir Hussain, Topological groups by L.Pontrjagin.

CHAPTER 1

Preliminaries

For the sake of completnesss, the present chapter is dedicated to the exposition of the foundamentals concepts of topology and group theory that will enable us to explore topological groups and rigorously develop their basic theory.

This chapter consisting of three section notation and terminology,background on topology and group theory

1.1 Basic notation and terminology

This section we establish the terminology and notation that will be used throughout the thesis.these noteshave been adapted mostly from [11]

We denote by $\mathbb{P}$, $\mathbb{N}$, and $\mathbb{N}_+$ respectively the set of primes, the set of natural numbers, and the set of positive integers. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will denote the integers, the rationals, the reals, and the complex numbers, respectively.

The notation $a \in M$ means that the element a belongs to the set$M$ .if the set M in finite or countable it will sometimes be given by enumerating its constituent elements.In symbols we would write

$$M = a_1, a_2, \ldots, a_n, \ldots$$

which means that the set M is composed of the elements $a_1, a_2, \ldots, a_n, \ldots$

The symbols $\subset$ and $\supset$ mean ordinary inclusion between sets ; they do not exclude the possibility of equality . the void set is denoted by $\emptyset$.

Frequently the sets we deal with are subsets of some universal set say E. in this case we denote by $A^c$the compliment of the set A

$$A^c = \{x \mid x \in E \text{ and } x \notin A\}$$

In every case it will be clear what the set E is .for sets A and B , the *symmetric difference* $A \triangle B$ is defined as the set

$$(A \cap B^c) \cup (A^c \cap B)$$

A Family $\mathcal{A}$ of sets has the *finite intersection property* if

$$\{A \mid A \in \mathcal{F}\} \neq \emptyset$$

For all finite subfamilies $\mathcal{F}$ of $\mathcal{A}$. in particular , no set in $\mathcal{A}$ is void if $\mathcal{A}$ has the finite intersection property .
A family $\{A_i\}_{i \in I}$ of sets is said to *partition* a set X if

$$\bigcup_{i \in I} A_i = X,$$

, each A_i is nonvoid, and the sets A_i are pairwise disjoint.
the family of countable sets includes finite sets and the void set .if a set is *infinite countable* we denote its cardinal by $\aleph_0$ and the real line has a cardinal number $2^{\aleph_0}$

We will write c for $2^{\aleph_0}$. the cardinal of an arbitrary set is denoted by $\overline{\overline{A}}$
The terms *mapping*, *transformation*, and *correspondence* are synonymous with *function*. A function f will often be defined by an expression

$$x \to f(x)$$

Where x denotes a generic element of the domain of the function and $f(x)$ denotes its image under f. For a function f and a subset A of its domain, $f|A$ denotes the *restriction* of f to A.

If f is a function on X into Y and g is a function on Y into Z, then the *composition* of g by f is the function $g \circ f$ on X into Z defined by

$$(g \circ f)(x) = g(f(x)) \quad \text{for } x \in X.$$

Let X be a set and A any subset of X. The symbol ξ_A will denote the function defined on X such that

$$\xi_A(x) = \begin{cases} 1 & \text{for } x \in A. \\ 0 & \text{for } x \in A^c. \end{cases}$$

The function ξ_A is called the *characteristic function* of A.

1.2 Background on topology

The purpose of these notes is to give a mostly self contained topological background for the study of topological groups .our primary source for this section are ([19] [27]. [15]. [14].)

Topological spaces

Definition 1.1. A topological space is a pair (X, τ) consisting of a set X and a collection τ of subsets of X, called **open sets**, satisfying the following axioms:

O_1: The union of open sets is an open set.

O_2: The finite intersection of open sets is an open set.

O_3: X and the empty set $\emptyset$ are open sets.

The collection τ is called a **topology** for X. The topological space (X, τ) is sometimes referred to as the **space** X when it is clear which topology X carries.

Example 1. Let a X be a set , and let τ be the collection of all subsets of X .then τ is clearly a topology on X, it is called the **discrete topology**
let a X be any set and let τ The collection consisting of X *and* $\emptyset$ only is also a topology on X; we call it the **indiscrete topology**, *or the* **trivial topology**.

Definition 1.2. If τ_1 and τ_2 are topologies for a set X, τ_1 is said to be **coarser** than τ_2 if every open set of τ_1 is an open set of τ_2. τ_2 is then said to be **finer** than τ_1, and the relationship is expressed as $\tau_1 \leq \tau_2$. Of course, as sets of sets, $\tau_1 \subseteq \tau_2$.

On a set X the coarsest topology is the *indiscrete topology* , and the finest topology is the *discrete topology*

Definition 1.3. In topological space (X, τ) a subset of X is said to be **closed** if its complement is an open set (if its complement is an element of τ) .

It's possible that a subset be both open and closed or that subset be neither open nor closed

1.2.1 Neighborhood system

Definition 1.4. Closely related to the concept of an open set is that of a **neighborhood** in space (X, τ) a Neighborhood of a point (or a set) is any subset of X which countains an open set contains the point (or the set) the collection of neighborhoods of point x is denoted by $\mathcal{N}_x$.

1.2.2 Limit point

Definition 1.5. A point p is a **limit point** of a set A if every open set containing p contains at least one point of A distinct from p.

The concept of limit point may also be defined for sequences of not necessarily distinct points. A point p is said to be a **limit point of a sequence** $\{x_n\}, n = 1, 2, 3, \ldots$ if every open set containing p contains all but finitely many terms of the sequence. The sequence is then said to **converge** to the point p. A weaker condition on p is that every open set containing p contains infinitely many terms of the sequence. In this case, p is called an **accumulation point of the sequence**. It is possible that a sequence has uncountably many **limit points**.

1.2.3 The closure of a set

Definition 1.6. The **closure** of a set A is the set together with its limit points, denoted by $\overline{A}$. Since a set which contains its limit points is closed, the closure of a set may be defined equivalently as the smallest closed set containing A. Allowing $\overline{A}$ to be A plus its ω-accumulation points or condensation points would permit $\overline{A}$.

1.2.4 The interior

Definition 1.7. We define the **interior** of a set A, denoted by A°, to be the largest open set contained in A, or equivalently, the union of all *open* sets in A. Clearly, the interior of A equals the complement of the closure of the complement of A.

1.2.5 Density

Definition 1.8. A set A is said to be **dense** in a space X if every point of X is a point or a limit point of A, that is, if $X = \overline{A}$.

A subset A of X is said to be **nowhere dense** in X if no nonempty open set of X is contained in $\overline{A}$. In other words, the interior of the closure of a nowhere dense set is empty.

1.2.6 Categories

Definition 1.9. A set is said to be of **first category** (or **meager**) in X if it is the union of a countable collection of nowhere dense subsets of X. Any other set is said to be of **second category**.

1.2.7 Basis and subbasis

Definition 1.10. In a topological space X a collection $\mathcal{B}$ of subset of τ is a **basis** for τ if each open set inτ can be written as a unions of elements in $\mathcal{B}$
subbasis for a topology τ on a set X is a collection $\mathcal{S}$ of subsets of X such that the collection of all finite intersections of elements of $\mathcal{S}$ forms a basis for τ.

1.2.8 Hausdorff space

Definition 1.11. A topological space (X, τ) is called a **Hausdorff** space when distinct points can be separated by open sets.

In symbols:

$$\forall x, y \in X \text{ with } x \neq y, \exists U, V \in \tau \text{ with } x \in U, y \in V \text{ and } U \cap V = \emptyset.$$

A space is said to be **separable** if it has a countable dense subset.

1.2.9 Metric space

Definition 1.12. Let X be a set. Suppose there exists a mapping d of $X \times X$ into the nonnegative real numbers , satisfying the following axioms:

$$(m_1) \quad d(x,y) \geq 0, \quad \text{for all } x,y \in E.$$
$$(m_2) \quad d(x,y) = 0 \quad \text{if and only if } x = y.$$
$$(m_3) \quad d(x,y) = d(y,x).$$
$$(m_4) \quad d(x,y) + d(y,z) \geq d(x,z), \quad x,y,z \in E.$$

We call $d(x,y)$ the **distance** between x and y, It is possible to use a metric to define a topology on X by taking as a basis all **open balls**:

$$B(r,x) = \{y \in \mathrm{X} \mid d(x,y) < r\}.$$

A topological space together with a metric giving its topology is called a **metric space**.

If each cauchy sequence converges in a metric space X Then X is said to be **a complete metric space**.

Theorem 1.1. *Every complete metric space X is of the second category or is baire space .*

1.2.10 Nets and Filters

Definition 1.13. Let A be a nonempty set such that there is a binary relation "$\geq$" defined for the elements of A satisfying the following conditions:

(i) "$\geq$" is reflexive, i.e., $a \geq a$ for each $a \in A$.

(ii) "$\geq$" is transitive, i.e., if $a \geq b$ and $b \geq c$, then $a \geq c$, for all $a,b,c \in A$.

(iii) If $a,b \in A$, then there exists $c \in A$ such that $c \geq a$ and $c \geq b$.

Then, $(A, \geq)$, or simply A, is said to be a **directed set**.

Let X be any set. A subset $\{x_\alpha\}_{\alpha \in A}$ of X is said to be a **net** if A is a directed set of indices. A net $\{x_\alpha\}_{\alpha \in A}$ in a topological space X **converges** to $x \in X$ if for each neighborhood U of x, there exists a $\beta \in A$ such that for all $\alpha \geq \beta$, we have $x_\alpha \in U$. Nets extend the concept of sequences to general topological spaces, where sequences may not be sufficient. In metric spaces, limits of sequences determine closures and topology, and nets play a similar role in general topology. However, limits of nets in topological spaces may not always be unique unless the space is Hausdorff.

Apart from the theory of convergence by means of nets , theres another the theory of convergence based on concepts of filters as defined bellow :

Definition 1.14. Let X be a given set and $\mathcal{F} = \{F_\alpha\}$ a nonempty family of subsets of X. $\mathcal{F}$ is said to be a **filter** on X if the following conditions hold:

(F_1) Any subset of X containing any F_α is in $\mathcal{F}$.

(F_2) The intersection of any finite number of F_α's is also in $\mathcal{F}$.

(F_3) The empty set $\emptyset$ does not belong to $\mathcal{F}$.

A nonempty subfamily $\mathcal{G} = \{G_\alpha\}$ of a filter $\mathcal{F} = \{F_\alpha\}$ on X is called a **base of** $\mathcal{F}$ if:

(i) The intersection of any two members in $\mathcal{G}$ contains a member of $\mathcal{G}$.

(ii) Each F_α contains a G_α.

A nonempty family $\mathcal{F} = \{F_\alpha\}$ on X satisfying (i) and not including $\emptyset$ is said to be a **filter-base**.

Definition 1.15. Let X be a topological space and $\mathcal{F}$ a filter on X. $\mathcal{F}$ is said to **converge** to $x \in X$ if for each neighborhood U of x, there exists an F_α such that $F_\alpha \subset U$.

The following facts are easily deduced:

(a) A subset U of X is open if, and only if, U is a member of each filter that converges to a point of U.

(b) If a filter $\mathcal{F}$ converges to x_0, then every other filter $\mathcal{F}'$ that is finer than $\mathcal{F}$ (i.e., every member of $\mathcal{F}'$ is contained in some member of $\mathcal{F}$) also converges to x_0.

1.2.11 Covering

Definition 1.16. A **cover** (or **covering**) of a set X is a collection of subsets whose union contains X. Formally, a collection $\mathcal{C} = \{C_i\}_{i \in I}$ of subsets of X is a cover of X if:

$$X \subseteq \bigcup_{i \in I} C_i.$$

This means that every element of X belongs to at least one of the sets C_i in the collection.

An **open cover** of a topological space X is a collection of open sets whose union contains X. Formally, a collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open sets in X is called an open cover of X if:

$$X \subseteq \bigcup_{i\in I} U_i.$$

1.2.12 Compactness and local compactness

Definition 1.17. A topological space X is **compact** if every open cover contains a finite subcover equivalently .X is compact if it satisties the finite intersection axiom.

Definition 1.18. Let (X, τ) be a compact topological space and A a closed subset of X. Then A is a compact set .

Definition 1.19. A topological space is called **locally compact** if each point is contained in a compact neighborhood every compact space X is locally compact since X itself is a compact neighborhood of each of its points .

1.2.13 Mapping

Definition 1.20. Mapping on spaces are important tools for studying properties of spaces and for constructing new spaces from previously existing ones.
A function f from a space (X, τ) to a space (Y, σ) is said to be **continuous** if the inverse image of every open set is open. this is equivalent to requiring that the The inverse image of closed sets is closed, or that for each subset A of X,

$$f(\bar{A}) \subseteq \overline{f(A)}.$$

Another equivalent condition is that for each x in X and each neighborhood N of $f(x)$, there exists a neighborhood M of x such that $f(M) \subseteq N$

If this last condition holds at a particular point p, the function is said to be **continuous at the point** p.

The composition $g \circ f$ is continuous whenever $f : X \to Y$ and $g : Y \to Z$ are both continuous, since the inverse image under g of an open set in Z is an open set in Y, and the inverse image of that open set under f is again an open set in X.

A function f from (X, τ) to (Y, σ) is said to be **open** if the image under f of each open set is open, and **closed** if the image under f of every closed set is closed. For bijective (one-to-one and onto) functions, the conditions of being open and of being closed are equivalent, although in general they are not equivalent. It is not difficult to see that f is an open bijective function if and only if f^{-1} is a continuous bijective function.

1.2.14 Homeomorphism

Definition 1.21. A bijective function f from a space (X, τ) to a space $(Y, \sigma)$is a **homeomorphism** if f and f^{-1} are continuous, or equivalently, if f is both continuous and open, or if

$$f(A) = f(\bar{A}) \text{ for all } A.$$

X and Y are then **topologically equivalent** or **homeomorphic**. Such spaces are indistinguishable from a topological point of view.

1.3 Basic notions from group theory

The definitions presented below are based on several standard references, including [5], [3], [21], [23].

1.3.1 Groups

Definition 1.22. A set G is called a group if there is defined in G an operation associating with each pair of elements a, b in G a definite element c in G in such a way that conditions 1) , 2) , 3) formulated belon and known as the group axioms, are satisfied the operation it self is usually referred to as multiplication and it's result is indicated by ab, $c = ab$ (the product ab may depend upon the order of the factors a and b : generally speaking, ab is not equal to ba)

1) Associativity: for each triple of elements a, b, c in G the relation $(ab)c = a(bc)$ is satisfied.

2) G possesses a left identity, i,e, an element e such that $ea = ae = a$ for every element a in G.

3) Every element a in G possesses a left inverse, i.e, an element a^{-1} such that $a^{-1}a = e$.
The set G may be either finite or infinite, if G is finite then the group is said to be finite and the number of elements in the group is its order, otherwise the group is said to be infinite.
If, in addition to the three axioms listed above, the group also satisfied the commutative law.The group is abelian

Example 2. Each of the following is a group:

- The set $\mathbb{Z}$ of all integers with the usual addition in the role of multiplication.
- Every ring is abelian group under addition.
- $(\mathbb{R}, \cdot)$ is not a group, since 0 has no multiplicative inverse. Similarly $(\mathbb{Q}, \cdot)$, $(\mathbb{C}, \cdot)$ are not groups.

1.3.2 Subgroups

Definition 1.23. Let G be a group. A subset H of G is called a subgroup of G if :
1) H is nonempty (often ensured by requiring that the identity element e of G is in H).
2) H is closed under products, meaning that for all $x, y \in H$, their product xy is also in H.
3) H is closed under inverses, meaning that for all $x \in H$, the inverse x^{-1} is also in H.
subgroups of G are just subsets of G which are themselves groups with respect to the operation defined in G, i.e, the binary operation on G restricts to give a binary operation on H which is associative, has an identity in H, and has inverses in H for all the elements of H.

Example 3. The following is a subgroup:
1) $\mathbb{Z} \leq \mathbb{Q}$ and $\mathbb{Q} \leq \mathbb{R}$ with the operation of addition.
2) Any group G has two subgroups: H=G and H=$\{1\}$, the latter is called the trivial subgroup and will henceforth be denoted by 1.

1.3.3 Normal subgroups

Definition 1.24. A subgroup N of G is said to be a normal subgroup of G if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Equivalently, if by gng^{-1} we mean the set of all $gng^{-1}, n \in N$, then N is a normal subgroup of G if and only if $gNg^{-1} \subseteq N$." for every $g \in G$.

1.3.4 Homomorphisms

Definition 1.25. A mapping ϕ from a group G into a group H is said to be a homomorphism if for all $a, b \in$G :

$$\phi(ab) = \phi(a)\ \phi(b)$$

Notice that on the left side of this relation, namely, in the term $\phi(ab)$, the product ab is computed in G using the product of elements of G,whereas on the right side of this relation, namely, in the term $\phi(a)\ \phi(b)$, the product is that of elements in H.

Example 4. $\phi(x) = e$ all $x \in G$. This is trivially a homomorphism. Likewise $\phi(x) = x$ for every $x \in G$ is a homomorphism.

1.3.5 Isomorphisms

Definition 1.26. Let G_1 and G_2 be groups. A bijective function $f : G_1 \mapsto G_2$ with the property that for any two elements a and b in G_1,

$$f(ab) = f(a)f(b)$$

is called an isomorphism from G_1 to G_2.
If there exists an isomorphism from G_1 to G_2, we say that G_1 is isomorphic to G_2.
If there exists an isomorphism f from G_1 to G_2, in other words, if G_1 is isomorphic to G_2, we symbolize this fact by writing :

$$G_1 \cong G_2$$

to be read, "G_1 is isomorphic to G_2."

Example 5. Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by :

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

Then T is an isomorphism.

1.3.6 Mappings

Definition 1.27. If S and T are nonempty sets, then a mapping from S to T is a subset, M, of $S \times T$ such that for every $s \in S$ there is a unique $t \in T$ such that the ordered pair (s, t) is in M.

Example 6. In this example the sets are assumed to be nonempty.
Let S and T be any sets, define $\tau : S \times T \to S$ by $(a, b)\tau = a$ for any $(a, b) \in S \times T$. This τ is called the projection of $S \times T$ on S. We could similarly define the projection of $S \times T$ on T.

1.3.7 Direct product

Definition 1.28. If H and K are groups, then their *direct product*, denoted by $H \times K$, is the group with elements all ordered pairs (h, k), where $h \in H$ and $k \in K$, and with operation

$$(h, k)(h', k') = (hh', kk').$$

It is easy to check that $H \times K$ is a group: the identity is $(1, 1)$; the inverse $(h, k)^{-1}$ is (h^{-1}, k^{-1}). Notice that neither H nor K is a subgroup of $H \times K$, but $H \times K$ does contain isomorphic replicas of each, namely,

$$H \times \{1\} = \{(h, 1) : h \in H\} \quad \text{and} \quad \{1\} \times K = \{(1, k) : k \in K\}.$$

1.3.8 Quotient groups

Definition 1.29. Let G be a group and H a normal subgroup of G. The quotient group G/H is defined to be the set of all left cosets of H in G, in symbols: $G/H = gH \mid g \in G$. The operation on G/H is given by $(xH) * (yH) = (x * y)H$.

CHAPTER 2

Topological groups

Topological groups at the intersection of algebra and topology, forming a rich framework where group operations interact continuously with the underlying topological structure. These objects-groups endowed with a topology compatible with their algebraic operations-arise naturally in areas ranging from harmonic analysis to dynamical systems and functional analysis. The study of topological groups not only generalizes classical group theory but also provides deep insights into the geometric and analytic properties of symmetry-bearing spaces.

In this chapter, we will delve into the central theme in the general theory is the interplay between algebraic properties and topological properties.

2.1 Definition of a topological group

Theorem 2.1. *Let G be an algebraic group equipped with a topology τ. Then (G, τ) is a topological group if and only if the two mapping :*

- *The multiplication map $p : G \times G \to G$, defined by $(x, y) \mapsto xy$*
- *The inversion map $t : G \to G$, defined by $x \mapsto x^{-1}$*

are jointly continuous in both variables. We remarke that to (i) is equivalent to the statement that, whenever $U \subseteq G$ is neighborhood of $g_1 g_2$, then there exsists two neighborhoods V_1, V_2such that V_1 is a neighborhood of g_1 and V_2 is a neighborhood of g_2 and $V_1 V_2 = \{h_1 h_2, h_1 \in V_1, h_2 \in V_2\} \subseteq U$

The latter two mappings are equivalent to saying that the mapping $\varphi(x, y) \mapsto xy^{-1}$ is continuous .

Theorem 2.2. *[14] Let G be a topological group. The following maps are homeomorphisms from G to G for all $a \in G$.*

1. *the right and left translations maps : r_a and l_a.*
2. *the inverse map : $x \mapsto x^{-1}$.*
3. *the inner automorphism map : $x \mapsto axa^{-1}$.*

Proof. Since G is a topological group, the group operation $\mu : G \times G \to G$, defined by $\mu(x, y) = xy$, is continuous. Fixing $a \in G$, the left translation map $L_a(x) = ax$ is the composition $\mu \circ f$, where $f(x) = (a, x)$, and similarly, $R_a(x) = xa$ is $\mu \circ g$, where $g(x) = (x, a)$. Both f and g are continuous, so L_a and R_a are continuous. Each map is bijective, with inverses $L_{a^{-1}}$ and $R_{a^{-1}}$,which are also translation maps and hence continuous by the same reasoning.Therefore, L_a and R_a are homeomorphisms. For the inverse mapping we know it is a bijection since G is a group (every element has a unique inverse), it is continuous by definition of a topological group. The inverse map of the inverse map is itself, so it has a continuous inverse. Thus the inverse map is a topological group. Lastly, we have the automorphism map. This is a composition of two homeomorphisms, $x \mapsto ax$ and $x \mapsto xa^{-1}$, and is thus a homeomorphism. □

Corollary 2.1. *Let G be a topological group, let $a \in G$, and let F, P, A be subsets of G where F is closed, P is open and A is arbitrary. Then, aF, Fa and F^{-1} are closed; and aP, Pa,P^{-1}, AP, and PA are open.*

Proof. Since the mapping $r_a : x \mapsto xa$ and $l_a : x \mapsto ax$ are homeomorphisms.
Therefore since F is closed, Fa and aF are also closed.
By the same argument, since P is open, Pa, aP are also open. Now since $AP = \cup aP$, $PA = \cup Pa$, and we know that union of open set is open. Therefore AP and PA is also open. □

Example 7. The following are some of the most useful and well-known examples of topological group, drawn from [**?**] and [12]:

1. Let G be an arbitrary group, and let $\mathcal{O}$ be the family of all subsets of G (the discrete topology). With this topology, G is a topological group. We shall often refer to such a G as a discrete group.

2. $G = R \sim \{0\}$, the set of all nonzero real numbers with multiplication as the group operation and with the topology induced from the set of real numbers. One verifies that G is a multiplicative abelian topological group.

3. Let G be any group. When G is given the discrete topology, the topology where the collection of open sets is $P(G)$ the power set of G, then G is a topological group. In a similar manner, the indiscrete topology on G, the topology where the collection of open sets is $\{\emptyset, G\}$,makes G into a topological group.

4. The sets of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ are all topological groups under addition when they are given their standard topologies. Similarly, if n is any positive integers, $\mathbb{Z}^n, \mathbb{Q}^n, \mathbb{R}^n, \mathbb{C}^n$ are all topological groups under addition when given their standard topologies.

5. $G = R^n$, the n-dimensional Euclidean space with addition as the coordinate addition. G is an additive abelian group. The topology is the usual metric topology defined by the metric: $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, in which $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$. This is also a noncompact, locally compact topological group satisfying the second axiom of countability.

6. Let G be any algebraic group with the indiscrete topology. Since $GG^{-1} = G$, for any group G, the group operations are continuous and hence G, endowed with the indiscrete topology, is a topological group. An indiscrete space is not even a T_0-space and so there is a topological group which is not a T_0-space and, hence, not Hausdorff.

2.2 Neighborhood systems of the identity

Before discussing neighborhood systems in topological groups, we first recall the definition of fundamental system of neighborhoods of a point in a general topological space.

Definition 2.1. [13] Let X be a topological space and $x \in X$. A collection $\mathcal{U}$ of neighborhoods of x is called a **fundamental system of neighborhoods** (or *neighborhood base*) at x if, for every neighborhood V of x, there exists $U \in \mathcal{U}$ such that

$$U \subseteq V.$$

Proposition 2.1. *[13][18] Now in a topological group G Let $\mathcal{U}$ be a fundemental system of open neighborhoods of the identity element e then we have the following properties :*

1. *The intersection of every element in $\mathcal{U}$ is e (i.e $\bigcap_{U \in \mathcal{U}} U = \{e\}$).*
2. *If $U, V \in \mathcal{U}$, then $\exists W \in \mathcal{U}$ such that $W \subseteq U \cap V$.*
3. *If $W \in \mathcal{U}$, then $\exists V \in \mathcal{U}$ such that $VW \subseteq U$.*
4. *If $U \in \mathcal{U}$, then $\exists V \in \mathcal{U}$ such that $V^{-1}V \subseteq U$.*
5. *If $U \in \mathcal{U}, x \in G$, then $\exists V \in \mathcal{U}$ such that $x^{-1}Vx \subseteq U$.*

Proof. Let us brievely prove these properties :

- In (2) and (3) $U \cap V, Ua^{-1}$ are clearly are open neighborhoods of the e thus they containt an element of $\mathcal{U}$.
- In (4) The mapping $f : G \times G \to G$ defined by $f(a, b) = a^{-1}b$ is continuous. Let $U \in \mathcal{U}$ be an open neighborhood of $e \in G$. $f^{-1}(U)$ is open and contains the point (e, e), Therefore, there exist open sets $A, B \subseteq G$ such that $e \in A$, $e \in B$, and $A \times B \subseteq f^{-1}(U)$. Let $\exists V\mathcal{U}$ such that $V \in A \cap B$, we have $V \times V \subseteq f^{-1}(U))$ which this prove $V^{-1}V \subseteq U$
- in (5): The mapping $f : G \times G \to G$ given by $a \mapsto x^{-1}ax$ is continuous, so $f^{-1}(U)$ is open and contains e, hence contains some $V \in \mathcal{U}$. For this V we have $x^{-1}Vx \subseteq U$.

□

Proposition 2.2. *[18] Let G be a topological group with identity element e and let $\mathcal{U}$ be fundemental system of open neighborhoods of the identity element e .then $\mathcal{U}_x^L = \{xU : U \in \mathcal{U}\}$ and $\mathcal{U}_x^R = \{Ux : U \in \mathcal{U}\}$ are fundemental system of open neighborhoods of x. for any $x \in G$.*

Proof. We show that $\mathcal{U}_x^L$ is a fundemental system of open neighborhoods of x we already proved that the left multipication by x is a homeomorphism ,and so $\mathcal{U}_x^L$ is a collection of open sets containing x. if W is a neighborhood of x, then x^-1W is a neighbourhood of e, so $U \subseteq x^{-1}W$ for some $U \in \mathcal{U}$. but then $xU \in \mathcal{U}_x^L$ and $xU \subseteq x(x^{-1}W) = W$. □

Theorem 2.3 ([11]). *Let G be an abstract group, and let $\mathcal{U}$ be a family of subsets with the finite intersection property for which properties (1)–(5) in Proposition 2.1 hold. Then the family of sets $\{xU\}$ where U runs through $\mathcal{U}$ and x runs through G is an open subbasis for a topology on G. With this topology, G is a topological group.*

A complete proof of this result can be found in [13, Chapter 2,proposition 2]

Definition 2.2. [4] A subset A of G which coincides with its image under the inverse mapping $x \mapsto x^{-1}$ (ie $A = A^{-1}$) is said to be symmetric . Simalarly the intersection $A \cap A^{-1}$is symmetric .

From the the previous definition We can state a basic property of topological groups concerning symmetric neighborhoods of the identity element.

Proposition 2.3. *[4] If W is a neighborhoodof the identity element e . then $W \cup W^{-1}$, $W \cap W^{-1}$, $W \cdot W^{-1}$ are symmetric neighborhoods of the identity element e.*

Proposition 2.4. *[4] In a topological group the the symmetric neighbourhoods form a fundamental system of neighbourhoods of the identity element .*

For the proof one can look [14, Chapter 3,proposition 1] .

Theorem 2.4. *[18] In topological group G , let U be a neighborhood of the identity e .then there exists a symmetric neighborhood V of the identity e such that $V \cdot V = V^2 \subseteq U$.*

Proof. [18] Since $e = e \cdot e$ and U is neighborhood if e . we can find neighborhoods N and W such that $MN \subseteq U$. Now let $V = (M \cap N) \cap (M \cap N)^{-1}$. Note that $M \cap N$ is a neighborhood of e since both M and N are . and that since $M \cap N$ is a neighborhood of e , $(M \cap N)^{-1}$ is a neighborhood of $e^{-1} = e$ and So V is a neighborhood of e . but then $V^2 \subseteq MN \subseteq U$. This completes the proof. □

Corollary 2.2. *[11] Let G be a topological group. for every neighborhood U of e there is , there is a neighborhood V of e such that $V^{-1} \subset U$.*

Proof. [11] Let V be a symmetric neighborhood of e such that $V^2 \subset U$ then if $x \in V^{-1}$ we have $(xV) \cap V \neq \emptyset$, hence $xv_1 = v_2$ for some $v_1 and v_2 \in V$ and thus $x = v_2 v_1^{-1} \in VV^{-1} = V^2 \subset U$. □

Corollary 2.3. *[18] Let G be a topological group .if $x \neq e$, there exists a neighborhood W of e such that $W \cap xW$ is empty .*

Proof. [18] Since G is a topological space there is a neighborhood V_1 of e not containing x or there is a neighborhood V_2 of x not containing e. In the second alternative , xV_2^{-1} is a neighborhood of e not containing x.in either case there is a neighborhood U of e not containig x .Let W be a symmetric neighborhood of e , $W^2 \subset U$. if $W \cap xW$ were not vacuous there would exist $w, w_1 \in W$ with $w_1 = xw$ .But this gives $x = w_1 w^{-1} \in W^2 \subset U$ which is false . □

Theorem 2.5. *[1] Every topological group G is a regular space.*

Proof. [1] Let U be an open neighborhood of the identity e in G. there is a an open symmetric neighborhood V of e such that $V^2 \subset U$. then if $x \in \bar{V}$ .we have $Vx \cap V\emptyset$ .hence $a_1 x = a_2$ for some $a_1, a_2 \in V$ and thus $x = a_1^{-1} a_2 \in V^{-1}V = V^2 \subset U$ this implies that $\bar{V} \subset U$. Since G is homogeneous space . the regularity of G is now immediate □

Theorem 2.6. *[11] Let G be a topological group, let U be any neighborhood of the identity element e, and let F be any compact subset of G. Then there exists a neighborhood V of e such that $xVx^{-1} \subset U$ for all $x \in F$..*

Proof. [11]

Let W be a symmetric neighborhood of e such that $W^3 \subset U$. Since $F \subset \bigcup_{x \in F} Wx$ and F is compact, there exist $x_1, \ldots, x_n \in F$ such that

$$F \subset \bigcup_{k=1}^{n} Wx_k.$$

Let

$$V = \bigcap_{k=1}^{n} x_k^{-1} W x_k.$$

Clearly V is a neighborhood of e, and $x_k V x_k^{-1} \subset W$ for $k = 1, \ldots, n$. If $x \in F$, then $x \in Wx_k$ for some k. Thus $x = wx_k$ for some $w \in W$, and hence

$$xVx^{-1} = wx_k V x_k^{-1} w^{-1} \subset wWw^{-1} \subset W^3 \subset U.$$

. □

Theorem 2.7. *[11] Let G be a topological group with identity e, F a compact subset of G, and U an open subset of G such that $F \subset U$. Then there is a neighborhood V of e such that $(FV) \cup (VF) \subset U$. If G is locally compact, then V can be chosen so that $\overline{(FV) \cup (VF)}$ is compact.*

Proof. [11] For each $x \in F$, there are a neighborhood W_x of e such that $xW_x \subset U$ and a neighborhood V_x of e such that $V_x^2 \subset W_x$. Since $F \subset \bigcup_{x \in F} xV_x$, there exist $x_1, \ldots, x_n \in F$ such that

$$F \subset \bigcup_{k=1}^{n} x_k V_{x_k}.$$

Let $V_1 = \bigcap_{k=1}^{n} V_{x_k}$. Then

$$FV_1 \subset \bigcup_{k=1}^{n} x_k V_{x_k} V_1 \subset \bigcup_{k=1}^{n} x_k V_{x_k}^2 \subset \bigcup_{k=1}^{n} x_k W_{x_k} \subset U.$$

Similarly, there is a neighborhood V_2 of e such that $V_2 F \subset U$. Letting $V = V_1 \cap V_2$, we obtain

$$(FV) \cup (VF) \subset U.$$

If G is locally compact, then V can be chosen so that $\overline{V}$ is compact. It follows that $F\overline{V}$ is closed and compact (4.4). Since $FV \subset F\overline{V}$ and $F\overline{V}$ is closed, we have $\overline{FV} \subset F\overline{V}$, and hence $\overline{FV}$ is compact. Similarly, $\overline{VF}$ is compact, so that $\overline{(FV) \cup (VF)}$ is compact. □

Corollary 2.4. *[11] A topological group is first-countable if and only if there is a countable neighbourhood base at the identity (or any point). A topological group is locally compact (respectively, locally connected) if and only if there is a neighbourhood base of compact (respectively, connected) sets at the identity (or any point) .*

2.3 Separation axioms in topological groups

The separation properties of topological groups are quite strong. We will show that every topological group is completely regular and that a T_0 topological group is Tychonoff . Some of the terms for separation axioms are not completely standard, so we clarify them here.

Definition 2.3. A space X is a T_0 space iff it satisfies the T_0 axiom, i.e, for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subset X$ so that U contains one of x and y but not the other.

Obviously the property T_0 is a topological property. An arbitrary product of T_0 spaces is T_0. Discrete spaces are T_0 but indiscrete spaces of more than one point are not T_0.

Definition 2.4. A space X is a T_1 space or Frechet space iff it satisfies the T_1 axiom, i.e, for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subset X$ so that $x \in U$ but $y \notin U$.

T_1 is obviously a topological property and is product preserving. Every T_1 space is T_0.

Example 8. The set $\{0, 1\}$ furnished with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$ is called Sierpinski space. It is T_0 but not T_1.

Proposition 2.5. *X is a T_1 space iff for each $x \in X$, the singleton set $\{x\}$ is closed.*

Definition 2.5. A space X is a T_2 space or Hausdorff space iff it satisfies the T_2 axiom, i.e, for each $x, y \in X$ such that $x \neq y$ there are open sets $U, V \subset X$ so that $x \in U, y \in V$ and $U \cap v = \emptyset$.

T_2 is a product preserving topological property, Every T_2 space is T_1.

Theorem 2.8. *Let G be a topological group, the following statements are equivalent:*

(a) G is a T_0-space.

(b) G is a T_1-space.

(c) G is a Hausdorff space.

(d) $\cap U = \{e\}$, where U is a fundamental system of neighborhood of e.

Proof. We shall show that $(a) \implies (b) \implies (c) \implies (d) \implies (a)$.
For $(a) \implies (b)$, that is let G be a T_0 space, let $x \neq y,\ x, y \in G$. By (a) for at least one (say, x) of x and y, there exists an open neighborhood P of x such that $y \notin P$. since $x^{-1}P = V$ is an open neighborhood of e, $V \cap V^{-1}$ is an open symmetric neighborhood of e and therefore yQ is a neighborhood of y. Now $x \notin yQ$ because otherwise $x^{-1} \in Qy^{-1}$ and, hence, $x^{-1} \in Qy^{-1} \subset Vy^{-1} \subset x^{-1}Py^{-1}$. But from this we get that $e = xx^{-1} \in xx^{-1}Py^{-1} = Py^{-1}$. or $y \in P$, which is a contradiction. Therefore we get a open neighborhood P of x such that $y \notin P$ and a neighborhood yQ of y such that $x \notin yQ$. Therefore G is a T_1-space.

For $(b) \implies (c)$, let $x \neq y$, $x, y \in G$. By (b), $\{x\}$ is a closed set and therefore $P = G \sim \{x\}$ is an open neighborhood of y and hence $y^{-1}P$ is an open neighborhood of e. Let V be an open neighborhood of e such that $VV^{-1} \subset y^{-1}P$.Then yV is an open neighborhood of y. Let $Q = G - yV$ which is an open set, and $x \in Q$. For otherwise $x \in yV$ and hence $xV \cap yV \neq \emptyset$.
But this shows that $x \in yVV^{-1} \subset y(y^{-1}P) = P$, which is a contradiction.
because $x \notin P$. Clearly $Q \cap yV = \emptyset$, $y \in yV$, and $x \in Q$, yV and Q are open sets. This proves that G is a T_2-space.
for $(c) \implies (d)$, let $x \in U$ for each U in $\{U\}$ and assumes $x \neq e$. Then (c) implies that there exists a neighborhood P of e such that $x \notin P$, but then there exists a U in $\{U\}$ such that $U \subset P$, we have the contradiction $x \in U \subset P$ and $x \notin P$. Hence $x = e$ and (d) is established.
for $(d) \implies (a)$, let $x \neq y$ then$xy^{-1} \neq e$ and hence, by (d) there exists a U in U such that $xy^{-1} \notin U$. Thus Uy being the neighborhood of y an $x \notin Uy$, (a) is proved. □

Lemma 2.1. *Let G be a topological group and let :*

$$U_0 \supseteq U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq ...$$

be symmetric neighborhoods of 1 with $U_n^3 \subseteq U_{n-1}$ for every $n \in \mathbb{N}$. Then there exists a continuous left invariant pseudo-metric d on G such that $U_n \subseteq B_{1/n}(e_G) \subseteq U_{n-1}$ for every n.

Theorem 2.9. ***(Birkho-Kakutani)*** *A topological group is metrizable iff it has a countable base of neighborhoods of e_G.*

Proof. The necessity is obvious as every point x in a metric space has a countable base of neighborhoods. Suppose now that G has countable base of neighborhoods of e_G. Then one can build a chain (2) of neighborhoods of e_G as in Lemma 3.1 that form a base of $V(e_G)$, in particular, $\cap_{n=1}^{\infty} U_n = \{e_G\}$. Then the pseudo-metric produced by the lemma is a metric that induces the topology of the group G because of the inclusions $U_n \subset B_{1/n} \subset U_n - 1$. □

2.4 Uniform structures on a topological group

In a topological group G we can define 'a uniform structure ' by using translations as fellow :given any two points x and y of G, we apply to both points the translation which sends one of them, say x, to the identity element e ; the" "proximity" of x and y is then evaluated in some sense by the neighbourhood U of e into which y is translated. This translation, which consists of multiplying both x and y by x^{-1}, can be carried out on the right or on the left, as fellow

$$U_r(U) = \{(x, y) \in G \times G \mid x^{-1}y \in U\}$$

$$U_l(U) = \{(x, y) \in G \times G \mid yx^{-1} \in U\}$$

.

We shall see that in either case we obtain effectively a uniformity on a which is compatible with the topology of G .

Let's consider the case where translations are performed on the right; then for each neighbourhood U of e we define the set U_r .and Let τ_r be the collection of such sets as U runs through the system of all neighbourhood of e .Then U_r is a fundamental system of entourages For since $e \in U$.

- The diagonal $\Delta = (x, x), x \in G$ of $G \times G$ is contained in U_d for each U, So τ_d satisfies axiom U_1;
- For $(x, y), (y, z)$ since the relations $yx^{-1} \in U$ and $xy^{-l} \in U{-1}$ are equivalent, we have U_r^{-1} = $(U^{-1})_r$ hence $U_r^{-1} \in \tau_r$, so that U_2 is satisfied .
- And finally, the relations $zx^{-1} \in U$ and $yz^{-1} \in U$ imply$yx^{-1} \in U$. ; hence $U_r \circ U_r$ is contained in $(U \cdot U)_r$, and that shows that τ_r satisfies U_3.

Hence ,τ_r defines a uniform structure on G compatible with its topology ,
for the relations $y \in U_r$ and $y \in U \cdot x$ are equivalent by definition; in other words $U_r = U \cdot x$.

Proposition 2.6. *[4] Let G be a topological group equipped with the right uniformity. Then, for every element $a \in G$, the left translation $l_a : x \mapsto ax$ and the right translation $r_a : x \mapsto xa$ are uniform homeomorphisms of G. In other words, the left and right translations are automorphisms of the right uniformity.*

Proof. [4] As to the right translations, the result is clear .since $yx^{-1} \in U$ is equivalent to $(ya)(xa)^{-1} \in U$. for the left translations $yx^{-1} \in U$ if and only if $ay)(ax)^{-1} \in aUa^{-1}$ hence $x \mapsto ax$is uniformly continuous on G_r.

□

A similar argument shows that both left and right translations are isomorphisms of the left uniformity onto itself as well.

Finally, every inner automorphism $x \mapsto axa^{-1}$is therfore an automorphism of the group structure of G, the topology of G, and both the left and right uniformities of G.

Proposition 2.7. *[4] The invertion mapping is an isomorphism of the right uniformity onto the left uniformity.*

Proposition 2.8. *[4] Every continuous homomorphism g of a topological group G into a topological group T is uniformly continuous when considered as a mapping of G_r into T_r for if U' is is a neighbourhood of the identity element in T and $U = g^{-1}(U')$.then the relation $yx^{-1} \in T$ implies $g(y)(g(x))^{-1} = g(yx^{-1}) \in U$.*

2.5 Subgroups

Definition 2.6. Let G be a topological group. A set H of elements of G is called a subgroup of the topological group G if :

1. H is a subgroup of the abstract group G.

2. H is a closed subset of the topological space G.

A subgroup N of a topological group G is called a normal subgroup of G if N is a normal subgroup of the abstract group G.

Proposition 2.9. *Let H be a subgroup of a topological group G. Then H is a topological group with respect to the subspace topology. Moreover, the closure $\overline{H}$ is also a subgroup of G. If H is normal in G, then $\overline{H}$ is also normal.*

Proof. Let H be a subgroup of G, endowed with the subspace topology. The product topology on $H \times H$ coincides with the subspace topology on $H \times H \subseteq G \times G$. Therefore the map $H \times H \mapsto H$ that maps (g, h) to $g^{-1}h$ is continuous. Hence H is a topological group. The continuity of the map $k(g, h) = \ g^{-1}h$ ensures that :

$$k(\overline{H} \times \overline{H}) = k(\overline{H \times H}) \subseteq \overline{k(H \times H)} = \overline{H} \tag{2.1}$$

Thus $\overline{H}$ is a subgroup. Suppose in addition that $H \trianglelefteq G$ is normal. For $a, g \in G$ we put $\gamma_a(g) = aga^{-1}$. Since the conjugation map $\gamma_a : G \mapsto G$ is continuous, we have :

$$\gamma_a(\overline{H}) \subseteq \overline{\gamma_a(H)} = \overline{H} \tag{2.2}$$

for all $a \in G$. This shows that $\overline{H}$ is normal in G. □

Lemma 2.2. *Let G be a topological group and suppose that $U \subseteq G$ is an open subset. If $X \subseteq G$ is any subset, then UX and XU are open subsets. In particular, the multiplication map $m : G \times G \mapsto G, (g,h) \mapsto gh$ and the map $k : (g,h) \mapsto g^{-1}h$ are open.*

Proof. For each $x \in X$, the sets $U_x = \rho_{x^{-1}}(U)$ and $xU = \lambda_x(U)$ are open. Hence $UX = \cup\{U_x \backslash x \in X\}$ and $XU = \cup\{xU \backslash x \in X\}$ are open as well. □

Proposition 2.10. *Let G be a topological group and let $H \subseteq G$ be a subgroup.*

(i) The subgroup H is open if and only if it contains a nonempty open set.

(ii) If H is open, then H is also closed.

(iii) The subgroup H is closed if and only if there exists an open set $H \subseteq G$ such that $U \cap H$ is nonempty and closed in U.

Proof. **for** (i), suppose that H contains the nonempty open set U. Then $H = UH$ is open by Lemma 3.2. Conversely, if H is open then it contains the nonempty open set H.

for (ii) suppose that $H \subseteq G$ is open. Then$G - H = \cup\{aH \backslash a \in G - H\}$ is also open.

for (iii) suppose that $U \cap H$ is nonempty and closed in the open set U. Then $U \cap H$ is also closed in the smaller set $U \cap H \subseteq U$. Upon replacing G by $\overline{H}$, we may thus assume in addition that H is dense in the ambient group G, and we have to show that $H = G$. The set $U - H = U - (U \cap H)$ is open in U and hence open in G. On the other hand, H is dense in G. Therefore $U - H = \emptyset$ and thus $U \subseteq H$. By (i) and (ii), H is closed in G, whence $H = G$.

Conversely, if H is closed, then H is closed in the open set G. □

Corollary 2.5. *Let G be a topological group. For every neighborhood U of e, there is a neighborhood V of e such that $V^{-1} \subset U$.*

Proof. Let V be a symmetric neighborhood of e such that $V^2 \subset U$ then if $x \in V^{-1}$, we have $(xV) \cap V \neq \emptyset$. Hence $xv_1 = v_2$ for some $v_1, v_2 \in V$, and thus :

$$x = v_2 v_1^{-1} \in VV^{-1} = V^2 \subset U \tag{2.3}$$

□

Theorem 2.10. *Let G be a topological group and H a subgroup of G such that $U^{-1} \cap H$ is closed in G for some neighborhood U of e in G. then H is closed.*

Proof. Let U be a neighborhood of e in G such that $U^{-1} \cap H$ is closed in G. Let V be a symmetric neighborhood of e in G such that $V^2 \subset U$. Now let x be any point in H^{-1}, and let $x_\alpha, \alpha \in D$, be a net in H such that x_α converges to x. Since $x^{-1} \in H^{-1}$, there is an element y in $Vx^{-1} \cap H$. There is an $\alpha_0 \in D$ such that $x_\alpha \in xV$ for all $\alpha \geq \alpha_0$. Thus, if $\alpha \geq \alpha_0$, we have $yx_\alpha \in (Vx^{-1})(xV) = V^2 \subset U$ and hence $yx_\alpha \in U^{-1} \cap H$. Since the net $yx_\alpha, \alpha \geq \alpha_0$, converges to yx and $U^{-1} \cap H$ is closed, we have $yx \in U^{-1} \cap H$. Hence $x = y^{-1}yx \in H$ and therefore $H^{-1} \subset H$ that's mean that H is closed. □

Theorem 2.11. *Every discrete subgroup H of a T_0 group G is closed.*

Proof. Let U be a neighborhood of e in G such that $U \cap H = \{e\}$. By corollary 3.2. there is a neighborhood V of e such that $V^{-1} \subset U$. Then $V^{-1} \cap H = \{e\}$, which is closed since G is Hausdorff. Now implies that H is closed. □

Theorem 2.12. *Let G be a Hausdorff topological group and H a subgroup of G. Then :*

(a) H is hausdorff.

(b) H is a compact subgroup if G is compact and H is closed.

(c) H is a locally compact subgroup if G is locally compact and H is closed.

(d) H is a metrizable if G is.

(e) H satisfies the second axiom of countability if G does.

(f) H is a complete subgroup if G is and H is closed.

Proof. **For**(a) and (b) are generally known for any topological space.
For(c), let U be a neighborhood of e in G such that $\overline{U}$ is compact. Then $U \cap H$ is a neighborhood of e in H. And the closure of $U \cap H$ taken in H is equal to the closure of $U \cap H$ taken in G, since H is closed. But $\overline{U}$ being compact, $\overline{U \cap U}$ is also compact because the latter is a closed subset of $\overline{U}$ this proves (c).
For(d) and (e) are immediate from relativization, and **for**(f) is well-known in any uniform space.
All parts from (a) to (e) of this theorem are true for semitopological groups as well. □

2.6 Quotient groups

Definition 2.7. Let G be a topological group , and H subgroup of G.Then G/H is the set of all left cosets xH of H in G, and we call G/H the quotient of G by H. let π be the canonical projection $\pi : G \to G/H$ that maps x to the to the left coset xH.

We endow G/H with the quotient topology induced by the projection map $\pi : G \to G/H$; that is, a subset $U \subseteq G/H$ is declared open if and only if $\pi^{-1}(U)$ is open in G. in other words the quotient topology on G/H consists of the sets $\pi(U)$ where U runs over the open subsets of G.

One could also consider right cosets instead of left cosets as the elements of G/H . The theory is entirely analogous.

Theorem 2.13. *[11] Under the quotient topology on G/H, the projection mapping $\pi : G \to G/H$ is continuous, and the quotient topology is the finest topology on G/H under which the projection mapping is continuous.*

Proof. [11] Let $\{aH \mid a \in U_i\}_{i\in I}$ be a family of open subsets of G/H where each U_i is open in G. then $\bigcup_{i\in I} U_i$ is open in$G$. similarly , the intersection of any finite number of sets in $\pi(G/H)$ is again in $\pi(G/H)$. clearly $\emptyset$ and G/H are in $\pi(G/H)$ and therefore is a topology on G/H the remaining statement of the theorem are easy to verify .

□

Theorem 2.14. *[29] The projection mapping of G onto G/H is open .*

Proof. [29] Let $U \neq \emptyset$ be an open set in G . we check that $\pi^{-1}(\pi(U)) = UH = \bigcup_{h\in H} Uh$ is open .since each Uh is open .therefore $\pi(U)$ is open in G/H ; hence π is an open mapping . □

Theorem 2.15. *[29] If G is a topological group and H a compact subgroup of G . Then the projection mapping of G onto G/H is closed mapping .*

Proof. As in the proof of the preceding theorem we are reduced to showing that if $F \subset G$ is closed then FH is closed . But H is compact and FH is closed. then π is closed mapping. □

Proposition 2.11. *[7] Let G be a topological group , and suppose H is a subgroup of G then we have the following properties :*

1. *if H is closed , then G/H Hausdorff.*

2. *if G is locally compact , so is G/H.*

3. *if H is normal , G/H is a topological group .*

Proof. [7]

(1) Suppose $\bar{x} = \pi(x), \bar{y} = \pi(y)$ are distinct points of G/H. If H is closed, xHy^{-1} is a closed set that does not contain e, so there is a symmetric neighborhood U of e with $UU \cap xHy^{-1} = \emptyset$. Since $U = U^{-1}$ and $H = HH$, we have: $1 \notin UxH(Uy)^{-1} = (UxH)(UyH)^{-1}$, so $(UxH) \cap (UyH) = \emptyset$. Thus $\pi(Ux)$ and $\pi(Uy)$ are disjoint neighborhoods of $\bar{x}$ and $\bar{y}$.

(2) If U is a compact neighborhood of e in G, $\pi(Ux)$ is a compact neighborhood of $\pi(x)$ in G/H.

(3) If $x, y \in G$ and U is a neighborhood of $\pi(xy)$ in G/H, by continuity of multiplication in G at (x, y), there are neighborhoods V, W of x, y such that $VW \subset q^{-1}(U)$. Then $\pi(V)$ and $\pi(W)$ are neighborhoods of $\pi(x)$ and $\pi(y)$ such that $\pi(V)\pi(W) \subset U$, So multiplication is continuous on G/H. Similarly, inversion is continuous.

□

Corollary 2.6. *[7] If G is T_1 then G is Hausdorff. If G is not T_1 then $\{e\}$ is a closed normal subgroup, and $G/\{e\}$ is a Hausdorff topological group.*

Proof. [7] The first assertion follows by taking $H = \{e\}$ in Proposition (2.11(1)). $\{1\}$ is a subgroup by theorem (2.12c); it is clearly the smallest closed subgroup of G. It is therefore normal, for otherwise one would obtain a smaller closed subgroup by intersecting it with one of its conjugates. The second assertion therefore follows from Proposition (2.2 (1,3)) by taking $H = \{e\}$. □

CHAPTER 3

Applications of topological groups

Topological groups occupy a central position in modern mathematics, serving as a unifying framework that merges the algebraic structure of groups with the analytical properties of topological spaces. Formally, a topological group is a group endowed with a topology such that the group operations multiplication and inversion are continuous with respect to this topology. This interplay between algebraic and topological structures enables the study of symmetry, continuity, and transformation in a cohesive manner, making topological groups indispensable in both theoretical and applied mathematics.

Since their formalization in the early 20th century, topological groups have become fundamental tools across diverse mathematical disciplines. In pure mathematics, they play a pivotal role in: **Harmonic Analysis**, where locally compact abelian groups provide the setting for Fourier analysis and Pontryagin duality. **Lie Theory**, in which smooth (differentiable) topological groups, known as Lie groups, describe continuous symmetries in geometry and differential equations. and also in **Functional Analysis**, where topological groups model spaces of operators and serve as the foundation for representation theory.

Beyond pure mathematics, topological groups have profound applications in physics and technology: **Classical Mechanics**, they are used to describe symmetries and physical laws, especially through Lie groups. **Quantum Mechanics**, they help represent quantum states and particle behavior using symmetry groups. **Computer Science**, topological groups are useful in cryptography, coding theory, and studying data structures

with symmetry.
The versatility of topological groups stems from their ability to encode both discrete and continuous phenomena, making them essential in studying limits, convergence, and deformation of algebraic structures. Moreover, their applications extend to real-world problems, such as signal processing and even data science.
This chapter provides a structured exploration of the applications of topological groups, beginning with foundational mathematical examples accessible to advanced undergraduates. We then proceed to their role in physics and engineering, concluding with concrete real-world case studies that demonstrate their practical significance.

3.1 Applications in Mathematics

3.1.1 Harmonic analysis and fourier theory

Harmonic analysis and topological groups are strongly connected. By extending Fourier analysis to locally compact abelian groups using the Haar measure, it becomes possible to study functions and their frequency components within a more general and abstract framework. This connection provides powerful tools for understanding the structure of functions beyond classical settings.

Haar measure

Let G be a topological group. A left Haar measure (resp. right Haar measure) on G is a nonzero regular Borel measure μ on G such that $\mu(gA) = \mu(A) \quad (\text{resp. } \mu(Ag) = \mu(A))$ for all $g \in G$ and all measurable subsets $A \subseteq G$.
This condition means that the measure μ is invariant under left (resp. right) translation by elements of the group. Haar measures allow us to integrate functions on G in a way that respects the group structure. For example, in the case of $\mathbb{R}$ under addition, the Haar measure corresponds to the usual Lebesgue measure[9].

Remark. Haar's theorem guarantees the existence and uniqueness (up to scalar multiplication) of a left (or right) Haar measure on every locally compact group.

Fourier transform on locally compact abelian groups

We can define the Fourier Transform and relate it to the structure space. For a locally compact abelian (LCA) group G, every function f defines a function, its Fourier transform, as:

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\,dx \tag{3.1}$$

where $\hat{f} : \widehat{G} \to \mathbb{C}$.
We first want to identify the Fourier transform with the Gelfand transform from $L^1(G)$ to $\Delta_{L^1(G)}$. Consider the map $\psi : \widehat{G} \to \Delta_{L^1(G)}$ where $\psi(\chi)(f) = \hat{f}(\chi)$. We will show that ψ is a homomorphism, so that each function on the structure space determines a unique character on $\widehat{G}$, and so the two transforms are identical [24].

Pontryagin duality

Pontryagin duality is an important result in topological group theory. It says that every locally compact abelian group G has a dual group $\widehat{G}$, which is the set of all continuous group homomorphisms from G to the circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. These homomorphisms are called characters. The dual group $\widehat{G}$ is also a locally compact abelian group, and the dual of $\widehat{G}$, denoted $\widehat{\widehat{G}}$, is isomorphic to the original group G. This property is called reflexivity.
Some simple examples :

- The dual of $\mathbb{Z}$ is the circle group $\mathbb{T}$.
- The dual of $\mathbb{R}$ is $\mathbb{R}$.
- The dual of $\mathbb{T}$ is $\mathbb{Z}$.
- The dual of $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Pontryagin duality is not just a theoretical result, it has many applications in Fourier analysis, abstract harmonic analysis, and number theory. It provides a powerful way to study the structure of topological groups by translating group properties into properties of functions and vice versa [6].
Note. The concepts of Pontryagin duality and harmonic analysis on topological groups are deeply connected to modern number theory. They are particularly useful in the

study of characters on local fields, class field theory, and the analysis of arithmetic functions.

3.1.2 Lie groups

Lie groups constitute a fundamental class of topological groups that also possess a smooth manifold structure. By definition, a Lie group is a group G that is simultaneously a differentiable manifold, such that the group operations(multiplication $\mu : G \times G \to G$ and inversion $\iota : G \to G$)are smooth maps. This allows Lie groups to be studied using both algebraic and differential geometric tools.
Some classical examples of Lie groups include:

- The general linear group $GL(n, \mathbb{R})$, consisting of all invertible $n \times n$ real matrices. It forms an open subset of $\mathbb{R}^{n^2}$, hence a smooth manifold.
- The real numbers $\mathbb{R}$ under addition.
- The circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Every Lie group has an associated Lie algebra, which is the tangent space at the identity element, equipped with a bilinear operation called the Lie bracket. This algebra encodes the local structure of the group and plays a key role in its classification and representation.
The significance of Lie groups lies in their ability to describe continuous symmetries in mathematics. Their theory connects topology, geometry, and algebra in a unified framework, and underpins many areas of modern mathematical analysis and beyond. The material presented here follows the treatment of Lie groups in Hall's book [10].

3.1.3 Functional analysis

We know that **topological group** is a group G equipped with a topology such that the group operation $(x, y) \mapsto xy^{-1}$ is continuous with respect to the product topology on $G \times G$, and the inverse map $x \mapsto x^{-1}$ is also continuous. These structures appear naturally in functional analysis, especially within the framework of topological vector spaces [19].
In particular, every **topological vector space** V over a field $\mathbb{K}$ (typically $\mathbb{R}$ or $\mathbb{C}$) is an

abelian topological group under vector addition. If V is also complete and normed, then it is called a **Banach space** [25]. The underlying additive group $(V, +)$ is a Hausdorff, locally convex, metrizable topological group. This allows the use of group-theoretic tools such as dual groups, continuous characters, and convolution operations in the analysis of linear operators on V.

Moreover, **locally compact topological groups** serve as a setting for abstract harmonic analysis. The Haar measure, which exists uniquely (up to a scalar multiple) on every locally compact group, allows the definition of integration and convolution on groups. These notions are essential for understanding operator algebras and representations of groups in Hilbert spaces [6].

An important example is the group $L^2(G)$, the space of square-integrable functions on a compact group G, which becomes a Hilbert space where G acts by left translation. This representation theory of topological groups is deeply connected to the spectral theory of unitary operators [22].

Thus, topological groups provide a natural and powerful language to express symmetry, continuity, and transformation in infinite-dimensional analysis.

3.2 Applications in Physics and other Sciences

3.2.1 Classical mechanics

In classical mechanics, symmetries of physical systems are often modeled by topological groups, particularly Lie groups, which describe continuous transformations such as rotations and translations. A central concept is that of the **configuration space** of a mechanical system, which is typically a smooth manifold M. The group of symmetries G acts continuously on M, preserving the physical structure of the system [17].

For instance, the group $SO(3)$, the rotation group in three dimensions, is a compact Lie group that represents the rotational symmetries of rigid bodies. As a topological group, $SO(3)$ is not simply connected, and this topological property has consequences in both classical and quantum settings [2].

The use of topological groups allows one to formalize conservation laws via Noether's theorem, where every continuous symmetry corresponds to a conserved quantity (e.g., rotational symmetry $\Rightarrow$ conservation of angular momentum). The group structure

ensures that these symmetries form a mathematical framework suitable for Hamiltonian and Lagrangian mechanics.
In this way, topological groups provide an elegant and powerful language to express the geometric and algebraic properties of classical systems.

Everyday Examples of Topological Groups in Classical Mechanics

Topological groups, especially Lie groups, appear naturally in many physical systems and even in everyday situations involving motion and rotation. Below are some intuitive examples that illustrate how classical mechanics often relies on the mathematical structure of topological groups:

- **Rotating a wheel:** When a car wheel spins, it undergoes a continuous rotation around its central axis. The set of all such rotations forms the group $SO(2)$, which is both a Lie group and a topological group. Each angle of rotation corresponds to an element in this group, and the group operation is the addition of angles.

- **Opening a door:** A door rotating around its hinge is another example of a physical system with a continuous symmetry. The angles through which the door turns are elements of $SO(2)$, and the continuous nature of the motion makes this a topological group action.

- **Rotating a rigid body (like a smartphone):** When we move a smartphone in space, the orientation changes continuously. The group $SO(3)$ captures all possible rotations in three-dimensional space. This group is widely used in gyroscopes and motion sensors in mobile devices.

- **Spinning top (toy):** A spinning top performs a complex rotational motion governed by the group $SO(3)$. The stability and precession observed in its motion are directly related to the properties of the rotation group.

- **Robot or drone motion:** The movement of robots and drones in three-dimensional space involves both rotation and translation. This combined motion is described by the group $SE(3)$, the special Euclidean group. It is a Lie group and a topological group, and its structure is fundamental in robotics and control theory.

These examples demonstrate how topological groups are essential for modeling continuous symmetries and motions in classical mechanics, providing a rigorous mathematical foundation for real-world physical behaviors.

3.2.2 Quantum mechanics

Topological groups, particularly Lie groups, play a central role in quantum mechanics by formalizing the symmetries of quantum systems and determining the structure of their physical states. These groups act continuously on Hilbert spaces and help describe the invariance of physical laws under transformations such as rotations, translations, and time evolution. The content of this part is largely based on the works of Hall [10], Tinkham [30], and Sakurai and Napolitano [26].

- **Spin and the group $SU(2)$:** The intrinsic angular momentum of quantum particles, known as spin, is mathematically described using the Lie group $SU(2)$. This group acts on two-dimensional complex vector spaces, and its representations classify particles as fermions or bosons based on their spin behavior.
- **Rotational symmetry and $SO(3)$:** Physical systems that are invariant under spatial rotation are associated with the group $SO(3)$, which reflects classical rotational symmetry. However, in quantum mechanics, the double cover $SU(2)$ is often used instead, as it accounts for spin-$\frac{1}{2}$ particles and their unique transformation properties.
- **Translation and the Heisenberg group:** The continuous group of translations in space and momentum is captured by the Heisenberg group. It forms the mathematical basis of the canonical commutation relations between position and momentum operators in quantum mechanics.
- **Time evolution and the unitary group $U(1)$:** The time evolution of a closed quantum system is governed by unitary transformations, which are elements of the group $U(1)$ or more generally $U(n)$. These ensure the conservation of probability and preserve the structure of the Hilbert space.

Topological groups are therefore fundamental in quantum theory: they encode conservation laws through Noether's theorem, classify particles via group representations, and ensure the consistency of quantum dynamics.

Everyday Examples Related to Topological Groups in Quantum Mechanics

Although quantum mechanics is not directly observable in everyday life, its principles based on topological groups underlie many modern technologies and phenomena:

- **Magnetic Resonance Imaging (MRI):** The functioning of MRI machines relies on the quantum spin of protons, which is governed by the group $SU(2)$. The manipulation and detection of these spin states are central to producing medical images.
- **Electron spin in hard drives:** Quantum spin, described by $SU(2)$, is used to encode information in magnetic storage devices. Changes in spin states enable writing and reading data in modern hard disks.
- **Quantum tunneling in semiconductors:** The phenomenon of tunneling, essential in transistors and microprocessors, depends on the symmetry of the wave function and involves translations described by topological group actions.
- **Laser operation and coherence:** The phase symmetry of electromagnetic waves in laser systems is related to the group $U(1)$, which models the conservation of probability and phase in quantum systems.

These examples show how abstract structures like topological groups have concrete implications in everyday technologies rooted in quantum mechanics.

3.2.3 Computer Sciences

Although topological groups originate in pure mathematics and theoretical physics, they have found meaningful applications in computer science, particularly in fields that involve symmetry, continuous transformations, and geometric modeling [28].

- **Computer Graphics and Animation:** The rotation of 3D objects in computer graphics is modeled using the Lie group $SO(3)$, which describes smooth rotations in three-dimensional space. These group operations ensure realistic animation and object manipulation [8].

- **Robotics and Motion Planning:** Topological groups like $SE(3)$, combining rotation and translation, are used to model the configuration space of robotic arms and autonomous drones. Efficient path planning algorithms utilize the group structure to compute feasible trajectories [20].

- **Computer Vision and Pose Estimation:** Tasks such as object recognition and camera tracking rely on transformations governed by topological groups to estimate the pose (position and orientation) of objects in 3D space [16].

- **Cryptography and Data Security:** While more theoretical, some approaches in post-quantum cryptography investigate group-based cryptographic protocols, where algebraic structures with topological or Lie group properties may offer alternative key exchange methods [8].

Everyday Examples

- **Video Games:** The smooth rotation of game characters or camera views in 3D environments is governed by the group $SO(3)$, enabling realistic motion and spatial interactions [8].

- **Autonomous Vehicles:** Self-driving cars rely on accurate motion planning and pose estimation, often modeled using $SE(2)$ or $SE(3)$ for navigation and localization [20].

- **Face Unlock in Smartphones:** Facial recognition systems apply geometric transformations derived from group theory to match features under various orientations and lighting conditions [16].

Topological groups thus provide a powerful framework for modeling motion, transformation, and symmetry in modern computing technologies.

Conclusion

this thesis has has provided an introduction to topological a mathematical structure that elegantly unifies algebraic and topological . After reviewing the necessary foundations from general topology and group theory, we explored the defining features of topological groups in depth, focusing on their structural and continuity properties, neighborhood systems, and uniform structures. These concepts form the core of the theory and are essential to understanding how topological groups behave and interact with other mathematical objects. We then considered selected applications of topological groups in both pure and applied settings. In particular, we discussed their role in harmonic analysis, Lie theory, and functional analysis, as well as their appearance in areas of physics and computer science. These applications highlight the wide relevance and depth of the theory. While this thesis is introductory in nature, it reflects the richness of the subject and opens the door to further study, including more advanced topics such as representation theory, non-abelian harmonic analysis, and topological group cohomology. The results presented here aim to build a solid foundation for such future exploration.

Bibliography

[1] A. V. Arhangel'skii and M. G. Tkachenko. *Topological Groups and Related Structures: An Introduction to Topological Algebra*, volume 1 of *Atlantis Studies in Mathematics*. Atlantis Press / World Scientific, Paris & Amsterdam, 2008.

[2] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer, 1989.

[3] M. Artin. *Algebra*. Prentice Hall, 1991.

[4] N. Bourbaki. *General Topology: Chapters 1–4*. Elements of Mathematics. Springer, Berlin, 1995.

[5] D. S. Dummit and R. M. Foote. *Abstract Algebra*. Wiley, 3rd edition, 2004.

[6] G. B. Folland. *A Course in Abstract Harmonic Analysis*. CRC Press, 1995.

[7] G. B. Folland. *A Course in Abstract Harmonic Analysis*, volume 29 of *Textbooks in Mathematics*. CRC Press, 2016.

[8] J. Gallier. *Geometric Methods and Applications: For Computer Science and Engineering*. Springer, 2011.

[9] J. Gleason. Existence and uniqueness of haar measure. *preprint*, 2010.

[10] B. C. Hall. *Lie groups, Lie algebras, and representations*. Springer, 2013.

[11] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis*, volume 1 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, Heidelberg, 2nd edition, 1979.

[12] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis, Volume I*. Springer, 2nd edition, 1994.

[13] P. J. Higgins. *An Introduction to Topological Groups*, volume 15 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1975.

[14] T. Husain. *Introduction to Topological Groups*. W.B. Saunders Company, Philadelphia, 1966.

[15] J. L. Kelley. *General Topology*, volume 27 of *Graduate Texts in Mathematics*. Springer, New York, 1975.

[16] Y. Ma, S. Soatto, J. Kosecka, and S. S. Sastry. *An Invitation to 3-D Vision: From Images to Geometric Models*. Springer, 2004.

[17] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17. Springer, 1999.

[18] D. Montgomery and L. Zippin. *Topological Transformation Groups*. Interscience Publishers, New York, 1955.

[19] J. R. Munkres. *Topology*. Prentice Hall, Upper Saddle River, NJ, 2nd edition, 2000.

[20] R. M. Murray, Z. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.

[21] L. S. Pontrjagin. *LS Pontryagin selected works*, volume 2. Gordon and Breach Science Publishers, 1986.

[22] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups*. Oxford University Press, 2nd edition, 2000.

[23] J. J. Rotman. *An Introduction to the Theory of Groups*, volume 148. Springer Science & Business Media, 2012.

[24] W. Rudin. *Fourier Analysis on Groups*. Interscience Publishers, 1962.

[25] W. Rudin. *Functional Analysis*. McGraw-Hill, 2nd edition, 1991.

[26] J. J. Sakurai and J. Napolitano. *Modern Quantum Mechanics*. Cambridge University Press, 2020.

[27] L. A. Steen and J. A. Seebach. *Counterexamples in Topology*. Springer, New York, 2nd edition, 1978.

[28] J. Stillwell. *Naive Lie Theory*. Springer, 2008.

[29] M. Stroppel. *Locally Compact Groups*, volume 3 of *EMS Textbooks in Mathematics*. European Mathematical Society, Zürich, 2006.

[30] M. Tinkham. *Group Theory and Quantum Mechanics*. Courier Corporation, 2003.