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Study of some generalized fractional differential equations

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Introduction

Fractional differential equations (FDEs) extend the concept of ordinary differential equations by incorporating derivatives of non-integer (fractional) order, this topic considered ancient, with most researchers in the history of mathematics agreeing that originated in the late 17^{th} century, in September 30, 1695, L'Hopital sent a letter to Leibniz asking about an observation related to his research on differentiation, the question was " What is the result of differentiation of order n for the function $f(x) = x$, where $n = \frac{1}{2}$ ". Thus this question constituted the actual starting point for fractional calculus.

Fractional differential equations are increasingly prevalent across diverse research domains and engineering disciplines. They provide effective descriptions for a multitude of physical phenomena, including viscoelastic behavior, diffusive processes, relaxation oscillations, electrochemical kinetics, among others see [9, 11, 15]. These equations involve derivatives of non-integer orders, capturing intricate dynamics that traditional integer-order derivatives may overlook.

Even though the applications of FDEs are quite broad, it cannot be applied to all systems. The researchers have found that certain phenomena related to material heterogeneities cannot be well-modeled using fractional derivatives. Due to this fact a solution to this problem was proposed by Caputo in 1967, who defined a fractional derivative allowing the application of initial conditions with physical meaning. In his researches, new FDs are defined, termed henceforth generalized fractional derivatives (GFDs), for a more extensive collection of fractional calculus, readers can refer to [2].

Due to the appearance of (GFDs), the existence and uniqueness naturally become an important topic. Many experts have proposed some methods to explore the existence and uniqueness to (GFDs) like Banach's fixed point theorem, Schaefer theorem see [1, 2, 14].

The stability properties of all kinds of equations have attracted the attention of many mathematicians. Particularly, the Ulam-Hyers stability was taken up by a number of mathematicians and the study of this area has the grown to be one of the central subjects in the mathematical analysis [10, 16] .

This memorandum is devoted to the study existence, uniqueness and stability for some generalized fractional boundary value problems, and supporting each study with some examples. This work has been divided into three chapters as follows:
Chapter 1: States some essential definitions, lemmas and theorems that we use to prove the main results.
Chapter 2 is devoted to study the existence, uniqueness and stability of the following second-type hybrid generalized fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)+g\left(t,u(t),D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)\right)=0, & t\in J,\\ \lim_{t\to 0}(\phi(t)-\phi(0))^{2-\alpha}(u(t)-f(t,u(t)))=0,\\ u\left(1\right)=\eta+f\left(1,u\left(1\right)\right), \end{cases}$$

where

- $D_{0^+}^{\alpha,\phi}$ is the ϕ -Riemann-Liouville fractional derivative.
- $J=[0,1]$.
- $f: J\times\mathbb{R}\longrightarrow\mathbb{R}$ and $g: J\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are given functions.

In chapter 3 we consider the question of existence, uniqueness and stability of the following nonlocal fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha,\phi}\left(z(t)\right)+f(t,z(t))=0, & t\in J,\\ z\left(0\right)=0,\\ z\left(1\right)=\beta z\left(\eta\right), \end{cases}$$

where:

- $D^{\alpha,\phi}$ is the ϕ-Riemann-Liouville fractional derivative.
- $J=[0,1]$.
- $f: J\times\mathbb{R}\longrightarrow\mathbb{R}$: is a given function.

CHAPTER 1

Preliminaries

In this chapter, we will refine certain definitions, theorems, lemmas, and results from functional analysis, fractional calculus, and Ulam-Hyers stability, along with the theory of fixed points in Banach spaces, that we have utilized in the following two chapters.

1.1 Generalities

Let $J = [a, b], (-\infty < a < b < +\infty)$ be a finite interval of the real axis $\mathbb{R} =]-\infty, +\infty[$.

1.1.1 Functional space

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the Chebyshev norm:

$$\|u\|_{C(J;\mathbb{R})} = \underset{t \in J}{max} |u(t)|.$$

we denote by $C^m(J, \mathbb{R})$ the space of functions f that are m times continuously differentiable on J.
the readers can refer to [10, 11].

1.1.2 Functional analysis tools

Let $(\mathbb{E}, \|.\|)$ be a Banach space

Definition 1.1. [11] The operator $A : \mathbb{E} \to \mathbb{E}$ is said to be continuous if and only if for any sequence $(u_n) \subset \mathbb{E}$,

$$\left(u_n \xrightarrow{n\to\infty} u\right) \implies \left(A(u_n) \xrightarrow{n\to\infty} A(u)\right).$$

Definition 1.2. [11] An operator $A : \mathbb{E} \to \mathbb{E}$ is called a contraction on $\mathbb{E}$ if there exists a real number k with $0 < k < 1$ such that

$$\forall x, y \in \mathbb{E}, \quad \|Ax - Ay\| \leq k\|x - y\|.$$

Definition 1.3. [11] An operator$A : \mathbb{E} \longrightarrow \mathbb{E}$ is called compact if the image of each bounded set $B \subset \mathbb{E}$ is relatively compact ($\overline{A\,(B)}$ is compact).

Definition 1.4. [11] An operator $A : \mathbb{E} \longrightarrow \mathbb{E}$ is called completely continuous operator if it is continuous and compact.

Theorem 1.5. *(Arzela-Ascoli theorem) [10] Let H be a subset of $C(J, \mathbb{R})$ equipped with the Chebyshev norm. Then H is relatively compact in $C(J, \mathbb{R})$ if and only if, H is equicontinuous and uniformly bounded.*

We recall that a family H of continuous functions on J into $\mathbb{R}$ is called:

Uniformly bounded: There exists a constant $M > 0$ such that:

$$\|f\|_{C(J;\mathbb{R})} = \underset{t\in J}{max}|f(t)| \leq M, \forall f \in H$$

Equicontinuous: For every $\epsilon > 0$, there exists $\eta > 0$ such that for all $f \in H$ and all $t_1, t_2 \in J$, with $|t_1 - t_2| < \eta$, we have:

$$|f(t_1) - f(t_2)| < \epsilon.$$

Theorem 1.6. *(The mean value theorem) [8] Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, where $a < b$. Then there exists $c \in [a, b]$ such that*

$$f(b) - f(a) = (a - b)f'(c).$$

1.2 Fractional Calculus

1.2.1 Riemann-Liouville fractional integral

Definition 1.7. [11, 15] The left Riemann-Liouville fractional integral $I_{a^+}^{\alpha}$ of order $\alpha \in R^+$ of the function f defined by

$$(I_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-s)^{\alpha-1} f(s) ds, \qquad t > a.$$

$\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int\limits_0^{\infty} t^{\alpha-1} e^{-t} dt, \qquad (\alpha > 0).$$

1.2.2 Riemann-Liouville fractional derivative

Definition 1.8. [11] The left Riemann-Liouville fractional derivative $D_{a^+}^{\alpha}$ of order $\alpha \in \mathbb{R}_+^*$ of the function f is defined by

$$\begin{aligned} (D_{a^+}^{\alpha} f)(t) &= \left(\frac{d}{dt}\right)^n (I_{a^+}^{n-\alpha} f)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int\limits_t^b (s-t)^{n-\alpha-1} f(s) ds, \qquad t \in J \end{aligned}$$

wher $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α.

When $\alpha = n \in N$, then $(D_{a^+}^{\alpha} f)(t) = (D_{a^+}^{n} f)(t) = f^{(n)}(t)$.

1.2.3 ϕ-Riemann-Liouville fracional integral

Definition 1.9. [1, 3] Let $f : [a,b] \to \mathbb{R}$ be an integrable function and $\phi : [a,b] \to \mathbb{R}$ be an increasing function such that, for all $t \in [a,b]$, $\phi'(t) \neq 0$. The left-sided ϕ-Riemann-Liouville fractional integral of a function f is defined as follows:

$$I_{a^+}^{\alpha,\phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1} f(s)\, ds.$$

1.2.4 ϕ-Riemann-Liouville fracional derivative

Definition 1.10. [1, 3] The left-sided ϕ-Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}_+^*$, with $n = [\alpha] + 1$, of a function f corresponding to ϕ-Riemann-Liouville fractional integral is defined as follows:

$$D_{a^+}^{\alpha,\phi} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\phi'(t)} \frac{d}{d(t)} \right)^n \int\limits_a^t \phi'(s)(\phi(t) - \phi(s))^{n-\alpha-1} f(s) ds.$$

We notice that the Riemann-Liouville fractional derivatives are just particular cases of ϕ-Riemann-Liouville fractional derivative (when $\phi(t) = t, t \in J$).

Lemma 1.11. *[4, 14] Let $z : C(J, \mathbb{R})$, with, $1 < \alpha < 2$, then*

- $I_{0^+}^{\alpha,\phi} D_{0^+}^{\alpha\phi} z(t) = z(t) + C_0(\phi(t) - \phi(0))^{\alpha-1} + C_1(\phi(t) - \phi(0))^{\alpha-2}$, *where* $C_0, C_1 \in \mathbb{R}$.
- $D_{0^+}^{\alpha\phi} I_{0^+}^{\alpha,\phi} z(t) = z(t)$.

1.3 Ulam-Hyers stability

We consider the following fractional differential equation:

$$D_{0^+}^{\alpha,\phi}(u(t)) = f(t, u(t)), \quad t \in J, \tag{1.1}$$

and the following fractional differential inequality:

$$|D_{0^+}^{\alpha,\phi}(y(t)) - f(t, y(t)))| \leq \epsilon, \quad t \in J. \tag{1.2}$$

Definition 1.12. [16] The equation (1.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C(J, \mathbb{R})$ of the inequality (1.2) there exists a solution $u \in C(J, \mathbb{R})$ of the equation (1.1) with

$$|y(t) - u(t)| \leq c_f \epsilon, \quad t \in J.$$

Remark 1.13. A function $y \in C(J, \mathbb{R})$ is a solution of (1.2) if and only if there exists a function $\psi \in C(J, \mathbb{R})$ (which depends on y) such that:

1. $|\psi(t)| \leq \epsilon, \quad t \in J.$

2. $D_{0^+}^{\alpha,\phi}(y(t) - f(t, y(t))) = g\left(t, y(t), D_{0^+}^{\alpha,\phi}(y(t) - f(t, y(t)))\right) + \psi(t), \quad t \in J.$

1.4 Some fixed point theorems

Theorem 1.14. *(Banach fixed point theorem) [5] Let* $\mathbb{E}$ *be a non-empty closed subset of a Banach space , then any contraction mapping* A *of* $\mathbb{E}$ *into itself has a unique fixed point, i.e.*

$$\exists! x \in \mathbb{E} : A(x) = x.$$

Theorem 1.15. *(Schaefer fixed point theorem) [5] Let* $(\mathbb{E}, \|\cdot\|)$ *be a non-empty Banach space. Let* $A : \mathbb{E} \to \mathbb{E}$ *be a completely continuous mapping. If the set* $\chi = \{y \in \mathbb{E} : y = \lambda A(y), \lambda \in [0,1]\}$ *is bounded in* $\mathbb{E}$*, then* A *admits at least one fixed point in* $\mathbb{E}$*.*

CHAPTER

2

Existence, uniqueness and stability of a second-type hybrid generalized fractional differential equation

2.1 Introduction

In recent years fractional differential equations play a very important role in scientific research chemistry, physics, and engineering. Many researchers focused on developing the methods of solution of the hybrid fractional differential equations by using deferent kinds of fixed point theorems, like in [1] the researchers studied the existence of solutions for hybrid fractional integral differential equations, involving ϕ-Caputo derivative. They use an hybrid fixed point theorem for a some of three operators due to Dhage for proving the main results. An example is provided to illustrate main results.

Motivated by this researches, in this chapter we will study the existence, uniqueness and stability of solution to the following generalized fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)+g\left(t,u(t),D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)\right)=0, & t\in J,\\ \lim_{t\to 0}(\phi(t)-\phi(0))^{2-\alpha}(u(t)-f(t,u(t)))=0, & \\ u\left(1\right)=\eta+f\left(1,u\left(1\right)\right), & \end{cases} \tag{Q}$$

where

- $D_{0^+}^{\alpha,\phi}$ is the ϕ -Riemann-liouville fractional derivative with $1 < \alpha < 2$.
- $J = [0, 1]$, $\eta \in \mathbb{R}$.
- $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $g \in C(J \times \mathbb{R}^2, \mathbb{R})$ tow continuous nonliner functions.
- the function $\phi : J \to J$ is a strictly increasing function such that $\phi \in C^2(J, \mathbb{R})$ and $\phi'(t) \neq 0$ for all $t \in J$.

For our purpose, we convert the differential problem into equivalent integral equations via constructing Green function then we established certain properties for it and we assumed some sufficient conditions through which we prove the existance of the solution using Schaefer's fixed point theory and the uniqueness of the solution using the Banach fixed point theory, we also studied the stability of this solution, and supported the study with an illustrative examples.

2.2 Existence, uniqueness and Ulam-Hyers stability results

The following section is devoted to stating and proving the existence, uniqueness and Ulam stability results for problem (Q).

Definition 2.1. The function u from $C(J, \mathbb{R})$ is a solution to the problem (Q) if it satisfies the equation:

$$D_{0^+}^{\alpha,\phi}\left(u(t) - f(t,(t))\right) = -g\left(t, u(t), D_{0^+}^{\alpha,\phi}\left(u(t) - f(t, u(t)\right)\right), \tag{2.1}$$

and the conditions:

$$\lim_{t\to 0}\left[\phi(t) - \phi(0))^{2-\alpha}(u(t) - f(t, u(t))\right] = 0, \tag{2.2}$$

$$u\left(1\right) = \eta + f\left(1, u\left(1\right)\right). \tag{2.3}$$

Lemma 2.2. *Let $h : J \longrightarrow \mathbb{R}$ be a continuous function, u is solution for the second-type hybrid fractional differential equation:*

$$D_{0^+}^{\alpha,\phi}\left(u(t) - f(t, u(t))\right) = -h(t), \quad t \in J$$

and satisfying the conditions (2.2)-(2.3), if and only if u is a solution for the integral equation via Green function:

$$u(t) = \eta\gamma(t) + f(t, u(t)) + \int_0^1 G(t,s)\phi'(s)h(s)ds, \ t \in J \tag{2.4}$$

where:

$$G(t,s) = \frac{\gamma(t)}{\Gamma(\alpha)} \begin{cases} (\phi(1)-\phi(s))^{\alpha-1} - \dfrac{1}{\gamma(t)}(\phi(t)-\phi(s))^{\alpha-1}, & 0 \le s \le t \le 1 \\ (\phi(1)-\phi(s))^{\alpha-1}, & 0 \le t \le s \le 1 \end{cases} \tag{2.5}$$

with:

- $K(t) = \phi(t) - \phi(0), t \in J.$
- $\gamma(t) = \dfrac{(K(t))^{\alpha-1}}{(K(1))^{\alpha-1}}, t \in J.$

Proof. We have u as a solution of the problem (Q),

$$I_{0^+}^{\alpha,\phi}\left(D_{0^+}^{\alpha,\phi}(u(t) - f(t,u(t)))\right) = -I_{0^+}^{\alpha,\phi}(h(t)) + C_0(\phi(t)-\phi(0))^{\alpha-1} + C_1(\phi(t)-\phi(0))^{\alpha-2},$$

$$u(t) - f(t,u(t)) = -I_{0^+}^{\alpha,\phi}(h(t)) + C_0(\phi(t)-\phi(0))^{\alpha-1} + C_1(\phi(t)-\phi(0))^{\alpha-2},$$

by using the conditions (2.2)-(2.3), we obtin $C_1 = 0$ and

$$C_0 = \frac{1}{(\phi(1)-\phi(0))^{\alpha-1}}\left(\eta + \frac{1}{\Gamma(\alpha)}\int_0^1 \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}h(s)ds\right)$$

By substitution:

$$\begin{aligned} u(t) &= f(t,u(t)) + \eta\gamma(t) - \frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}h(s)ds \\ &+ \frac{\gamma(t)}{\Gamma(\alpha)}\int_0^1 \phi'(s)(\phi(1)-\phi(s))^{\alpha-1}h(s)ds \\ &= f(t,u(t)) + \eta\gamma(t) + \int_0^1 G(t,s)\phi'(s)h(s)ds. \end{aligned}$$

The converse can be easily inferred from Lemma (1.11). □

Lemma 2.3. *The following conditions are satisfied by the Green function G defined by equation* (2.5)

(i) $G(t,s) \ge 0$, for all $t, s \in J$.
(ii) $G(t,s) \le \dfrac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)}$, for all $t, s \in J$.

Proof. **We prove (i)**: Since ϕ is a strictly increasing function, we have $\phi(1) \geq \phi(s)$ whenever $s \leq 1$. So one can easily conclude from equation (2.4) that, for $0 \leq t \leq s \leq 1$, $G(t,s) \geq 0$.

For $0 \leq s \leq t \leq 1$, we consider $\phi(1) - \phi(t) \geq 0$ Multiplying both sides by $\phi(s) - \phi(0) \geq 0$, we have:

$$(\phi(1) - \phi(t))(\phi(s) - \phi(0)) \geq 0,$$

which implies,

$$\phi(1)\phi(s) + \phi(0)\phi(t) \geq \phi(1)\phi(0) + \phi(s)\phi(t),$$

multiplying both sides of the above inequality by (-1), we get:

$$-\phi(1)\phi(0) - \phi(s)\phi(t) \geq -\phi(1)\phi(s) - \phi(0)\phi(t),$$

adding $\phi(1)\phi(t) + \phi(0)\phi(s)$ to both sides, we obtain:

$$(\phi(t) - \phi(0))(\phi(1) - \phi(s)) \geq (\phi(t) - \phi(s))(\phi(1) - \phi(0)),$$

raising both sides to the power $(\alpha - 1)$ and then dividing by $(\phi(t) - \phi(0))^{\alpha-1}$, we get:

$$(\phi(1) - \phi(s))^{\alpha-1} - \frac{(\phi(1) - \phi(0))^{\alpha-1}}{(\phi(t) - \phi(0))^{\alpha-1}}(\phi(t) - \phi(s))^{\alpha-1} \geq 0,$$

hence $G(t,s) \geq 0$ for all $t, s \in J$.

We prove (ii): Since ϕ is a strictly increasing function, we have $\phi(t) - \phi(0) \leq \phi(1) - \phi(0)$ whenever $t \in J$,

which implies $\gamma(t) \leq 1$, for $0 \leq t \leq s \leq 1$ we can easily concluded that:

$$\frac{\gamma(t)}{\Gamma(\alpha)}(\phi(1) - \phi(s))^{\alpha-1} \leq \frac{(\phi(1) - \phi(0))^{\alpha-1}}{\Gamma(\alpha)},$$

and for $0 \leq s \leq t \leq 1$:

$$\begin{aligned}\frac{\gamma(t)}{\Gamma(\alpha)}\left((\phi(1) - \phi(s))^{\alpha-1} - \frac{1}{\gamma(t)}(\phi(t) - \phi(s))^{\alpha-1}\right) &\leq \frac{1}{\Gamma(\alpha)}\Bigg((\phi(1) - \phi(0))^{\alpha-1} \\ &\quad - \frac{(\phi(1) - \phi(0))^{\alpha-1}(\phi(t) - \phi(s))^{\alpha-1}}{(\phi(t) - \phi(0))^{\alpha-1}}\Bigg) \\ &\leq \frac{1}{\Gamma(\alpha)}(\phi(1) - \phi(0))^{\alpha-1}\left(1 - \frac{(\phi(t) - \phi(s))^{\alpha-1}}{(\phi(t) - \phi(0))^{\alpha-1}}\right) \\ &\leq \frac{1}{\Gamma(\alpha)}(\phi(1) - \phi(0))^{\alpha-1},\end{aligned}$$

hence $G(t,s) \leq \dfrac{(\phi(1) - \phi(0))^{\alpha-1}}{\Gamma(\alpha)}$ for $t, s \in J$. □

Let us defin the operator $\mathcal{T} : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ by:

$$\mathcal{T}(u(t)) = f(t,u(t)) + \eta\gamma(t) + \int\limits_0^1 G(t,s)\phi'(s)\sigma_u(s)ds,$$

with $\sigma_u(t) = g\left(t, y(t), D_{0^+}^{\alpha;\phi}(u(t) - f(t,u(t)))\right)$.
Where $C(J,\mathbb{R})$ is equipped with the norm:

$$\|u\|_\infty = \underset{t\in J}{max}|u(t)|.$$

We note that the fixed point of this operator is a solution to the problem (Q).

2.2.1 Existence results

Using Schaefer's fixed point theory under certain conditions imposed on the function f and g, we will establish the existence of at least one solution to the problem (Q). assume that the functions $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous and satisfying the following condition:

(H_1) There exicts a constant $M_g \in \mathbb{R}_+^*$ so that

$$|g(t,u,v)| \leq M_g,$$

for all $u,v \in \mathbb{R}$ and $t \in J$.

(H_2) There exicts a constant $M_f \in \mathbb{R}_+^*$ so that

$$|f(t,u)| \leq M_f,$$

for all $u \in \mathbb{R}$ and $t \in J$.

Theorem 2.4. *We assume that the conditions $(H_1) - (H_2)$ are satisfied then the problem (Q) has at least one solution.*

Proof. The proof will be given in four steps

Step one: Proof that $\mathcal{T}$ is continuous, let (U_n) be a convergent sequence towards

$(u_n) \subset C(J,\mathbb{R})$ therefore, for all the $t \in J$, we have

$$\begin{aligned}
|\mathcal{T}(u_n(t)) - \mathcal{T}(u(t))| &= \left| f(t,u_n(t)) - f(t,u(t)) + \eta\gamma(t) - \eta\gamma(t) + \int_0^t G(t,s)\phi'(s)(\sigma_{u_n})(s) - \sigma_u(s))ds \right| \\
&\leq \Big| f(t,u_n(t)) - f(t,u(t)) \Big| + \int_0^1 G(t,s)\phi'(s)\Big|(\sigma_{u_n})(s) - \sigma_u(s))\Big| ds \\
&\leq \Big| f(t,u_n(t)) - f(t,u(t)) \Big| + \frac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \phi'(s)\Big|(\sigma_{u_n})(s) - \sigma_u(s))\Big| ds \\
&\leq \|f(.,u_n(.)) - f(.,u(.))\|_\infty + \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)} \|(\sigma_{u_n}(.) - \sigma_u(.))\|_\infty,
\end{aligned}$$

since, the functions f and g are continuous, we get

$$\lim_{n\to\infty} \|\mathcal{T}(u_n(.)) - \mathcal{T}(u(.))\|_\infty = 0,$$

hence, $\mathcal{T}$ is continuous.

Step two: The image of every bounded set of $C(J,\mathbb{R})$ under $\mathcal{T}$ is a uniformly bounded in $C(J,\mathbb{R})$, To establish this, it suffices to demonstrate that for any given $r > 0$, there exists a positive constant $l > 0$ therefore, for evry $u \in B_r$ we have $\|\mathcal{T}u(.)\|_\infty \leq l$ with

$$B_r = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq r\}.$$

For evry $t \in J$, and by using the conditions (H1) and (H2) we get

$$\begin{aligned}
|\mathcal{T}(u(t))| &\leq |\eta|\gamma(t) + |f(t,u(t))| + \int_0^1 G(t,s)\phi'(s)|\sigma_u(s)|ds \\
&\leq |\eta|\gamma(t) + M_f + \frac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)} M_g \int_0^1 \phi'(s)ds \\
&\leq |\eta| + M_f + \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)} M_g = l,
\end{aligned}$$

hence $\mathcal{T}(B_r)$ is uniformly bounded.

Step three: The image of every bounded set of $C(J,\mathbb{R})$ under $\mathcal{T}$ is a equicontinuous set in $C(J,\mathbb{R})$ for each $u \in B_r$ and $t_1, t_2 \in J, t_1 < t_2$, we have

$$\begin{aligned}
|\mathcal{T}(u(t_2)) - \mathcal{T}(u(t_1))| &= |f(t_2,u(t_2)) - f(t_1,u(t_1)) + \eta(\gamma(t_2) - \gamma(t_1)) \\
&\quad + \int_0^1 (G(t_2,s) - G(t_1,s))\phi'(s)\sigma_u(s)ds| \\
&\leq |f(t_2,u(t_2)) - f(t_1,u(t_1))| + |\eta||\gamma(t_2) - \gamma(t_1)| \\
&\quad + \int_0^1 |G(t_2,s) - G(t_1,s)|\phi'(s)\sigma_u(s)ds,
\end{aligned}$$

and

$$|G(t_2,s)-G(t_1,s)| = \left|\frac{(\phi(1)-\phi(s))^{\alpha-1}}{(\phi(1)-\phi(0))^{\alpha-1}\Gamma(\alpha)}\left[(\phi(t_2)-\phi(0))^{\alpha-1}-(\phi(t_1)-\phi(0))^{\alpha-1}\right]\right.$$

$$\left.+\frac{1}{\Gamma(\alpha)}\left[(\phi(t_1)-\phi(s))^{\alpha-1}-(\phi(t_2)-\phi(s))^{\alpha-1}\right]\right|,$$

by applying the mean value theorem, we obtain:

$$|G(t_2,s)-G(t_1,s)| = |t_2-t_1|\left[\frac{(\phi(1)-\phi(s))^{\alpha-1}}{(\phi(1)-\phi(0))^{\alpha-1}\Gamma(\alpha)}h_1(\xi)+\frac{1}{\Gamma(\alpha)}h_2(\theta)\right].$$

with

$$h_1(\xi) = (\alpha-1)\phi'(\xi)(\phi(\xi)-\phi(0))^{\alpha-2},$$

$$h_2(\theta) = (\alpha-1)\phi'(\theta)(\phi(\theta)-\phi(s))^{\alpha-2},$$

therefore, as $t_1 \longrightarrow t_2$, $|\mathcal{T}(u(t_2))-\mathcal{T}(u(t_1))| \longrightarrow 0$.

Hence by the Arzela Ascoli theorem, $\mathcal{T} : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is completely continuous.

Step four: Prove that the set χ is bounded

$$\chi = \Big\{u(t) \in C(J,\mathbb{R}) : u(t) = \lambda\mathcal{T}(u(t)), \lambda \in [0,1]\Big\}.$$

Let $u \in \chi$ for all $t \in J$ we have:

$$\begin{aligned} u(t) &= \lambda\left[\eta\gamma(t)+f(t,u(t))+\int_0^1 G(t,s)\phi'(s)\sigma_u(s)ds\right] \\ |u(t)| &\le |\eta|\gamma(t)+M_f+M_g\int_0^1 G(t,s)\phi'(s)ds \\ &\le |\eta|+M_f+\frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}M_g = L, \end{aligned}$$

hence χ is bounded.

Then using Schaefer's fixed point theory we found that the problem (Q) has at least one solution. ☐

Example 2.5. Let the problem with the following general fractional differential equaions:

$$\begin{cases} D_{0^+}^{\frac{7}{4},\frac{e^t}{7}}(u(t)-f(t,u(t)))+g\left(t,u(t),D_{0^+}^{\frac{7}{4},\frac{e^t}{7}}(u(t)-f(t,u(t)))\right)=0, t\in J \\ \lim_{t\to 0}(\phi(t)-\phi(0))^{\frac{7}{4}}(u(t)-f(t,u(t)))=0, \\ u(1)=1+f(1,u(1)), \end{cases} \tag{Q$_1$}$$

where

$$f(t,u(t)) = \frac{t}{2}\cos(u(t)),$$

$$g\left(t, u(t), D_{0^+}^{\frac{7}{4},\frac{e^t}{7}}\left(u(t)-f(t,u(t))\right)\right)=\frac{1}{3}(1+t)\cos(u(t))+\frac{1}{9}\sin(D_{0^+}^{\frac{7}{4},\frac{e^t}{7}}\left(u(t)-f(t,u(t))\right)).$$

Let's put:

$$f(t,u)=\frac{t}{2}\cos(u),$$

$$g(t,u,v)=\frac{1}{3}(1+t)\cos(u)+\frac{1}{9}\sin(v).$$

For $u, v \in \mathbb{R}$ and $t \in J$ then:

$$|f(t,u)| \leq \frac{1}{2},$$

$$|g(t,u,v)| \leq \frac{7}{9}.$$

Hence, all the conditions of theorem (2.5) are satisfied, with: $M_f = \frac{1}{2}$, $M_g = \frac{7}{9}$, therefore there exists at least one solution to the problem (Q_1) .

2.2.2 Existence and uniqueness results

In what followes we will establish the existence of a unique solution to the problem (Q) using the Banach Fixed-point theorem under centain conditions imposed on the functions f and g.

We impose the following conditions

(H_3) There exists the constents $k_1, k_3 \in \mathbb{R}_+^*$ and $k_2 \in]0,1[$ such that

$$|g(t,u,v)-g(t,\bar{u},\bar{v})| \leq k_1|u-\bar{u}|+k_2|v-\bar{v}|,$$

$$|f(t,u)-f(t,\bar{u})| \leq k_3|u-\bar{u}|,$$

for every $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

Theorem 2.6. *We assume that the condition*(H_3) *is satisfied, if*

$$p = k_3 + \frac{(\phi(1)-\phi(0))^{\alpha} k_1}{\Gamma(\alpha)(1-k_2)} < 1, \tag{2.6}$$

then the problem (Q) *admits a unique solution in* $C(J,\mathbb{R})$ *.*

Proof. We consider the previously defined operator $\mathcal{T}$, for all $x, y \in C(J,\mathbb{R})$ and $t \in J$, according to the condition (H3), we have

$$\begin{aligned}|\mathcal{T}(x(t))-\mathcal{T}(y(t))| &= |f(t,x(t))-f(t,y(t))+\eta\gamma(t)-\eta\gamma(t)+\int_0^1 G(t,s)\phi'(s)\left(\sigma_x(s)-\sigma_y(s)\right)ds \\ &\leq |f(t,x(t))-f(t,y(t))|+\int_0^1 G(t,s)\phi'(s)|\left(\sigma_x(s)-\sigma_y(s)\right)|ds,\end{aligned}$$

$$\leq k_3|x-y| + \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\int_0^1 |\left(\sigma_x(s)-\sigma_y(s)\right)|ds. \tag{2.7}$$

On the other hand

$$\begin{aligned}|\left(\sigma_x(s)-\sigma_y(s)\right)| &= |g\left(t, x(t), D^{\alpha,\phi}\left(x(t)-f(t,x(t))\right)\right) - g\left(t, y(t), D^{\alpha,\phi}\left(y(t)-f(t,y(t))\right)\right)| \\ &\leq k_1|x-y| + k_2|D^{\alpha,\phi}\left(x(t)-f(t,x(t)\right) - D^{\alpha,\phi}\left(y(t)-f(t,y(t)\right)| \\ &\leq k_1|x-y| + k_2|\left(\sigma_x(t)-\sigma_y(t)\right)|,\end{aligned}$$

$$(1-k_2)|\left(\sigma_x(t)-\sigma_y(t)\right)| \leq k_1|x-y|,$$

$$|\left(\sigma_x(t)-\sigma_y(t)\right)| \leq \frac{k_1}{(1-k_2)}|x-y|, \tag{2.8}$$

by substituting (2.8) in (2.7) we get

$$\begin{aligned}|\mathcal{T}(x(t)) - \mathcal{T}(y(t))| &\leq k_3|x-y| + \frac{(\phi(1)-\phi(0))^{\alpha}k_1}{\Gamma(\alpha)(1-k_2)}|x-y| \\ &\leq \left[k_3 + \frac{(\phi(1)-\phi(0))^{\alpha}k_1}{\Gamma(\alpha)(1-k_2)}\right]|x-y|,\end{aligned}$$

thus

$$\|\mathcal{T}x(.) - \mathcal{T}y(.)\|_{\infty} \leq p\|x-y\|_{\infty},$$

and according to the condition (2.6) the operator $\mathcal{T}$ is contraction, then according to Banach's fixed point theorem it admits a unique fixed point, and it is the unique solution of the problem (Q). ☐

Example 2.7. Let the problem (Q_1), according to the condition (H_3) we have for $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$

$$|f(t,u) - f(t,\bar{u})| \leq \frac{1}{2}|u-\bar{u}|,$$

$$|g(t,u,v) - g(t,\bar{u},\bar{v})| \leq \frac{2}{3}|u-\bar{u}| + \frac{1}{9}|v-\bar{v}|.$$

Hence all the conditions of theorem (2.7) are satisfied, and $p = 0.5698569$,
with $k_1 = \frac{2}{3}$, $k_2 = \frac{1}{9}$, $k_3 = \frac{1}{2}$.
Therefore, there exists a unique solution of the problem (Q_1).

2.2.3 Ulam-Hyers stability results

In this subsection we investigate Ulam-Hyers stability for the problem (Q).

Lemma 2.8. *If y is a solution for the following fractional differential inequality for $\epsilon > 0$:*

$$|D^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)+g\left(t,u(t),D^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)\right)|<\epsilon, \tag{2.9}$$

then y is a solution to the following inequality:

$$|y(t)-\mathcal{T}(y(t))| \leq \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\epsilon. \tag{2.10}$$

Proof. Let y be a solution to the inequality (2.10), for $\epsilon > 0$ and by using Theorem (2.7) and Remark (1.13) which related the continuous function ψ; $|\psi(t)| < \epsilon$, $t \in J$ and according to (2.9), we have

$$y(t)=\eta\gamma(t)+f(t,y(t))+\int_0^1 G(t,s)\phi'(s)\left[\sigma_y(s)+\psi(s)\right]ds,$$

then

$$\begin{aligned}|y(t)-\mathcal{T}(y(t))| =&\Big|\eta\gamma(t)+f(t,y(t))+\int_0^1 G(t,s)\phi'(s)\left[\sigma_y(s)+\psi(s)\right]ds\\ &-\eta\gamma(t)-f(t,y(t))-\int_0^1 G(t,s)\phi'(s)\sigma_y(s)ds\Big|\\ =&\left|\int_0^1 G(t,s)\phi'(s)\psi(s)ds\right|\\ \leq&\int_0^1 |G(t,s)|\phi'(s)|\psi(s)|ds\\ \leq&\frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\epsilon,\end{aligned}$$

□

Theorem 2.9. *We assume that the conditions (H_3) are satisfied, then the problem (Q) is stable according to Ulam-Hyers.*

Proof. Under conditions (H_3), there exists a unique solution for the problem (Q) in $C(J,\mathbb{R})$. Let $y \in C(J,\mathbb{R})$ be a solution to the inequality (2.10), therefore, for $t \in J$ we have

$$\begin{aligned}|y(t)-u(t)| =&|y(t)-\eta\gamma(t)-f(t,u(t))-\int_0^1 G(t,s)\phi'(s)\sigma_u(s)ds|\\ \leq&|y(t)-\mathcal{T}(y(t))+\mathcal{T}(y(t))-\mathcal{T}(u(t))|\\ \leq&|y(t)-\mathcal{T}(y(t))|+|\mathcal{T}(y(t))-\mathcal{T}(u(t))|\\ \leq&\frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\epsilon+p|y(t)-u(t)|,\end{aligned}$$

thus

$$\|y-u\|_\infty \leq \frac{(\phi(1)-\phi(0))^\alpha}{\Gamma(\alpha)(1-p)}\epsilon,$$

we put

$$c_g = \frac{(\phi(1)-\phi(0))^\alpha}{\Gamma(\alpha)(1-p)},$$

we get

$$|y(t)-u(t)| \leq c_g\epsilon.$$

□

Therefore, the problem (Q) is stable according to Ulam-Hyers.

Example 2.10. Let the problem(Q_1), according to the condition (H_3) and to the Theorem (2.10) we have $c_g = 0.2165387$, then the problem (Q_1) is stable according to Ulam-Hyers.

CHAPTER

3

Existence, uniqueness and stability for some generalized fractional boundary value problems

3.1 Introduction

As previously mentioned fractional differential equations play a very important role in basic sciences, for that the existence of solutions of the regular fractional boundary value problems (FBVPs), have been discussed in many papers, such as [7, 12]. In recent years many mathematicians have studied deferent types of boundary conditions, to obtain new results of differential models, taking into consideration that fractional differential equations can be extended by creating different types of boundary conditions, see [6, 13]. Motivated by this researches, in this chapter we will study the existence and stability of a unique solution to the following nonlocal fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha,\phi}\left(z(t)\right) + f(t, z(t)) = 0, \quad t \in J \\ z\left(0\right) = 0, \\ z\left(1\right) = \beta z\left(\eta\right), \end{cases} \tag{S}$$

where:

- $D^{\alpha,\phi}$ is the ϕ-Riemann-Liouville fractional derivative with $1 < \alpha < 2$.
- $J = [0, 1]$, $\eta \in \mathbb{R}$.

- $f \in C(J \times \mathbb{R}, \mathbb{R})$ is a nonlinear function.

For our purpose, we convert the differential problem into equivalent integral equations via constructing Green function then we established certain properties for it and we assumed some sufficient conditions through which we prove the existence of a unique solution using the Banach fixed point theory, we also studied the stability of this solution, and concluded the chapter with an illustrative example.

3.2 Existence, uniqueness and Ulam-Hyers stability results

We consider the problem (S).

Definition 3.1. The function z from $C(J, \mathbb{R})$ is a solution to the problem (S) if it satisfies the equation

$$D_{0^+}^{\alpha;\phi}\left(z(t)\right) = -f\left(t, z(t)\right), \tag{3.1}$$

and the conditions

$$z\left(0\right) = 0, \tag{3.2}$$

$$z\left(1\right) = \beta z(\eta). \tag{3.3}$$

Lemma 3.2. *z is a solution for the fractional differential equation (3.1) and satisfying the conditions(3.2)-(3.3), if and only if z is a solution for the integral equation via Green function:*

$$z(t) = \int_0^1 H(t,s)\phi'(s) f(s, z(s))ds, \qquad t \in J \tag{3.4}$$

where

$$H(t,s) = (\mu\Gamma(\alpha))^{-1}\Pi(t,s), \tag{3.5}$$

$$\Pi(t,s) = \begin{cases} \left[k(t)(\phi(1)-\phi(s))\right]^{\alpha-1} - \beta\left[k(t)(\phi(\eta)-\phi(s))\right]^{\alpha-1} - \mu(\phi(t)-\phi(s))^{\alpha-1}, \\ 0 \le s \le t \le 1, \qquad s \le \eta, \\ \left[k(t)(\phi(1)-\phi(s))\right]^{\alpha-1} - \mu\left[\phi(t)-\phi(s)\right]^{\alpha-1}, \\ 0 \le \eta \le s \le t \le 1, \\ \left[k(t)(\phi(1)-\phi(s))\right]^{\alpha-1} - \beta\left[k(t)(\phi(\eta)-(\phi(s))\right]^{\alpha-1}, \qquad 0 \le t \le s \le \eta \le 1 \\ \left[k(t)(\phi(1)-\phi(s))\right]^{\alpha-1}, \qquad 0 \le t \le s \le 1, \qquad \eta \le s. \end{cases} .$$

with:

$$k(t) = \phi(t) - \phi(0),$$

$$\mu = [k(1)]^{\alpha-1} - \beta\,[k(\eta)]^{\alpha-1},$$

Proof. We have z as a solution of the problem (S).

$$I_{0^+}^{\alpha,\phi}\left(D_{0^+}^{\alpha,\phi}(z(t))\right) = -I_{0^+}^{\alpha,\phi}(f(t,z(t))) + C_0(\phi(t)-\phi(0))^{\alpha-1} + C_1(\phi(t)-\phi(0))^{\alpha-2},$$

$$z(t) = -I_{0^+}^{\alpha,\phi}(f(t,z(t))) + C_0(\phi(t)-\phi(0))^{\alpha-1} + C_1(\phi(t)-\phi(0))^{\alpha-2},$$

by using the giving conditions (3.2)-(3.3), we obtin $C_1 = 0$ and

$$C_0 = \frac{1}{\mu\Gamma(\alpha)}\left[\int_0^1 \phi'(s)(\phi(1)-\phi(s))^{\alpha-1}f(s,z(s))ds - \beta\int_0^1 \phi'(s)(\phi(\eta)-\phi(s))^{\alpha-1}f(s,z(s))ds\right],$$

by substitution:

$$\begin{aligned}
z(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}f(s,z(s))ds \\
&\quad + \frac{1}{\mu\Gamma(\alpha)}\left[\int_0^1 \phi'(s)(\phi(1)-\phi(s))^{\alpha-1}f(s,z(s))ds - \beta\int_0^1 \phi'(s)(\phi(\eta)-\phi(s))^{\alpha-1}f(s,z(s))ds\right][k(t)]^{\alpha-1} \\
&= \int_0^1 H(t,s)\phi'(s)f(s,z(s))ds.
\end{aligned}$$

The converse can be easily inferred from Lemma (1.11). □

Lemma 3.3. *The function H defined by (3.5) satisfies :*

(i) $H(t,s) > 0$, for all $t,s \in J$.

(ii) $H(t,s) \leq (\mu\Gamma(\alpha))^{-1}(\phi(1)-\phi(0))^{2\alpha-2}$, for all $t,s \in J$.

Proof. **We prove (i)**: For $0 \leq s \leq t \leq 1, s \leq \eta$, let $\Pi(t,s) = \dfrac{k(s)}{k(t)}$ and

$$h(t,s) = [k(t)\,(\phi(1)-\phi(s))]^{\alpha-1} - \beta\,[k((t)\,(\phi(\eta)-\phi(s))]^{\alpha-1} - \mu(\phi(t)-\phi(s))^{\alpha-1}.$$

Then we have

$$h(t,s) = (k(t))^{\alpha-1}\left[(\phi(1)-\phi(s))^{\alpha-1} - \beta(\phi(\eta)-\phi(s))^{\alpha-1} - \mu(1-\Pi(t,s))^{\alpha-1}\right],$$

consider

$$\omega(t,s) = \left[(\phi(1)-\phi(s))^{\alpha-1} - \beta(\phi(\eta)-\phi(s))^{\alpha-1} - \mu(1-\Pi(t,s))\right],$$

thus, we obtain

$$\frac{d\omega(t,s)}{dt} = -(\alpha-1)\mu(1-\Pi(t,s))^{\alpha-2}\frac{(k(s))\phi'(t)}{(k(t))^2} < 0,$$

which implies that $\omega(t,s)$ is a decreasing function for $0 \leq s \leq t \leq 1, t \leq \eta$.

Moreover, we note that for $t \in J$, we have:

$$\omega(1,s) = \left[(\phi(1)-\phi(s))^{\alpha-1} - \beta(\phi(\eta)-\phi(s))^{\alpha-1}\mu(1-\Pi(1,s))^{\alpha-1}\right]$$

$$= \left[-\beta(\phi(\eta)-\phi(s))^{\alpha-1} + \beta(k(\eta))^{\alpha-1}(1-\Pi(1,s))^{\alpha-1}\right],$$

by adding and subtracting $\phi(0)$ in the first term of the above equation, we get

$$\omega(1,s) = \beta(k(\eta))^{\alpha-1}\left[(1-\Pi(1,s))^{\alpha-1} - (1-\Pi(\eta,s))^{\alpha-1}\right] > 0,$$

hence, $h(s,t) > 0$.

For $0 \leq s \leq t \leq 1, s \leq \eta$ for $\eta \leq s \leq t$, we let :

$$h(t,s) = (k(t))^{\alpha-1}(\phi(1)-\phi(s))^{\alpha-1} - \left[(k(1))^{\alpha-1} - \beta(k(\eta))^{\alpha-1}\right](\phi(t)-\phi(s))^{\alpha-1},$$

it follows that

$$h(t,s) > (k(t))^{\alpha-1}(\phi(1)-\phi(s))^{\alpha-1} - (k(1))^{\alpha-1}(\phi(t)-\phi(s))^{\alpha-1},$$

adding and subtracting $\phi(0)$, we have :

$$h(t,s) = (k(t))^{\alpha-1}(\phi(1)-\phi(0)+\phi(0)-\phi(s))^{\alpha-1} - (k(1))^{\alpha-1}(\phi(t)-\phi(0)+\phi(0)-\phi(s))^{\alpha-1},$$

therefore, we obtain

$$h(t,s) = [k(1)k(t)]^{\alpha-1}\left[(1-\Pi(1,s))^{\alpha-1} - (1-\Pi(t,s))^{\alpha-1}\right] > 0.$$

For $t \leq s \leq \eta$, we let

$$\begin{aligned} h(t,s) &= (k(t))^{\alpha-1}(\phi(1)-\phi(s))^{\alpha-1} - \beta(k(t))^{\alpha-1}(\phi(\eta)-\phi(s))^{\alpha-1} \\ &= (k(t))^{\alpha-1}\left[(\phi(1)-\phi(s))^{\alpha-} - \beta(\phi(\eta)-\phi(s))^{\alpha-1}\right] \\ &= (k(t))^{\alpha-1}\left[(\phi(1)-\phi(0))^{\alpha-1}(1-\Pi(1,s))^{\alpha-1} - \beta(\phi(\eta)-\phi(0))^{\alpha-1}(1-\Pi(\eta,s))^{\alpha-1}\right], \end{aligned}$$

since $\mu > 0$, we obtain

$$h(t,s) > (k(t))^{\alpha-1}\mu(1-\Pi(\eta,s))^{\alpha-1} > 0.$$

Clearly, for $0 \leq t \leq s \leq 1, \eta \leq s, h(t,s) > 0$.

Hence $H(t,s) > 0$ for $s, t \in J$.

We prove (ii) Since ϕ is a strictly increasing function, we have:

$$\begin{aligned}
\phi(1) - \phi(s) &\leq \phi(1) - \phi(0) \\
\phi(\eta) - \phi(s) &\leq \phi(\eta) - \phi(0) \\
\phi(t) - \phi(0) &\leq \phi(1) - \phi(0)
\end{aligned}$$

whenever $t \in J$

which implies $\Pi(t,s) \leq 1, k(t) \leq \phi(t) - \phi(0)$.

For $0 \leq s \leq t \leq 1, s \leq \eta$ we get :

$$\begin{aligned}
\Pi(t,s) &= (k(t))^{\alpha-1}\left[(\phi(1)-\phi(s))^{\alpha-1} - \beta(\phi(\eta)-\phi(s))^{\alpha-1} - \mu(1-\Pi(t,s))^{\alpha-1}\right] \\
&\leq (\phi(1)-\phi(0))^{\alpha-1}\left[(\phi(1)-\phi(0))^{\alpha-1} - \beta(\phi(\eta)-\phi(0))^{\alpha-1} - \mu\right] \\
&\leq (\phi(1)-\phi(0))^{2\alpha-2}.
\end{aligned}$$

For $0 \leq \eta \leq s \leq t \leq 1$ we get :

$$\begin{aligned}
\Pi(t,s) &= (k(t))^{\alpha-1}\left[(\phi(1)-\phi(s))^{\alpha-1} - \mu(1-\Pi(t,s))^{\alpha-1}\right] \\
&\leq (\phi(1)-\phi(0))^{\alpha-1} - \mu \\
&\leq (\phi(1)-\phi(0))^{2\alpha-2}.
\end{aligned}$$

For $0 \leq t \leq s \leq \eta \leq 1$ we get :

$$\begin{aligned}
\Pi(t,s) &= (k(t))^{\alpha-1}\left[(\phi(1)-\phi(s))^{\alpha-1} - \beta(\phi(\eta)-\phi(s))^{\alpha-1}\right] \\
&\leq (\phi(1)-\phi(0))^{\alpha-1}\left[(\phi(1)-\phi(0))^{\alpha-1} - \beta(\phi(\eta)-\phi(0))^{\alpha-1}\right] \\
&\leq (\phi(1)-\phi(0))^{2\alpha-2}.
\end{aligned}$$

For $0 \leq t \leq s \leq 1, \eta \leq s$ we get :

$$\begin{aligned}
\Pi(t,s) &= (k(t))^{\alpha-1}(\phi(1)-\phi(s))^{\alpha-1} \\
&\leq (\phi(1)-\phi(0))^{\alpha-1}(\phi(1)-\phi(0))^{\alpha-1} \\
&\leq (\phi(1)-\phi(0))^{2\alpha-2}.
\end{aligned}$$

Hence

$$h(t,s) \leq (\mu\Gamma(\alpha))^{\alpha-1}(\phi(1)-\phi(0))^{2\alpha-2}, \qquad t, s \in J.$$

□

Let us defin the operator $\mathcal{P} : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ by:

$$\mathcal{P}(z(t)) = \int_0^1 H(t,s)\phi'(s)f(s,z(s))ds, \tag{3.6}$$

where $C(J,\mathbb{R})$ is equipped with the norm:

$$\|z\|_\infty = \underset{t\in J}{max}|z(t)|.$$

We note that the fixed point of this operator is a solution to the problem (S).

3.2.1 Existence and uniquness results

In what followes we will establish the existence of a unique solution to the problem (S) using the Banach Fixed-point theorem under certain conditions imposed on the function f.

We impose the following condition:

(H_1) There exists the constent $k_1 \in \mathbb{R}_+^*$ so that

$$|f(t,u) - f(t,\bar{z})| \leq k_1|z - \bar{z}|,$$

for $t \in J$ and $z, \bar{z} \in \mathbb{R}$.

Theorem 3.4. *We assume that the condition (H_1) is satisfied, if*

$$p = k_1(\mu\Gamma(\alpha))^{-1}\left[(\phi(1) - \phi(0))\right]^{2\alpha-1} < 1, \tag{3.7}$$

then the problem (S) admits a unique solution in $C(J,\mathbb{R})$.

Proof. We consider the operator $\mathcal{P}$ defined in (3.6), $x, y \in C(J,\mathbb{R})$, for every $t \in J$ according to the condition (H1), we have

$$\begin{aligned}|\mathcal{P}(x(t)) - \mathcal{P}(y(t))| &= \left|\int_0^1 H(t,s)\phi'(s)f(t,x(t))ds - \int_0^1 H(t,s)\phi'(s)f(t,y(t))ds\right| \\ &\leq \int_0^1 H(t,s)\phi'(s)|f(t,x(t)) - f(t,y(t))|ds \\ &\leq (\mu\Gamma(\alpha))^{-1}(\phi(1) - \phi(0))^{2\alpha-2}k_1|x - y|\int_0^1 \phi'(s)ds, \\ &\leq (\mu\Gamma(\alpha))^{-1}(\phi(1) - \phi(0))^{2\alpha-1}k_1|x - y|.\end{aligned}$$

Thus $\|\mathcal{P}(x(.)) - \mathcal{P}(y(.))\|_\infty \leq p\|x - y\|_\infty$. □

So according to the condition (3.7) the operator $\mathcal{P}$ is contraction, then according to Banach's fixed-point theorem it admits a unique fixed point, and it is the unique solution to the problem (S).

3.2.2 Ulam-Hyers stabilty results

In this subsection we investigate Ulam-Hyers stability for problem (S).

Lemma 3.5. *If y is a solution for the following franctional differential inequality for $\epsilon > 0$,*

$$|D_{0+}^{\alpha,\phi}\left(z(t)\right) + f\left(t, z(t)\right)| < \epsilon. \tag{3.8}$$

then y is a solution to the following inequality:

$$|y(t) - \mathcal{P}(y(t))| \leq (\mu\Gamma(\alpha))^{\alpha-1}(\phi(1) - \phi(0))^{2\alpha-1}\epsilon, \tag{3.9}$$

Proof. Let z be a solution to the inequality (3.8), for $\epsilon > 0$ and by using Theorem (3.5) and the Remark (1.13) which related the continuous function ψ; $|\psi(t)| < \epsilon$, $t \in J$, and according to (3.8), we have

$$y(t) = \int_0^1 H(t,s)\phi'(s)\left[f(s) + \psi(t)\right] ds,$$

$$\begin{aligned}
|y(t) - \mathcal{P}(y(t))| =& \left| \int_0^1 H(t,s)\phi'(s)\left[f(s) + \psi(t)\right] - \int_0^1 H(t,s)\phi'(s)f(s)ds \right| \\
=& \left| \int_0^1 H(t,s)\phi'(s)\psi(s)ds \right| \\
\leq& \int_0^1 |H(t,s)|\phi'(s)|\psi(s)|ds \\
\leq& (\mu\Gamma(\alpha))^{\alpha-1}(\phi(1) - \phi(0))^{2\alpha-1}\epsilon.
\end{aligned}$$

□

Theorem 3.6. *We assume that the condition (H_1) is satisfied, then the problem (S) is stable according to Ulam-Hyers.*

Proof. Under the condition (H_1), there exists a unique solution for the problem (S), in $C(J,\mathbb{R})$.

Let $y \in C(J,\mathbb{R})$, be a solution to the inequality (3.8), therefore for $t \in J$, we have:

$$\begin{aligned}
|y(t) - z(t)| =& |y(t) - \int_0^1 H(t,s)\phi'(s)f(s)ds| \\
&\leq |y(t) - \mathcal{P}(y(t)) + \mathcal{P}(y(t)) - \mathcal{P}(z(t))| \\
&\leq |y(t) - \mathcal{P}(y(t))| + |\mathcal{P}(y(t)) - \mathcal{P}(z(t))| \\
&\leq (\mu\Gamma(\alpha))^{\alpha-1}(\phi(1) - \phi(0))^{2\alpha-1}\epsilon + p|y(t) - z(t)|,
\end{aligned}$$

thus

$$\|y-z\|_\infty \leq \frac{(\mu\Gamma(\alpha))^{\alpha-1}(\phi(1)-\phi(0))^{2\alpha-1}}{(1-p)}\epsilon,$$

we put

$$c_f = \frac{(\mu\Gamma(\alpha))^{\alpha-1}(\phi(1)-\phi(0))^{2\alpha-1}}{(1-p)},$$

we get

$$|y(t)-z(t)| \leq c_f\epsilon.$$

□

Therefore, the problem (S) is stable according to Ulam-Hyers.

Example 3.7. Let the problem with the following general fractional differential equaions:

$$\begin{cases} D_{0^+}^{\frac{3}{2},\frac{t^2}{2}} z(t) + f(t,z(t)), \quad t \in J \\ z\left(0\right) = 0, \\ z\left(1\right) = \frac{1}{2} z\left(\frac{1}{3}\right), \end{cases} \tag{S$_1$}$$

where

$$f(t,z(t)) = e^{\frac{t}{2}}(1+z(t)).$$

Furthermore, we get

$$\mu = \left[k(1)\right]^{\alpha-1} - \beta\left[k(\eta)\right]^{\alpha-1} = \frac{5\sqrt{2}}{12} \approx 0.589255.$$

Let's put:

$$f(t,z) = e^{\frac{t}{2}}(1+z),$$

then for $z, \bar{z} \in \mathbb{R}$ and $t \in J$:

$$|f(t,z) - f(t,\bar{z})| \leq e^{\frac{1}{2}}|z-\bar{z}|.$$

Hence all the conditions of Theorem (3.5) are satisfied, and $p = 0.7892935$, with $k_1 = e^{\frac{1}{2}}$, therefore, there exists a unique solution of the problem (S$_1$) and according to the Theorem (3.7) with $c_g = 0.857406$, the problem (S$_1$) is Ulam-Hyers stable.

Conclusion

The topic of fractional order differential equations is considered one of the important and complex subjects in applied mathematics. This memorandum is just a small part of a much larger field, in which we have diligently tried to achieve the desired goal, which is to study the existence and uniqueness of the solution for some problems involving generalized fractional differential equations, as well as to study the stability of these problems according to Hyers-Ulam.

Finally, we hope to have contributed, even if just a little, in explaining how to study these issues. We hope to continue this research through the following topics:

- Studying these problems for another type of fractional derivatives in unbounded domains.
- Studying these problems in more general Banach spaces.

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