People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Higher Normal School of Technological Education -Skikda-

Department of Mathematics and Computer Science

End studies dissertation with a view to obtain a middle teaching diploma specializing in mathematics

Theoretical and numerical studies for a delayed swelling porous thermoelastic soils model

By the students

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2024-2025

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Dedication

“No one walks alone, and when you are walking on the journey of life…you have to start to thank those that joined you, walked beside you, and helped you along the way”

David H. Hooker

In the name of Allah, the Most Gracious, the Most Merciful.
All praise is due to Allah, by whose grace good things are accomplished.

I proudly and heartily dedicate this work to To the one whom God placed paradise beneath her feet, whose heart embraced me before her hands, and whose prayers eased my burdens. To the compassionate soul who illuminated my path, stood by me in the darkest nights, wiped my tears, and uplifted me. She was both a mother and a father to me …

My dearest mother.

To the one whose name I carry with pride and honor, to the absent yet ever present in my heart, to the one my soul mourns, to the pure spirit of my father, whose body was laid to rest far from my sight but never far from my prayers or my memory…

May God have mercy on you, my beloved father.

To my teachers and everyone who taught me during my academic career, with special thanks to my supervisor, **Dr Khochemane Houssem Eddine**, for the exceptional quality of his supervision, his patience, his rigor, and his availability throughout the preparation of this dissertation.
Finally, to the charming, smart and hardworking partners in this work **Amina** and **zineb**.

To all who believed in me, offered help, uphold and prayed for my success.
Thank you for being always in my back.

Abir Bourouila

Dedication

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-David H. Hooker -

In the name of Allah, the Most Gracious, the Most Merciful.
All praise is due to Allah, by whose grace good things are accomplished.

I proudly and heartily dedicate this work To the candle who always enlightened my life, who taught me the value of hard work, my source of happiness and success my dear mother. May Allah protect her.

To my lovely sister **Amal**, who is also my best friend, thank you for your constant support, your contagious humor, and your comforting presence. I am truly proud to have you in my life.

To my dear friends, who have been my pillars in difficult times and my party partners in times of joy, thank you for your sincere friendship, your unfailing support and your unconditional love.

To my teachers and everyone who taught me during my academic career, with special thanks to my supervisor, **Dr Khochemane Houssem Eddine**, for the quality of his supervision, his patience, his rigor and his availability throughout the preparation of this dissertation.

Finally, To the charming, smart, and hardworking partners in this work **Abir** and **Zineb**.
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Thank you for being always in my back

Amina Boudjallaba

Dedication

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David H. Hooker

In the name of Allah, the Most Gracious, the Most Merciful.

All praise is due to Allah, by whose grace good things are accomplished.

I proudly and heartily dedicate this work to:

To the pillars of my life-my dear parents.

To my beloved mother, whose endless love, prayers, and warm heart have been my greatest source of strength-thank you for being my light through every dark moment.

To my dear father, your support, patience and quiet strength have taught me resilience. I owe this success to your constant encouragement.

To my dear brothers, **Oussama** and **Wassim** thank you for being the kind of brothers every sister dreams of. I carry your love with me in every success.

To my precious friends, your love, laughter, kindness and support turned difficult moments into cherished memories. I'm deeply grateful to have shared this journey with you.

To my teachers and everyone who taught me during my academic career, with special thanks to my supervisor, **Dr Khochemane Houssem Eddine**, for the quality of his supervision, his patience, his rigor and his availability throughout the preparation of this dissertation

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Finally, To the charming, smart, and hardworking partners in this work **Abir** and **Amina**.

To all who believed in me, offered help, uphold and prayed for my success

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Thank you for being always in my back.

Alhamdulillah for every step, every tear and every small victory.

Zineb Fellah

Acknowledgement

"No one who achieve success does so without acknowledging the help of the others"

-Alfred North Whitehead-

Above all, praise be to almighty Allah for giving us the strength and wisdom to carry out this work.

Also, and because "a good teacher can inspire hope, ignite the imagination, and instill a love of learning" (Brad Henry), we are grateful to our supervisor, **Mr. Khochemane Houssem Eddine** for his thoughtful comments and broad-minded supervision during our academic work, as well as for his constant and beneficial feedback. We consider ourselves extremely fortunate to have the opportunity to work under his ideal supervision.

Our infinite appreciation is addressed to our beloved teacher and the chairwomen, **Mrs. Seghier Fatma Zohra**, and the examiner, **Mr. Boucenna Djalal** and **Mr. Mansouri Bouzid** for agreeing to serve on the jury.

We also want to thank the teaching team of Mathematics department at the Higher Normal School of Technological Education -Skikda- for their dedication, support, and the knowledge they have imparted to us throughout our years of study.

ABSTRACT IN ENGLISH

In this work, we consider a one-dimensional swelling problem in porous elastic soils with second sound and a constant internal delay, where the heat conduction is given by Cattaneo's law. The system's well-posedness is established using the semigroup approach, ensuring the existence and uniqueness of the solution. Furthermore, through energy analysis and the construction of a suitable Lyapunov functional, it is proven that the dissipation provided by the second sound mechanism leads to exponential stability, regardless of the system parameters. Finally, we give some numerical tests to illustrate the theoretical results by carrying out an Euler scheme for time discretization and finite difference method for spatial discretization. Then, we introduce a fixed point algorithm to solve the discretized problem.

Key words : *Hyperbolic problems, Delay, Swelling porous system, Porous system, Heat conduction equations, Semigroup theory, Lyapunov functional, Exponential stability*.

ABSTRACT IN FRENCH

DAns ce travail, nous étudions un problème de gonflement uni-dimensionnel dans des sols élastiques poreux, avec la prise en compte du second son et d'un retard interne constant, où la conduction thermique est décrite par la loi de Cattaneo. On démontre que le système admet une solution unique, stable et bien définie, en utilisant la théorie des semi-groupes, ce qui garantit l'existence et l'unicité de la solution.Par ailleurs, à travers une analyse énergétique et la construction d'une fonctionnelle de Lyapunov appropriée, il est démontré que la dissipation induite par le mécanisme du second son conduit à une stabilité exponentielle, indépendamment des paramètres du système. Enfin, nous présentons quelques tests numériques afin d'illustrer les résultats théoriques, en appliquant un schéma d'Euler pour la discrétisation temporelle et la méthode des différences finies pour la discrétisation spatiale. Ensuite, nous introduisons un algorithme du point fixe pour résoudre le problème discrétisé..

Mots-clés : *Problèmes hyperboliques, Retard, Système poreux gonflant, Système poreux, Équations de conduction thermique, Théorie des semi-groupes, Fonctionnelle de Lyapunov, Stabilité exponentielle*.

ملخص

في هذا العمل، تم التطرق إلى دراسة نظام مسامي مرن حراري أحادي البعد، مع وجود الصوت الثاني وتأخير داخلي ثابت، حيث يتم وصف إنتقال الحرارة وفقا لقانون كاتانيو. نهدف إلى إثبات وجود ووحدانية الحل باستخدام نظرية أنصاف الزمر. إضافة إلى ذلك، ومن خلال تحليل الطاقة وبناء دالة ليابونوف مناسبة، يتم إثبات أن التخميد الناتج عن آلية الصوت الثاني يؤدي إلى إستقرار أسي.

في النهاية، نقدم بعض الإختبارات العددية لتأكيد النتائج النظرية من خلال تنفيذ مخطط أولر لتقدير الزمن وطريقة الفروق المنتهية للتقدير المكاني، ثم نقوم باعتماد خوارزمية النقطة الثابتة لحل المسالة بعد تفكيكها.

كلمات مفتاحية: مسائل زائدية، تأخير، نظام مسامي منتفخ، نظام مسامي، معادلة توصيل حراري، نظرية أنصاف الزمر، دالة ليابونوف، إستقرار أسي.

NOTATIONS

- PDEs: the abbreviation for " partial differential equations".
- $\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).
- Ω: the open domain in $\mathbb{R}^N$
- $L^p(\Omega)$: Lebesgue space.
- $W^{k,p}(\Omega)$: the Sobolev space.
- $C^k(\Omega)$: the space of k times continously differentiable functions.
- $C^\infty(\Omega) = \cap_{k\in\mathbb{N}} C^k(\Omega)$.
- $C_c^\infty(\Omega)$: the space of $C^\infty(\Omega)$ functions with compact support.
- $< .,. >$: the scalar product.
- $\|.\|$: the norm.
- C_0: the abbreviation for "CesÃ ro summable of order 0".

GENERAL INTRODUCTION

Swelling porous soils are among the most important topics of interest in geotechnical engineering and soil mechanics due to their significant and often hazardous impact on structural safety and stability particularly in regions with high concentrations of active clay minerals such as montmorillonite. When these minerals absorb water, the interlayer spacing between clay particles increases, resulting in considerable volumetric expansion. This swelling can lead to cracks, uneven settlement, and foundation failure. Therefore, understanding the behavior of such soils is crucial for preventing structural damage and improving construction design.

In this context, we consider a mathematical framework which is a type of model that describes the coupled dynamics of mechanical displacement, thermal conduction and delayed response within a porous medium. Here's a breakdown of the model ([3])

$$\begin{cases} \rho_z z_{tt} = \alpha_1 z_{xx} + \alpha_2 u_{xx}, & \text{in } (0, L) \times (0, \infty) \\ \rho_u u_{tt} = \alpha_3 u_{xx} + \alpha_2 z_{xx} + \beta \theta_x - \mu u_t(x, t-\tau), & \text{in } (0, L) \times (0, \infty) \\ c\theta_t = -q_x + \beta u_{tx}, & \text{in } (0, L) \times (0, \infty) \\ \tau_0 q_t = -q - k\theta_x, & \text{in } (0, L) \times (0, \infty) \end{cases} \tag{1.1}$$

Where z, u, θ, q represent respectively solid matrix displacement, pore fluid displacement, temperature field and the heat flux.

The system parameters:

ρ_z : Density of the solid.

ρ_u : Density of the fluid.

c : Specific heat.

τ_0 : Relaxation time in Cattaneo's law.

α_1 : Elastic stiffness of solid.

α_2 : Coupling coefficient (solid-fluid).

α_3 : Fluid stiffness / viscosity.

β : Thermoelastic coupling coefficient.

μ : Damping or memory coefficient related to the delay response.

k : Thermal conductivity.

τ : Delay time.

The first equation describes the motion of the solid matrix z in a porous medium. The second equation describes the fluid motion u within the pores.The third equation models the energy balance (heat equation). The fourth equation describes the heat conduction via Cattaneo's law which introduces finite propagation speed for heat (Hyperbolic thermal model).

In this dissertation, we investigate the existence and uniqueness of solution of the system(1.1) subject to the following initial and boundary conditions

$$\begin{cases} z(x,0) = z_0(x) \quad , \; z_t(x,0) = z_1(x) \;\; , \;\; \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,L) \\ u(x,0) = u_0(x) \quad , \;\; u_t(x,0) = u_1(x) \;\; , \;\; q(x,0) = q_0(x) \quad , \quad x \in (0,L) \\ u_t(x,-t) = f_0(x,t) \quad \text{in } (0,L) \times (0,\tau) \\ z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = \theta_x(0,t) = \theta(L,t) = q(0,t) = 0, t > 0 \end{cases} \tag{1.2}$$

To ensure the physical consistency and mathematical well-posedness of the system, we impose the following conditions on the material parameters:

$$\alpha_1\alpha_3 > \alpha_2^2. \tag{1.3}$$

Our study is based on the semi-group theory, more precisely Lumer Phillips theorem or Hille-Yosida theorem ([7],[13]). Then, we establish an exponential decay result of the solution by using the multipliers method to construct a Lyapunov function which is equivalent to the energy's system (see ([3],[10])). Moreover, we give a numerical simulation by introducing a fixed point algorithm to solve the discretized problem ([3]). Finally, we illustrate the theoretical result by choosing different choices of the coefficients of the system and initial conditions.

This dissertation is structured into four chapters as follows:

• Chapter 1 presents essential mathematical definitions and theoretical tools that are used throughout the following chapters, including functional analysis concepts and preliminary results.

• Chapter 2 focuses on the existence and uniqueness of the solution to the delayed swelling porous thermoelastic system, using the theory of semigroups of linear operators.

• Chapter 3 is devoted to the study of energy estimates and stability analysis. Lyapunov functionals are constructed to demonstrate the asymptotic behavior of the solution.

• Chapter 4 presents a numerical test that illustrates the theoretical results and simulates the behavior of the model under specific physical conditions.

CHAPTER 1

Preliminaries

1.1 Partial differential equations (PDEs)

Definition 1.1.1. *[16] A partial differential equation (PDE) is relation involving an unknown function u defined from $\mathbb{R}^n$ to $\mathbb{R}$, the variables $x, y, ...,$ and their partial derivatives $u_x, u_y, ..., u_{xx}, u_{yy}, ...$It is generally written as*

$$F(x, y, ..., u, u_x, u_y, ..., u_{xx}, u_{xy}, u_{yy}, ...) = 0, \tag{1}$$

where

$$u_x = \frac{\partial u}{\partial x}, \qquad u_y = \frac{\partial u}{\partial y}.$$

Equation (1) is considered in a domain $\Omega \subset \mathbb{R}^n$.

The solutions of equation (1) are functions that satisfy it in Ω.

Definition 1.1.2. *[16] The order of a partial differential equation is the highest order of partial derivative appearing in the equation.*

Definition 1.1.3. *[16] A partial differential equation is called **linear** if the function F is linear with respect to the function u and all its partial derivatives. In other words, if u and its partial derivatives appear separately and to the power 1, the equation is said to be linear.*

Definition 1.1.4. *[16] A **homogeneous** partial differential equation is satisfied for $u = 0$. (That is, if* *written in the usual form, the right-hand side containing neither u nor its partial derivatives is identically zero.)*

1.1.1 First-order Linear PDE

The most general form of a first-order linear PDE with two variables is:

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = D(x,y).$$

where A, B, C, D are functions.

Example 1.1.1.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

1.1.2 Classification of Second-order Linear PDEs with Constant Coefficients

A second-order linear PDE with constant coefficients can be written as:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0.$$

The first three terms correspond to the **principal part**. The constants A, B, C, D, E, F, G are constants.

1.1.3 The type of the PDE depends on the sign of $B^2 - 4AC$

- If $B^2 - 4AC > 0$, the PDE is said to be **hyperbolic.**
- If $B^2 - 4AC = 0$, the PDE is said to be **parabolic.**
- If $B^2 - 4AC < 0$, the PDE is said to be **elliptic.**

1.1.4 Initial and Boundary conditions

PDEs problems typically involve initial conditions, which define the value of the solution at a given initial time, such problems also require boundary conditions which are imposed on a partial differential equation to determine a unique solution. These conditions are specified at the boundaries of the domain on which the *PDE* is defined.

Boundary conditions can be classified into different types, depending on the nature of the problem and the physical or mathematical requirements. Here are some commonly encountered types of boundary conditions

• **Dirichlet boundary condition:** Specifies the values of the unknown function at the boundaries.

• **Neumann boundary condition**: Which can be defined as the derivative of the solution with respect to the normal on a part of the boundary.

• **Mixed condition:** There are several ways to obtain mixed conditions:

The first is to impose Dirichlet conditions in some directions and Neumann conditions in others at the same point on the boundary.

The second way is a linear combination of a Dirichlet condition and a Neumann condition which called Robin conditions.

1.2 Functional spaces

1.2.1 Banach space

Standing assumptions for this chapter

• F denotes $\mathbb{R}$ or $\mathbb{C}$.

• V denotes vector space over F.

Definition 1.2.1. *[5]* ***(Inner product):*** *An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in F$ and has the following properties*

• *Positivity*

$$\langle v, v\rangle \geq 0 \textit{ for all } v \in V.$$

• *Definiteness*

$$\langle v, v\rangle = 0 \textit{ if and only if } v = 0.$$

• *Additivity in first slot*

$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \textit{ for all } u, v, w \in V.$$

• *Homogeneity in first slot*

$$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \text{ for all } \lambda \in F \text{ and all } u, v \in V.$$

• *Conjugate symmetry*

$$\langle u, v \rangle = \overline{\langle v, u \rangle} \text{ for all } u, v \in V.$$

Definition 1.2.2. *[15]* ***(Norm):*** *For $v \in V$, the norm of V denoted by $\|v\|$ is defined by $\|v\| = \sqrt{\langle v, v \rangle}$ satisfying the following properties*

• $\|v\| = 0$ *if and only if* $v = 0$.

• $\|\lambda v\| = |\lambda| \, \|v\|$ *, for all* $\lambda \in F$.

• $\|u + v\| \leq \|u\| + \|v\|$ *,* $\forall u \in V$.

Definition 1.2.3. *[5]* ***(Banach space)*** *Let $(V, \|.\|)$ be a normed space, we said that V is a Banach space if V is a complete space.*

1.2.2 Hilbert space

Definition 1.2.4. *[5]* ***(Hilbert space)*** *A Hilbert space is a vector space H equipped with an inner product such that H is complete for the norm $\|u\| = \langle u, u \rangle^{\frac{1}{2}}$.*

1.2.3 Lebesgue space $L^P(\Omega)$

Definition 1.2.5. *[5] Let $p \in \mathbb{R}$ with $1 \leq p < \infty$ and Ω open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < +\infty \right\},$$

with

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Definition 1.2.6. *[5] For $p = \infty$, we have*

$$L^\infty(\Omega) = \left\{ \begin{array}{c} f : \Omega \to \mathbb{R} : f \text{ is measurable and there exists a positive constant constant } C \\ \text{such that } |f(x)| \leq C \text{ a.e on } \Omega \end{array} \right\},$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \sup ess\,|f(x)| = \inf\left\{C \in \mathbb{R}^+ \text{ such that } |f(x)| \leq C \text{ a.e in } \Omega\right\}.$$

Remark 1.2.1. *For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space equipped with the inner product*

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx,$$

hence

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2\,dx\right)^{\frac{1}{2}}.$$

1.2.4 Distribution spaces

Definition 1.2.7. *[2] Let Ω be a non-empty open set in $\mathbb{R}^n$. The support of a function $f : \Omega \to \mathbb{R}$ denoted as $supp(f)$, is the closed subset defined by*

$$\sup p(f) = \overline{\{x \in \Omega : \; f(x) \neq 0\}},$$

which means that

$$x_0 \notin \sup p(f) \Leftrightarrow \exists V \in \mathcal{V}(x_0) : f(x) = 0, \; \forall x \in V.$$

Definition 1.2.8. *[1] Let Ω be an open set of $\mathbb{R}^n$. We denote by $C_c^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ the space of functions of class $C^\infty(\Omega)$ with compact support in Ω, such that*

$$\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : \sup p(\varphi) \text{ is compact}\}.$$

Example 1.2.1. *Let the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ given by*

$$x \to \varphi(x) = \begin{cases} \exp(\dfrac{-1}{1-|x|^2}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}.$$

Then $\varphi \in D(\Omega)$ and its support is exactly the closed unit ball

$$\overline{B(0,1)} = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Definition 1.2.9. *[5] Let Ω be an open set of $\mathbb{R}^n$. A distribution on Ω is defined as a continuous linear* *form on $\mathcal{D}(\Omega)$. We denoted $\mathcal{D}'(\Omega)$ the set of distributions on Ω.*

In general, we use the notation $\langle T, \varphi\rangle$ or $\langle T, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$ rather than $T(\varphi)$ for the image of the test function $\varphi \in \mathcal{D}(\Omega)$ by the distribution $T \in \mathcal{D}'(\Omega)$.

Lemma 1.2.1. *[11] Let $f \in L^2(\Omega)$, if for any function $\varphi \in D(\Omega)$ we have*

$$\int_\Omega f(x)\varphi(x) = 0,$$

then $f(x) = 0$ almost every where in Ω.

1.2.4.1 Weak derivative

Definition 1.2.10. *[4] Let $u \in L^2(\Omega)$ and $g \in L^2(\Omega)$, we say that g is a weak derivative of u in $L^2(\Omega)$ if for every test function $\varphi \in C_c^\infty(\Omega)$, we have*

$$\int_a^b u(x)\varphi'(x)dx = -\int_a^b g(x)\varphi(x)dx.$$

Theorem 1.2.1. *[4] If such a weak derivative exists, it is unique (almost everywhere), and it is denote it by $\partial_x u$.*

1.2.5 Sobolev space $W^{m,p}(\Omega)$

Definition 1.2.11. *[5] Let $m \geq 0$ be an integer and p a real number with $1 \leq p \leq \infty$, we define the space $W^{m,p}(\Omega)$ by*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega);\ D^\alpha f \in L^p(\Omega),\ \forall\alpha \in \mathbb{N}^n;\ |\alpha| \leq m\},$$

such that $|\alpha| = \sum_{j=1}^n \alpha_j \leq m$ and $D^\alpha f = \dfrac{\partial^{\alpha_1+\alpha_2+......+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}......\partial x_n^{\alpha_n}} f$, where $D^\alpha f$ represents a weak partial derivative of f (in the sense of distributions).

Theorem 1.2.2. *[2] The space $W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}},$$

Remark 1.2.2. *If $p = 2$, Sobolev spaces become particularly due to their property of being Hilbert spaces, denoted as $H^m(\Omega)$ and $W^{m,2}(\Omega) = H^m(\Omega)$.*

Definition 1.2.12. *[1] For an integer $m \geq 1$, the Sobolev space $H^m(\Omega)$ is defined by*

$$H^m(\Omega) = \left\{ f \in L^2(\Omega) \text{ such that } \forall \alpha \in \mathbb{N}^n; |\alpha| \leq m,\ \partial^\alpha f \in L^2(\Omega) \right\},$$

where the partial derivative $\partial^\alpha f$ is taken in the weak sense.

Proposition 1.2.1. *[5] equipped with the scalar product*

$$\langle f, g \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha f \partial^\alpha g dx.$$

and with the norm $\|f\|_{H^m(\Omega)} = \sqrt{\langle f, f \rangle}$, the sobolev space $H^m(\Omega)$ is a Hilbert space.

Definition 1.2.13. *Given an integer $m \geq 2$ and a real number $1 \leq p < \infty$, the space $W_0^{m,p}(\Omega)$ is defined as the closure of $C_c^m(\Omega)$ in $W^{m,p}(\Omega)$. One shows that*

$$W_0^{m,p}(\Omega) = \left\{ u \in W^{m,p}(\Omega);\ u = Du = ... = D^{m-1}u = 0 \text{ on } \partial\Omega \right\}.$$

It is essential to notice the distinction between

$$W_0^{2,p}(\Omega) = \left\{ u \in W^{2,p}(\Omega);\ u = Du = 0 \text{ on } \partial\Omega \right\},$$

and

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = \left\{ u \in W^{2,p}(\Omega);\ u = 0 \text{ on } \partial\Omega \right\}.$$

Theorem 1.2.3. *[2] Let $u \in W^{1,p}(\Omega)$ then $u \in W_0^{1,p}(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.*

Definition 1.2.14. *[4] The Sobolev space $H^1(\Omega)$ is defined as the set of functions in $L^2(\Omega)$ that admit a weak derivative in $L^2(\Omega)$*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega),\ \exists \partial_x u \in L^2(\Omega) \right\},$$

and it is equipped with the norm :

$$\|u\|_{H^1} = \sqrt{\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2}.$$

The key point to retain is the following "integration by parts" formula, which is in fact the definition of the *weak derivative*

$$\forall \varphi \in C_c^{\infty}(\Omega), \ \int_a^b u(x)\varphi'(x)dx = -\int_a^b \partial_x u(x)\varphi(x)dx.$$

1.3 Some useful inequalities

Lemma 1.3.1. *[5] (Young's inequality) Let $a, b \in \mathbb{R}$ and $p, q \in]1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. So*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

in addition, if $f, g \in L^2(\Omega)$ and $f.g \in L^1(\Omega)$, then

$$\int_\Omega |fg|\, dx \leq \varepsilon \int_\Omega |f|^2\, dx + \frac{1}{4\varepsilon} \int_\Omega |g|^2\, dx, \ \forall \varepsilon > 0.$$

Lemma 1.3.2. *[5] (Poincaré's inequality) For all $f \in H_0^1(\Omega)$, the following inequality holds*

$$\|f\|_{L^2(\Omega)} \leq C_p \|\partial_x f\|_{L^2(\Omega)}.$$

Lemma 1.3.3. *[2] (Holder's inequality) Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p,\ q \leq +\infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$f.g \in L^1(\Omega) \text{ and } \int_\Omega |fg|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Lemma 1.3.4. *[5] (Cauchy-Shwarz's inequality) We put $p = q = 2$ in Holder's inequality, we obtain the Cauchy-Shwarz's inequality*

$$\int_\Omega |fg|\, dx \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

1.4 Lax-Milgram Theorem

Definition 1.4.1. *[10] A bilinear form $a : H \times H \to \mathbb{R}$ is said to be*

• Continuous if there exists a constant C such that

$$|a(u,v)| \leq C \|u\|_H \|v\|_H, \ \forall u,v \in H.$$

• Coercive if there exists a constant $\alpha > 0$ such that

$$|a(v,v)| \geq \alpha \|v\|_H^2, \forall v \in H.$$

Theorem 1.4.1. *[14] (Lax-Milgram) Assume that $a(u,v)$ is a continuous coercive bilinear form on H and $L : H \to \mathbb{R}$ a continuous linear form.*

Then, there exists a unique element $u \in H$ such that

$$a(u,v) = L(v), \ \forall v \in H.$$

1.5 Definitions and notions

Definition 1.5.1. *[14] An unbounded linear operator on X is a pair $(A, D(A))$, where $D(A)$ is a vector subspace of X, referred to as the domain of A and A is a linear mapping from $D(A)$ into X.*

Definition 1.5.2. *[14] An unbounded linear operator $(A, D(A))$ in X, is closed if its graph*

$$G(A) = \{(x, Ax) \mid x \in D(A)\} \text{ is closed in } X \times X.$$

Definition 1.5.3. *[14] Let $(A, D(A))$ an unbounded linear operator in X. If the domain $D(A)$ is dense in X, then $(A, D(A))$ is said to be densely defined in X.*

1.5.1 The m-dissipative operators in a Banach space

Definition 1.5.4. *[7] A linear unbounded operator A is dissipative if and only if*

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \text{ for all } x \in D(A) \text{ and } \lambda > 0.$$

Definition 1.5.5. *[9] An unbounded linear operator $(A, D(A))$ on X is said to be $m-$dissipative if :*

• *A is dissipative.*

• *The operator $(\lambda I - A)$ is surjective, that's*

$$\forall f \in X, \forall \lambda > 0, \exists\, x \in D(A) \text{ such that } \lambda x - Ax = f.$$

Theorem 1.5.1. *[9] If A is m-dissipative, then, for all $\lambda > 0$ the operator $(\lambda I - A)$ admits an inverse, $(\lambda I - A)^{-1} f$ belongs to $D(A)$ for all $f \in X$, and $(\lambda I - A)^{-1}$ is a bounded linear operator verifying*

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{1}{\lambda}.$$

Theorem 1.5.2. *[9] Let $(A, D(A))$ an unbounded dissipative linear operator in X. The operator A is m-dissipative if and only if*

$$\exists\, \lambda_0 \text{ such that } \forall f \in X,\ \exists\, x \in D(A) \text{ verifies } \lambda_0 x - Ax = f.$$

Theorem 1.5.3. *[9] Let $(A, D(A))$ an unbounded linear operator in X. If it exists $\lambda_0 > 0$ for which the operator $\lambda_0 I - A$ is bijection of $D(A)$ on X, and if $(\lambda_0 I - A)^{-1}$ is a bounded operator on X then A is closed.*

Definition 1.5.6. *[13] Let A be dissipative with $R(I - A) = X$. If X is reflexive then*

$$\overline{D(A)} = X.$$

1.5.2 The m-dissipative operators in a Hilbert space

Definition 1.5.7. *[9] An unbounded linear operator $A : D(A) \subset H \longrightarrow H$, is said to be dissipative if it satisfies*

$$(Ax, x) \leq 0 \ \forall x \in D(A).$$

A is maximal if $R\,(I - A) = H$, i.e

$$\forall f \in H, \exists\, x \in D(A) \text{ such that } x - Ax = f.$$

The operator A is said to be monotone if $-A$ is dissipative, i.e

$$(Ax, x) \geq 0 \textit{ for all } x \in D(A).$$

Proposition 1.5.1. *[5] Let $A : D(A) \subset H \to H$ (H denotes a Hilbert space) be a m-dissipative operator, then*

- *A is closed.*
- *$D(A)$ is dense in H.*

1.6 Strongly continuous semigroup of contractions of bounded linear operators

Let X be a Banach space over the field F ($\mathbb{C}$ or $\mathbb{R}$) and let $B(X)$ be the Banach algebra of bounded linear operators from X into X.

Definition 1.6.1. *[13] A family of operators $(T(t))_{t\geq 0} \subset B(X)$ is called a semigroup if it satisfies the following properties*

- *$T(0) = I$ (where I is the identity operator on X).*
- *$T(t+s) = T(t)\,T(s)$, for all $t, s \geq 0$.*

Definition 1.6.2. *[13] A semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t\to 0} T(t)x = x, \quad \forall x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0-semigroup.

1.6.1 The infinitesimal generator

Definition 1.6.3. *[13] The infinitesimal generator of the semigroup $(T(t))_{t\geq 0}$ is the linear operator A defined by*

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)\, x}{dt}\Big|_{t=0} \quad for\ any\ x \in D(A),$$

with

$$D\left(A\right)=\left\{x\in X:\lim_{t\to 0}\frac{T(t)x-x}{t}exists\right\}.$$

Theorem 1.6.1. *[7] Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. There exist constant $\omega\geq 0$ and $M\geq 1$ such that*

$$\left\|T\left(t\right)\right\|\leq M\exp(\omega t)\ for\ 0\leq t<\infty.$$

Theorem 1.6.2. *[7] A C_0-semigroup $(T(t))_{t\geq 0}$ on X is of contractions if*

$$\left\|T\left(t\right)\right\|\leq 1\ for\ 0\leq t<\infty.$$

Theorem 1.6.3. *[13] Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup$(T(t))_{t\geq 0}$ on X, satisfying $\|T(t)\|\leq M\exp(\omega t)$. If B is a bounded linear operator on X then $A+B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying $\|S(t)\|\leq M\exp((\omega+M\|B\|)t)$.*

Theorem 1.6.4. *[13] Let $(T(t))_{t\geq 0}$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

- *For $x\in X$*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}T\left(s\right)xds=T\left(t\right)x.$$

- *For $x\in X$, $\int_0^t T(S)xds\in D(A)$ and*

$$A\left(\int_0^t T\left(s\right)xds\right)=T\left(t\right)x-x.$$

- *For $x\in D(A)$, $T(t)x\in D(A)$ and*

$$\frac{d}{dt}T\left(t\right)x=AT\left(t\right)x=T\left(t\right)Ax.$$

- *For $x\in D(A)$*

$$T\left(t\right)x-T\left(s\right)x=\int_s^t T\left(\tau\right)Axd\tau=\int_s^t AT\left(\tau\right)xd\tau.$$

1.7 Hille-Yosida Theorem

Theorem 1.7.1. *[13] An unbounded linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of a contractions on X if and only if the following conditions are satisfied*

(a) *$D(A)$ is dense in X.*

(b) *A is a closed operator.*

(c) *For every $\lambda > 0$, the operator $(\lambda I - A)$ is bijective mapping from $D(A)$ into X and $(\lambda I - A)^{-1}$ is a bounded operator on X satisfying*

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{1}{\lambda}.$$

Theorem 1.7.2. *[13] An unbounded linear operator $(A, D(A))$ on X is the infinitesimal generator of a C_0-semigroup of contractions on X if and only if A is m-dissipative and of dense domaine in X.*

Theorem 1.7.3. *[5] Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A)),$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \ \ on \ \ [0, +\infty[\\ u\,(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0| \ \ and \ \ \left| \frac{du}{dt}\,(t) \right| = |Au\,(t)| \leq |Au_0|\,, \ \forall t \geq 0.$$

1.8 Lumer-Phillips

Theorem 1.8.1. *[13] Let A be a linear operator with dense domain $D(A)$ in X*

$a)$ *If A is dissipative and there exists a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

$b)$ *If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.*

1.9 Fundamental Concepts of Stability

Stability is one of the essential aspects in the study of solutions to differential systems, and its concept aims to formalize the properties of these systems. This means that the properties of differential systems are closely related to the behavior of the properties of these systems. This also means that the properties of differential systems are closely related to the behavior of the system's equilibrium points.

Definition 1.9.1. *[8] (equilibrium point) a point $x^*$is said to be an equilibrium point if and only if $f(x^*) = 0$, in other words x^* is a state solution of the equation*

$$x'(t) = f(x(t)).$$

i.e If $x(t_0) = x^*$ then, $x(t) = x^*$, $\forall t \geq t_0$.

1.9.1 Stability and Instability

Definition 1.9.2. *[8] (Stability of an equilibrium point) The equilibrium point $x^*(t)$ of the differential equation $x'(t) = f(x(t))$ is*

• *Stable: If, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $x(t)$ of the differential equation, if*

$$\|x_0 - x^*\| \leq \delta,$$

then

$$\|x(t) - x^*\| \leq \varepsilon, \text{ for all } t \geq 0.$$

• *Unstable: The equilibrium point is said to be unstable if it is not stable.*

1.9.2 Types of Stability

Definition 1.9.3. *[8] The equilibrium point x^* is said to be asymptotically stable if it is stable, and if there exists $r > 0$ such that for every solution $x(t)$ of $\dot{x}(t) = f(x(t))$, we have*

$$\|x_0 - x^*\| \leq r \Longrightarrow \lim_{t\to\infty} \|x(t) - x^*\| = 0$$

Example 1.9.1. *Consider the system*

$$\begin{cases} \dot{x} = \dfrac{-x}{1+t}, \\ t \geq t_0 \quad t_0 \in \mathbb{R}_+ \\ x(t_0) = x_0. \end{cases}$$

This system admits a solution of the form

$$\begin{aligned} X(t, t_0, x_0) &= x_0 \exp(\int_{t_0}^{t} \frac{-1}{1+s} ds) \\ &= x_0 \exp(- [\log(1+s)]_{t_0}^{t} \\ &= x_0 \left(\frac{1+t_0}{1+t} \right). \end{aligned}$$

Note that the equilibrium $x^* = 0$ is stationary, and moreover, we have:

$$\lim_{t \to \infty} X(t, t_0, x_0) = 0.$$

Therefore, the equilibrium point is asymptotically stable.

1.9.2.1 Exponential Stability

Definition 1.9.4. *[8] An equilibrium point x_e is exponentially stable for the equation $x'(t) = f(x)$ if there exist two positive scalars α and K such that:*

$$\|x(t) - x_e\| \leq K \|x(0) - x_e\| \exp(-\alpha t),$$

Example 1.9.2. *consider the following system:*

$$x = -(1 + \sin^2(x))x, with x(t) = x_0$$

- *It is clear that $x_e = 0$ is an equilibrium point.*
- *The solution of the system is given by*

$$x(t) = x(0) \exp(\int_0^t -(1 + \sin^2(s) ds), \ \forall t \geq 0.$$

Let

$$a(t) = (1 + \sin^2(t)).$$

We have exponential stability.

1.9.2.2 Stability (in the sense of Lyapunov)

Consider a dynamical system

$$\dot{x} = f(x),\ x \in \mathbb{R}^n,$$

with an equilibrium point x_e, typically $x_e = 0$.

Definition 1.9.5. *[8] (Lyapunov stability) The equilibrium point $x = 0$ is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$, there exists a real number $\delta > 0$ such that*

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon,\ \forall t \geq 0.$$

This means that any trajectory that starts "near" the equilibrium point remains close indefinitely.

1.9.2.3 Stability via Lyapunov function

Definition 1.9.6. *[8] A function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Lyapunov function for the system if there exists a neighborhood D of the origin such that*

- *$V(0) = 0$ and $V(x) > 0$ for all $x \in D \setminus \{0\}$ (positive definite),*

- *The derivative along trajectories $\dot{V}(x) = \nabla V(x)^T f(x) \leq 0$ (negative semi-definite).*

Interpretations

- if $\dot{V}(x) \leq 0$:The equilibrium is stable

- if $\dot{V}(x) < 0$:The equilibrium is asymptotically stable.

- if $V(x) \geq \alpha_1 \|x\|^2$ and $\dot{V}(x) \leq -\alpha_2 \|x\|^2$ for some positive constants $\alpha_1, \alpha_2 > 0$ the equilibrium is exponentially stable.

1.10 The delay differential equations (DDEs)

in the study of delay differential (DDEs), delays represent how the current rate of change of a system depends not only on the current state but also on its past states. Delays are common in biological, engineering and economic systems. There are three main types of delays: discrete, distributed, and neutral delays.

1.10.1 Discrete Delay

Definition 1.10.1. *[6] A discrete delay refers to a situation where the effect of the past state on the present is localizes at a single point in time.*

Mathematical form

$$\frac{dx(t)}{dt} = f(x(t), x(t-\tau)),$$

where $\tau > 0$ is a constant delay.

Example 1.10.1.

$$\frac{dx(t)}{dt} = -ax(t-\tau).$$

Applications

- Control systems.
- Population dynamics (e.g, delayed birth rate)

1.10.2 Distributed Delay

Definition 1.10.2. *[6] A distributed delay accounts for the influence of the state over an entire interval in the past, rather than at a single instant.*

Mathematical form

$$\frac{dx(t)}{dt} = f\left(x(t), \int_0^\tau k(s)\, x(t-s)\, ds\right),$$

where $k(s)$ is a kernel function describing the weight of the past states.

Example 1.10.2.

$$\frac{dx(t)}{dt} = -ax(t) + \int_0^\tau k(s)\, x(t-s)\, ds.$$

Applications

• Epidemiological models.

• Neural field equations.

• Thermal processes.

1.10.3 Neutral Delay

Definition 1.10.3. *[6] In neutral delay differential equations, the derivative of the delayed state also influences the present rate of change.*

Mathematical form

$$\frac{d}{dt}\left[x\left(t\right)-g\left(x\left(t-\tau\right)\right)\right]=f\left(x\left(t\right),x\left(t-\tau\right)\right).$$

Example 1.10.3.

$$\frac{d}{dt}\left[x\left(t\right)-bx\left(t-\tau\right)\right]=-ax\left(t\right).$$

Features

• More complex dynamics.

• Can introduce higher-order effects like oscillations or instability.

Application

• Transmission lines.

• Mechanical system with damping.

CHAPTER 2

Well-posedness

In this chapter, we prove the existence and uniqueness of solutions for system (1.1)-(1.2) based on the semigroup theory and more precisely the Lummer Phillips theorem.

First, we introduce the following change of variable

$$\varphi\,(x,\rho,t) = u_t(x,t-\rho\tau) \text{ in } (0,L)\times(0,1)\times(0,\infty), \tag{1.3}$$

which satisfies

$$\tau\varphi_t(x,\rho,t) + \varphi_\rho(x,\rho,t) = 0 \text{ in } (0,L)\times(0,1)\times(0,\infty).$$

Hence problem(1.1) takes the form

$$\begin{cases} \rho_z z_{tt} = \alpha_1 z_{xx} + \alpha_2 u_{xx} & \text{in } (0,L)\times(0,\infty) \\ \rho_u u_{tt} = \alpha_3 u_{xx} + \alpha_2 z_{xx} + \beta\theta_x - \mu\varphi\,(x,1,t) & \text{in } (0,L)\times(0,\infty) \\ c\theta_t = -q_x + \beta u_{tx} & \text{in } (0,L)\times(0,\infty) \\ \tau_0 q_t = -q - k\theta_x & \text{in } (0,L)\times(0,\infty) \\ \tau\varphi_t = -\varphi_\rho & \text{in } (0,L)\times(0,1)\times(0,\infty) \end{cases} \tag{1.4}$$

with the boundary and the initial data

$$\begin{cases} z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = \theta_x(0,t) = \theta(L,t) = q(0,t) = 0 \quad , \quad t > 0 \\ z(x,0) = z_0(x) \quad , \quad z_t(x,0) = z_1(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,L) \\ u(x,0) = u_0(x) \quad , \quad u_t(x,0) = u_1(x) \quad , \quad q(x,0) = q_0(x) \quad , \quad x \in (0,L) \\ \varphi(x,\rho,0) = f_0(x,\rho t) \quad \text{in } (0,L) \times (0,\tau) \end{cases} \tag{1.5}$$

Second, we start to transform the system (1.4)-(1.5) as a Cauchy problem. Indeed, we introduce the vector function $U = (z, v, u, \psi, \theta, q, \varphi)^T$, where $v = z_t$, and $\psi = u_t$. Then, the system (1.4)-(1.5) can be rewritten as follows

$$\begin{cases} U_t = (\mathcal{A} + \mathcal{B})U, \ t > 0, \\ U(x,0) = U_0(x) = (z_0, z_1, u_0, u_1, \theta_0, q_0, f_0)^T, \end{cases} \tag{2.1}$$

where

$$\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{\alpha_1}{\rho_z} z_{xx} + \dfrac{\alpha_2}{\rho_z} u_{xx} \\ \psi \\ \dfrac{\alpha_2}{\rho_u} z_{xx} + \dfrac{\alpha_3}{\rho_u} u_{xx} + \dfrac{\beta}{\rho_u}\theta_x - \dfrac{\mu}{\rho_u}\varphi(x,1,t) - \dfrac{|\mu|}{\rho_u}\psi \\ \dfrac{\beta}{c}\psi_x - \dfrac{1}{c} q_x \\ -\dfrac{k}{\tau_0}\theta_x - \dfrac{1}{\tau_0} q \\ -\dfrac{1}{\tau}\varphi_\rho \end{pmatrix}, \tag{2.2}$$

with $\mathcal{A}$ is unbounded linear operator on $\mathcal{H}$ defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ \dfrac{\alpha_1}{\rho_z}\dfrac{\partial^2}{\partial x^2}(.) & 0 & \dfrac{\alpha_2}{\rho_z}\dfrac{\partial^2}{\partial x^2}(.) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ \dfrac{\alpha_2}{\rho_u}\dfrac{\partial^2}{\partial x^2}(.) & 0 & \dfrac{\alpha_3}{\rho_u}\dfrac{\partial^2}{\partial x^2}(.) & -\dfrac{|\mu|}{\rho_u} I & \dfrac{\beta}{\rho_u}\dfrac{\partial}{\partial x}(.) & 0 & -\dfrac{\mu}{\rho_u} I \,|_{\rho=1} \\ 0 & 0 & 0 & \dfrac{\beta}{c}\dfrac{\partial}{\partial x}(.) & 0 & -\dfrac{1}{c}\dfrac{\partial}{\partial x}(.) & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{k}{\tau_0}\dfrac{\partial}{\partial x}(.) & -\dfrac{1}{\tau_0} I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\dfrac{1}{\tau}\dfrac{\partial}{\partial \rho}(.) \end{pmatrix},$$

and the operator $\mathcal{B} : \mathcal{D}(\mathcal{B}) = \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{B}U = \frac{|\mu|}{\rho_u}(0,0,0,\psi,0,0,0)^T .$$

We define the energy space as

$$\begin{aligned}\mathcal{H} = {} & \tilde{H}^1(0,L) \times L^2(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L) \\ & \times L^2((0,L)\times(0,1)) .\end{aligned}$$

The domain of A is given by

$$\begin{aligned}D(A) = \Big\{ & U \in \mathcal{H} \mid z,u \in \tilde{H}^2(0,L) ; v,\psi,q \in \tilde{H}^1(0,L) ; \\ & \theta \in \tilde{H}^1_*(0,L) ; \varphi,\varphi_\rho \in L^2((0,L)\times(0,1)) \\ & z_x(L) = u_x(L) = \theta_x(0) = 0 \} .\end{aligned}$$

Where

$$\begin{aligned}\tilde{H}^1(0,L) &= \Big\{ f \in H^1(0,L) ; f(0) = 0 \Big\} , \\ \tilde{H}^1_*(0,L) &= \Big\{ f \in H^1(0,L) ; f(L) = 0 \Big\} , \\ \tilde{H}^2(0,L) &= H^2(0,L) \cap \tilde{H}^1(0,L) .\end{aligned}$$

2.1 Energy space

We define the energy functional of the system (1.4)-(1.5) as follows

$$\begin{aligned}E(t) = \frac{1}{2}\int_0^L \Bigg(& \rho_z z_t^2 + \rho_u u_t^2 + \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 \\ & + c\theta^2 + \frac{\tau_0}{k} q^2 + |\mu| \tau \int_0^1 \varphi^2(x,\rho,t)\, d\rho \Bigg) dx, \end{aligned} \tag{2.3}$$

and we will demonstrate later in chapter 3 how to calculate this quantity. To assure that the energy functional is well defined, it's necessary that the following conditions hold

$$\int_0^L z_t^2 dx < \infty,$$

which implies that $z_t \in L^2(0,L)$, we have $z_t = v$, it follows that $v \in L^2(0,L)$. Also we

$$\int_0^L u_t^2 dx < \infty,$$

since $u_t \in L^2(0,L)$ and $u_t = \psi$, we conclude that $\psi \in L^2(0,L)$. In addition

$$\int_0^L u_x^2 dx < \infty,$$

and by Poincaré's inequality, we deduce that

$$\int_0^L u^2 dx < \infty.$$

Then, by taking into account the boundary conditions, we conclude that

$u \in \tilde{H}^1(0,L)$.

Furthermore

$$\int_0^L {z_x}^2 dx < \infty,$$

by applying the Poincaré's inequality, we also obtain

$$\int_0^L z^2 dx < \infty.$$

Then, by taking into account the boundary conditions, we conclude that

$z \in \tilde{H}^1(0,L)$.

Moreover

$$\int_0^L \theta^2 dx < \infty,$$

therefore, $\theta \in L^2(0,L)$. Then

$$\int_0^L q^2 dx < \infty.$$

This is implies that $q \in L^2(0,L)$. Finally

$$\int_0^L \int_0^1 \varphi^2(x,\rho,t)\, d\rho dx < \infty,$$

which confirm that $\varphi \in L^2\left((0,L)\times(0,1)\right).$

2.2 Domain of the operator A

Because $U \in \mathcal{H}$, we assume that the components

$$\begin{cases} z,u \in \tilde{H}^1(0,L), \\ v,\psi \in L^2(0,L), \\ \theta,q \in L^2(0,L), \\ \varphi \in L^2\left((0,L)\times(0,1)\right). \end{cases} \tag{2.4}$$

In order to $\mathcal{A}U$ belongs to $\mathcal{H}$, it's necessary to impose the following conditions

$$\begin{cases} v \in \tilde{H}^1(0,L), \\ \dfrac{\alpha_1}{\rho_z}z_{xx} + \dfrac{\alpha_2}{\rho_z}u_{xx} \in L^2(0,L), \\ \psi \in \tilde{H}^1(0,L), \\ \dfrac{\alpha_2}{\rho_u}z_{xx} + \dfrac{\alpha_3}{\rho_u}u_{xx} + \dfrac{\beta}{\rho_u}\theta_x - \dfrac{\mu}{\rho_u}\varphi(x,1,t) - \dfrac{|\mu|}{\rho_u}\psi \in L^2(0,L), \\ \dfrac{\beta}{c}\psi_x - \dfrac{1}{c}q_x \in L^2(0,L), \\ -\dfrac{k}{\tau_0}\theta_x - \dfrac{1}{\tau_0}q \in L^2(0,L), \\ -\dfrac{1}{\tau}\varphi_\rho \in L^2\left((0,L)\times(0,1)\right). \end{cases}$$

According to the definition of $D(\mathcal{A})$, we get

$$v,\psi \in \tilde{H}^1(0,L),$$

and

$$\frac{\alpha_1}{\rho_z}z_{xx} + \frac{\alpha_2}{\rho_z}u_{xx} \in L^2(0,L),$$

so

$$\int_0^L |(z_{xx}+u_{xx})|^2\,dx < \infty \Longrightarrow \int_0^L \left(z_{xx}^2 + u_{xx}^2 + 2z_{xx}u_{xx}\right)dx < \infty,$$

Young's inequality leads to

$$-\int_0^L z_{xx}u_{xx}dx \leq \frac{\varepsilon}{2}\int_0^L z_{xx}^2dx + \frac{1}{2\varepsilon}\int_0^L u_{xx}^2dx,$$

$$-2\int_0^L z_{xx}u_{xx}dx \leq \varepsilon\int_0^L z_{xx}^2dx + \frac{1}{\varepsilon}\int_0^L u_{xx}^2dx,$$

$$2\int_0^L z_{xx}u_{xx}dx \geq -\varepsilon\int_0^L z_{xx}^2dx - \frac{1}{\varepsilon}\int_0^L u_{xx}^2dx,$$

$$(1-\varepsilon)\int_0^L z_{xx}^2dx + \left(1-\frac{1}{\varepsilon}\right)\int_0^L u_{xx}^2dx \leq \int_0^L z_{xx}^2dx + \int_0^L u_{xx}^2dx + 2\int_0^L z_{xx}u_{xx}dx < \infty.$$

Which implies that

$$\int_0^L z_{xx}^2 < \infty,$$
$$\int_0^L u_{xx}^2 < \infty,$$

so, we deduce

$$z, u \in H^2(0,L)$$

and from (2.4), we obtain $z, u \in \tilde{H}^1(0,L)$. Then

$$z, u \in \tilde{H}^2(0,L).$$

Next, we consider the term

$$\frac{\beta}{c}\psi_x - \frac{1}{c}q_x \in L^2(0,L).$$

Similarly

$$\int_0^L q_x^2 < \infty.$$

Hence

$$q_x \in L^2(0,L) \text{ and } q \in L^2(0,L),$$

taking into account the boundary conditions, we have

$$q(0,t) = 0,$$

so

$$q \in \tilde{H}^1(0,L).$$

Moreover,

$$-\frac{k}{\tau_0}\theta_x - \frac{1}{\tau_0}q \in L^2(0,L) \text{ and } q \in L^2(0,L),$$

thus, $\theta_x \in L^2(0,L)$ and from (2.4), we deduce

$$\theta \in L^2(0,L),$$

and because

$$\theta(L,t) = 0,$$

then,

$$\theta \in \tilde{H}^1_*(0,L).$$

Finally, it is clear that

$$\varphi, \varphi_\rho \in L^2((0,L)\times(0,1)).$$

For any $U = (z,v,u,\psi,\theta,q,\varphi)^T \in \mathcal{H}$, $\tilde{U} = \left(\tilde{z},\tilde{v},\tilde{u},\tilde{\psi},\tilde{\theta},\tilde{q},\tilde{\varphi}\right)^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product

$$\begin{aligned}
\langle U,\tilde{U}\rangle_{\mathcal{H}} &= \rho_z \int_0^L v\tilde{v}dx + \rho_u \int_0^L \psi\tilde{\psi}dx + \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x\tilde{u}_x dx \\
&+ \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}\tilde{u}_x + \sqrt{\alpha_1}\tilde{z}_x\right)dx + c\int_0^L \theta\tilde{\theta}dx \\
&+ \frac{\tau_0}{k}\int_0^L q\tilde{q}dx + |\mu|\,\tau\int_0^L\int_0^1 \varphi(x,\rho,t)\,\tilde{\varphi}(x,\rho,t)\,d\rho dx.
\end{aligned}$$

2.3 Existence and uniqueness result

Theorem 2.3.1. *Let $U_0 \in \mathcal{H}$ and assume that (1.3) holds. Then, there exists a unique solution $U \in C(\mathbb{R}_+,\mathcal{H})$ for the problem (1.4)-(1.5). Moreover, if $U_0 \in D(A)$, then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+,\mathcal{H}).$$

Proof. We use the semi-group approach. So, we prove that $\mathcal{A}$ is a maximal dissipative operator and that $\mathcal{B}$ is Lipschitz continuous operator. First, we prove that $\mathcal{A}$ is dissipative. For this reason, we use formula (2.2.1). Let $U \in D(\mathcal{A})$, then we have

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} v \\ \frac{\alpha_1}{\rho_z} z_{xx} + \frac{\alpha_2}{\rho_z} u_{xx} \\ \psi \\ \frac{\alpha_2}{\rho_u} z_{xx} + \frac{\alpha_3}{\rho_u} u_{xx} + \frac{\beta}{\rho_u}\theta_x - \frac{\mu}{\rho_u}\varphi(x,1,t) - \frac{|\mu|}{\rho_u}\psi \\ \frac{\beta}{c}\psi_x - \frac{1}{c} q_x \\ -\frac{1}{\tau_0} q - \frac{k}{\tau_0}\theta_x \\ -\frac{1}{\tau}\varphi_\rho \end{pmatrix}, \begin{pmatrix} z \\ v \\ u \\ \psi \\ \theta \\ q \\ \varphi \end{pmatrix} \right\rangle_{\mathcal{H}},$$

then

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \int_0^L (\alpha_1 z_{xx} + \alpha_2 u_{xx}) v dx + \int_0^L (\alpha_3 u_{xx} + \alpha_2 z_{xx} + \beta\theta_x - \mu\varphi(x,1,t) \\ &\quad - |\mu|\psi)\psi dx + \int_0^L \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) u_x\psi_x dx \\ &\quad + \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}\psi_x + \sqrt{\alpha_1} v_x\right) dx + \int_0^L (-q_x + \beta\psi_x)\theta dx \\ &\quad + \int_0^L \left(-\frac{1}{k} q - \theta_x\right) q dx - |\mu| \int_0^L \int_0^1 \varphi\varphi_\rho(x,\rho,t)\, d\rho dx. \end{aligned}$$

Integrating by parts and taking into account the boundary conditions, we get

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\alpha_1 \int_0^L z_x v_x dx - \alpha_2 \int_0^L u_x v_x dx - \alpha_3 \int_0^L u_x\psi_x dx - \alpha_2 \int_0^L z_x\psi_x dx \\ &\quad - \beta \int_0^L \theta\psi_x dx - \mu \int_0^L \psi\varphi(x,1,t)\, dx - |\mu| \int_0^L \psi^2 dx + \alpha_3 \int_0^L u_x\psi_x dx \\ &\quad - \frac{\alpha_2^2}{\alpha_1} \int_0^L u_x\psi_x dx + \frac{\alpha_2^2}{\alpha_1} \int_0^L u_x\psi_x dx + \alpha_2 \int_0^L u_x v_x dx + \alpha_2 \int_0^L z_x\psi_x dx \\ &\quad + \alpha_1 \int_0^L z_x v_x dx + \int_0^L q\theta_x dx + \beta \int_0^L \theta\psi_x dx - \frac{1}{k}\int_0^L q^2 dx - \int_0^L q\theta_x dx \\ &\quad - \frac{|\mu|}{2} \int_0^L \varphi^2(x,1,t)\, dx + \frac{|\mu|}{2} \int_0^L \psi^2 dx. \end{aligned}$$

It follows that

$$\langle \mathcal{A}U,U\rangle_{\mathcal{H}}=-\mu\int_0^L\psi\varphi\left(x,1,t\right)dx-\frac{1}{k}\int_0^L q^2dx-\frac{|\mu|}{2}\int_0^L\varphi^2\left(x,1,t\right)dx-\frac{|\mu|}{2}\int_0^L\psi^2dx. \tag{2.6}$$

Also, using Young's inequality, we get

$$-\mu\int_0^L\psi\varphi\left(x,1,t\right)dx\leq\frac{|\mu|}{2}\int_0^L\psi^2dx+\frac{|\mu|}{2}\int_0^L\varphi^2\left(x,1,t\right)dx.$$

Finally, we obtain

$$\langle \mathcal{A}U,U\rangle_{\mathcal{H}}\leq-\frac{1}{k}\int_0^L q^2dx\leq 0.$$

Therefore, the operator $\mathcal{A}$ is dissipative. Next, we prove that the operator $(I-\mathcal{A})$ is surjective. For any $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T\in\mathcal{H}$, we prove that there exists $U\in D\left(\mathcal{A}\right)$ such that

$$\left(I-\mathcal{A}\right)U=F. \tag{2.7}$$

The problem (2.7), leads to solve the following system

$$\begin{cases} z-v=f_1\in\tilde{H}^1\left(0,L\right),\\ \rho_z v-\alpha_1 z_{xx}-\alpha_2 u_{xx}=\rho_z f_2\in L^2\left(0,L\right),\\ u-\psi=f_3\in\tilde{H}^1\left(0,L\right),\\ \left(\rho_u+|\mu|\right)\psi-\alpha_3 u_{xx}-\alpha_2 z_{xx}-\beta\theta_x+\mu\varphi\left(x,1,t\right)=\rho_u f_4\in L^2\left(0,L\right),\\ c\theta+q_x-\beta\psi_x=cf_5\in L^2\left(0,L\right),\\ \left(\tau_0+1\right)q+k\theta_x=\tau_0 f_6\in L^2\left(0,L\right),\\ \tau\varphi+\varphi_\rho=\tau f_7\in L^2\left(\left(0,L\right)\times\left(0,1\right)\right). \end{cases} \tag{2.8}$$

The last equation in (2.8) with $\varphi\left(x,0,t\right)=\psi\left(x,t\right)$ has a unique solution given by

$$\varphi\left(x,\rho,t\right)=\psi\left(x,t\right)\exp\left(-\tau\rho\right)+\tau\exp\left(-\tau\rho\right)\int_0^\rho\exp\left(\tau\sigma\right)f_7\left(x,\sigma,t\right)d\sigma. \tag{2.9}$$

From the sixth equation in (2.8), we obtain

$$\theta_x=-\frac{\left(\tau_0+1\right)}{k}q+\frac{\tau_0}{k}f_6, \tag{2.10}$$

integrating (2.10), we get

$$\theta(x,t) = \frac{\tau_0+1}{k}\int_x^L q(y)\,dy - \frac{\tau_0}{k}\int_x^L f_6(y)\,dy. \tag{2.11}$$

Inserting $v = z - f_1$, $\psi = u - f_3$, (2.9), (2.11) in $(2.8)_2$, $(2.8)_4$, $(2.8)_5$, we get

$$\begin{cases} \rho_z z - \alpha_2 u_{xx} - \alpha_1 z_{xx} = g_1 \in L^2(0,L), \\ -\alpha_3 u_{xx} + \lambda u - \alpha_2 z_{xx} + \dfrac{\beta(\tau_0+1)}{k} q = g_2 \in L^2(0,L), \\ q_x + \dfrac{c(\tau_0+1)}{k}\displaystyle\int_x^L q(y)\,dy - \beta u_x = g_3 \in L^2(0,L), \end{cases} \tag{2.12}$$

where

$$\begin{aligned} g_1 &= \rho_z(f_1+f_2) \in L^2(0,L), \\ g_2 &= \rho_u f_4 + \lambda f_3 + \frac{\beta\tau_0}{k} f_6 - \tau\mu\exp(-\tau)\int_0^1 \exp(\tau\sigma) f_7(x,\sigma,t)\,d\sigma \in L^2(0,L), \\ g_3 &= c f_5 + \frac{c\tau_0}{k}\int_x^L f_6(y)\,dy - \beta f_{3_x} \in L^2(0,L), \\ \lambda &= \rho_u + |\mu| + \mu\exp(-\tau). \end{aligned}$$

To solve (2.12), we consider

$$B((z,u,q);(\tilde{z},\tilde{u},\tilde{q})) = \mathcal{G}(\tilde{z},\tilde{u},\tilde{q}), \tag{2.13}$$

where $B : V^2 \to \mathbb{R}$ is the bilinear form and $\mathcal{G} : V \to \mathbb{R}$ is a linear form with

$$V = \tilde{H}^1(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L).$$

In order to obtain B and $\mathcal{G}$, we proceed by multiplying $(2.12)_1$, $(2.12)_2$, $(2.12)_3$ by $\tilde{z}, \tilde{u}, \dfrac{\tau_0+1}{k}\tilde{q}$ respectively, integrating over $(0,L)$ and summing them up, we find

$$\begin{aligned} B((z,u,q);(\tilde{z},\tilde{u},\tilde{q})) = {} & \rho_z\int_0^L z\tilde{z}dx - \alpha_1\int_0^L z_{xx}\tilde{z}dx - \alpha_2\int_0^L u_{xx}\tilde{z}dx \\ & - \alpha_3\int_0^L u_{xx}\tilde{u}dx + \lambda\int_0^L u\tilde{u}dx - \alpha_2\int_0^L z_{xx}\tilde{u}dx \\ & + \frac{\beta(\tau_0+1)}{k}\int_0^L q\tilde{u}dx + \frac{\tau_0+1}{k}\int_0^L q_x\left(\int_x^L \tilde{q}(y)\,dy\right)dx \end{aligned}$$

$$+\frac{c(\tau_0+1)^2}{k^2}\int_0^L\left(\int_x^L q(y)\,dy\right)\left(\int_x^L \tilde{q}(y)\,dy\right)dx$$
$$-\frac{\beta(\tau_0+1)}{k}\int_0^L u_x\left(\int_x^L \tilde{q}(y)\,dy\right)dx,$$

integrating by parts and taking into account the boundary conditions, we obtain

$$\begin{aligned} B((z,u,q);(\tilde{z},\tilde{u},\tilde{q})) &= \rho_z\int_0^L z\tilde{z}dx+\alpha_1\int_0^L z_x\tilde{z}_xdx+\alpha_2\int_0^L u_x\tilde{z}_xdx+\lambda\int_0^L u\tilde{u}dx \\ &+\alpha_3\int_0^L u_x\tilde{u}_xdx+\alpha_2\int_0^L z_x\tilde{u}_xdx+\frac{\tau_0+1}{k}\int_0^L q\tilde{q}dx \\ &+\frac{c(\tau_0+1)^2}{k^2}\int_0^L\left(\int_x^L q(y)\,dy\right)\left(\int_x^L \tilde{q}(y)\,dy\right)dx \\ &+\frac{\beta(\tau_0+1)}{k}\left(\int_0^L q\tilde{u}dx-\int_0^L u\tilde{q}dx\right), \end{aligned} \tag{2.14}$$

and

$$\mathcal{G}(\tilde{z},\tilde{u},\tilde{q})=\int_0^L g_1\tilde{z}dx+\int_0^L g_2\tilde{u}dx+\frac{\tau_0+1}{k}\int_0^L g_3\left(\int_x^L \tilde{q}(y)\,dy\right)dx. \tag{2.15}$$

Let V equipped with the norm

$$\|z,u,q\|_V^2=\|z\|_2^2+\|u\|_2^2+\|q\|_2^2+\|z_x\|_2^2+\|u_x\|_2^2,$$

we proceed to prove the boundedness of B.

Using Cauchy Schwarz's inequality, we get

$$\rho_z\left|\int_0^L z\tilde{z}dx\right|\le\rho_z\|z\|_2\|\tilde{z}\|_2\le\rho_z\|(z,u,q)\|_V\|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\lambda\left|\int_0^L u\tilde{u}dx\right|\le\lambda\|u\|_2\|\tilde{u}\|_2\le\lambda\|(z,u,q)\|_V\|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\frac{\tau_0+1}{k}\left|\int_0^L q\tilde{q}dx\right|\le\frac{\tau_0+1}{k}\|q\|_2\|\tilde{q}\|_2\le\frac{\tau_0+1}{k}\|(z,u,q)\|_V\|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\alpha_1\left|\int_0^L z_x\tilde{z}_x\right|\le\alpha_1\|z_x\|_2\|\tilde{z}_x\|_2\le\alpha_1\|(z,u,q)\|_V\|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\alpha_3\left|\int_0^L u_x\tilde{u}_xdx\right|\le\alpha_3\|u_x\|_2\|\tilde{u}_x\|_2\le\alpha_3\|(z,u,q)\|_V\|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\alpha_2 \left| \int_0^L (u_x \tilde{z}_x + z_x \tilde{u}_x)\, dx \right| \leq \alpha_2 \left(\|u_x\|_2 \|\tilde{z}_x\|_2 + \|z_x\|_2 \|\tilde{u}_x\|_2 \right)$$
$$\leq 2\alpha_2 \|(z,u,q)\|_V \|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\frac{c(\tau_0+1)^2}{k^2} \left| \int_0^L \left(\int_x^L q(y)\, dy \right) \left(\int_x^L \tilde{q}(y)\, dy \right) dx \right| \leq \frac{c(\tau_0+1)^2}{k^2} \|q\|_2 \|\tilde{q}\|_2$$
$$\leq \frac{c(\tau_0+1)^2}{k^2} \|(z,u,q)\|_V \|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

$$\frac{\beta(\tau_0+1)}{k} \left| \int_0^L (q\tilde{u} - u\tilde{q})\, dx \right| \leq \frac{\beta(\tau_0+1)}{k} \left(\|q\|_2 \|\tilde{u}\|_2 + \|u\|_2 \|\tilde{q}\|_2 \right)$$
$$\leq \frac{2\beta(\tau_0+1)}{k} \|(z,u,q)\|_V \|(\tilde{z},\tilde{u},\tilde{q})\|_V.$$

Summing them up, we obtain

$$|B((z,u,q);(\tilde{z},\tilde{u},\tilde{q}))| \leq C \|(z,u,q)\|_V \|(\tilde{z},\tilde{u},\tilde{q})\|_V,$$

where
$$C = \max\left(\rho_z, \lambda, \frac{\tau_0+1}{k}, \alpha_1, \alpha_3, 2\alpha_2, \frac{c(\tau_0+1)^2}{k^2}, \frac{2\beta(\tau_0+1)}{k} \right),$$

thus, B is bounded.

We establish that B is coercive

$$B((z,u,q);(z,u,q)) = \rho_z \int_0^L z^2 dx + \lambda \int_0^L u^2 dx + \frac{\tau_0+1}{k} \int_0^L q^2 dx + \alpha_1 \int_0^L z_x^2 dx$$
$$+ \alpha_3 \int_0^L u_x^2 dx + 2\alpha_2 \int_0^L u_x z_x dx$$
$$+ \frac{c(\tau_0+1)^2}{k^2} \int_0^L \left(\int_x^L q(y)\, dy \right)^2 dx.$$

Young's inequality leads to

$$-2\alpha_2 \int_0^L u_x z_x dx \leq 2\alpha_2 \left(\frac{\varepsilon}{2} \int_0^L u_x^2 dx + \frac{1}{2\varepsilon} \int_0^L z_x^2 dx \right)$$
$$= \alpha_2 \varepsilon \int_0^L u_x^2 dx + \frac{\alpha_2}{\varepsilon} \int_0^L z_x^2 dx.$$

Hence

$$2\alpha_2 \int_0^L u_x z_x dx \geq -\alpha_2 \varepsilon \int_0^L u_x^2 dx - \frac{\alpha_2}{\varepsilon} \int_0^L z_x^2 dx.$$

$$\begin{aligned} B\left((z,u,q);(z,u,q)\right) \geq \rho_z \int_0^L z^2 dx + \lambda \int_0^L u^2 dx + \frac{\tau_0+1}{k} \int_0^L q^2 dx + \left(\alpha_1 - \frac{\alpha_2}{\varepsilon}\right) \int_0^L z_x^2 dx \\ + \left(\alpha_3 - \alpha_2 \varepsilon\right) \int_0^L u_x^2 dx + \frac{c\left(\tau_0+1\right)^2}{k^2} \int_0^L \left(\int_x^L q\left(y\right) dy\right)^2 dx. \end{aligned}$$

We select ε as follows

$$\alpha_1 - \frac{\alpha_2}{\varepsilon} > 0 \iff \alpha_1 > \frac{\alpha_2}{\varepsilon} \iff \varepsilon > \frac{\alpha_2}{\alpha_1},$$

and

$$\alpha_3 - \alpha_2 \varepsilon > 0 \iff \alpha_3 > \alpha_2 \varepsilon \iff \varepsilon < \frac{\alpha_3}{\alpha_2}.$$

Therefore

$$\frac{\alpha_2}{\alpha_1} < \varepsilon < \frac{\alpha_3}{\alpha_2}.$$

It's sufficient to take

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left(\frac{\alpha_3}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right) \\ \varepsilon &= \frac{\alpha_1 \alpha_3 + \alpha_2^2}{2\alpha_1 \alpha_2}. \end{aligned}$$

Consequently, we obtain

$$B\left((z,u,q);(z,u,q)\right) \geq M_0 \left\|(z,u,q)\right\|_V^2.$$

Where

$$M_0 = \min\left\{\rho_z, \lambda, \frac{\tau_0+1}{k}, \alpha_1 - \frac{\alpha_2}{\varepsilon}, \alpha_3 - \alpha_2 \varepsilon\right\}.$$

Thus, B is coercive.

We now aim to prove that $\mathcal{G}$ is bounded

$$\begin{aligned} \mathcal{G}\left(\tilde{z}, \tilde{u}, \tilde{q}\right) &\leq \|g_1\| \|\tilde{z}\|_2 + \|g_2\| \|\tilde{u}\|_2 + \frac{\tau_0+1}{k} \|g_3\| \|\tilde{q}\|_2 \\ &\leq C' \left(\|\tilde{z}\|_2 + \|\tilde{u}\|_2 + \|\tilde{q}\|_2\right), \end{aligned}$$

where

$$C' = \max\left(\|g_1\|, \|g_2\|, \frac{\tau_0+1}{k}\|g_3\|\right).$$

Since

$$C'\left(\|\tilde{z}\|_2 + \|\tilde{u}\|_2 + \|\tilde{q}\|_2\right) \le 3C' \|(\tilde{z},\tilde{u},\tilde{q})\|_v,$$

assuming that $3C' = M$, it follows that

$$C'\left(\|\tilde{z}\|_2 + \|\tilde{u}\|_2 + \|\tilde{q}\|_2\right) \le M \ \|(\tilde{z},\tilde{u},\tilde{q})\|_V.$$

Then

$$|\mathcal{G}(\tilde{z},\tilde{u},\tilde{q})| \le M \ \|(\tilde{z},\tilde{u},\tilde{q})\|_V.$$

Thus, $\mathcal{G}$ is bounded. Consequently, by Lax-Milgram Lemma, system (2.12) has a unique solution $(z,u,q) \in V$ satisfying (2.13).
Substituting z and u in $(2.8)_1$ and $(2.8)_3$, respectively, we obtain

$$v, \psi \in \tilde{H}^1(0,L),$$

then, inserting q in (2.11) and $(2.8)_6$, we get

$$\theta \in \tilde{H}^1_*(0,L).$$

Similarly, inserting ψ in (2.9) and bearing in mind $(2.8)_7$, we obtain

$$\varphi, \varphi_\rho \in L^2\left((0,L)\times(0,1)\right).$$

Moreover, if we take $(\tilde{z},\tilde{u}) \equiv (0,0) \in \tilde{H}^1(0,L)\times\tilde{H}^1(0,L)$, then (2.13) reduces to

$$\begin{aligned}&\int_0^L q\tilde{q}dx + \frac{c(\tau_0+1)}{k}\int_0^L\left(\int_x^L q(y)dy\right)\left(\int_x^L \tilde{q}(y)dy\right)dx - \beta\int_0^L u\tilde{q}dx \\ &= \int_0^L g_3\left(\int_x^L \tilde{q}(y)dy\right)dx, \ \forall \tilde{q} \in L^2(0,L).\end{aligned} \tag{2.16}$$

That is

$$
\begin{aligned}
&-\int_0^L (q-\beta u)\left(\int_x^L \tilde{q}(y)dy\right)_x dx \\
&=\int_0^L \left[\frac{-c(\tau_0+1)}{k}\left(\int_x^L q(y)dy\right)+g_3\right]\left(\int_x^L \tilde{q}(y)dy\right)dx,\ \forall \tilde{q}\in L^2(0,L),
\end{aligned}
\tag{2.17}
$$

which implies

$$
q_x=\beta u_x-\frac{c(\tau_0+1)}{k}\left(\int_x^L q(y)dy\right)+g_3,\ \in L^2(0,L).
$$

We conclude that

$$
q\in H^1(0,L),
$$

and on the other hand, we have (2.17). Thus

$$
q(0)\int_0^L \tilde{q}(y)dy=0,\ \forall \tilde{q}\in L^2(0,L).
$$

Since $\tilde{q}\in L^2(0,L)$ is arbitrary. Then

$$
q(0)=0.
$$

Consequently

$$
q\in \tilde{H}^1(0,L).
$$

If we choose $(\tilde{u},\tilde{q})\equiv(0,0)\in H^1(0,L)\times L^2(0,L)$ in (2.13), we have

$$
\int_0^L (\alpha_1 z_x+\alpha_2 u_x)\,\tilde{z}_x dx=\int_0^L (g_1-\rho_z z)\,\tilde{z}dx,\ \forall \tilde{z}\in \tilde{H}^1(0,L).
\tag{2.18}
$$

This last is also true for any function $\phi\in C^1(0,L)$, $\phi(0)=0$ which is in $\tilde{H}^1(0,L)$, thus

$$
-\alpha_1 z_{xx}-\alpha_2 u_{xx}=g_1-\rho_z z,\ \in L^2(0,L).
$$

Similarly, if we select $(\tilde{z},\tilde{q})\equiv(0,0)\in H^1(0,L)\times L^2(0,L)$ in (2.13), we find

$$
\int_0^L (\alpha_3 u_x+\alpha_2 z_x)\,\tilde{u}_x dx=\int_0^L \left(g_2-\lambda u-\frac{\beta(\tau_0+1)}{k}q\right)\tilde{u}dx,\ \forall \tilde{u}\in \tilde{H}^1(0,L).
\tag{2.19}
$$

This last is also true for any function $\phi \in C^1(0, L)$, $\phi(0) = 0$ which is in $\tilde{H}^1(0, L)$, thus

$$-\alpha_3 u_{xx} - \alpha_2 z_{xx} = g_2 - \lambda u - \frac{\beta(\tau_0 + 1)}{k} q, \ \in L^2(0, L).$$

Therefore,

$$u_{xx}, z_{xx} \in L^2(0, L).$$

So,

$$z, u \in \tilde{H}^2(0, L).$$

Finally, from $(2.8)_6$, we get

$$\theta_x(0) = 0,$$

and from (2.18), (2.19), we find

$$\tilde{z}(L) \left[\alpha_1 z_x(L) + \alpha_2 u_x(L)\right] = 0,$$

$$\tilde{u}(L) \left[\alpha_3 u_x(L) + \alpha_2 z_x(L)\right] = 0.$$

Since $\tilde{z}$, $\tilde{u} \in \tilde{H}^1(0, L)$ are arbitrary. Then

$$z_x(L) = u_x(L) = 0.$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (2.7) is satisfied. Consequently, the operator $\mathcal{A}$ is dissipative. With this, we conclude that $\mathcal{A}$ is a maximal dissipative operator. On the other hand, it is obvious that operator $\mathcal{B}$ is Lipschitz continuous. Consequently, $\mathcal{A} + \mathcal{B}$ is the infinitesimal generator of a linear contraction C_0 semigroup on $\mathcal{H}$. Therefore, the well-posedness result follows from the Lumer Phillips theorem. $\square$

CHAPTER 3

Stability result

In this section, we present an exponential stability result of solution of the considered problem using the energy method (see [10], [12]). For this reason, we need some necessary results which help us to achieve our goal.

3.1 Energy decay

In this section, we state and prove some lemmas needed for the proof of our stability result.

Lemma 3.1.1. *Let $(z, u, \theta, q, \varphi)$ be the solution of (1.4)-(1.5). Then, the energy functional $E(t)$ defined by*

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_z z_t^2 + \rho_u u_t^2 + \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 + c\theta^2 \right.$$
$$\left. + \frac{\tau_0}{k} q^2 + |\mu| \tau \int_0^1 \varphi^2(x, \rho, t)\, d\rho \right) dx, \tag{3.1}$$

satisfies

$$E'(t) \leq -\frac{1}{k}\int_0^L q^2 dx + |\mu| \int_0^L u_t^2 dx. \tag{3.2}$$

Proof. Multiplying $(1.4)_1$, $(1.4)_2$, $(1.4)_3$, $(1.4)_4$ by z_t, u_t, θ, $\frac{1}{k}q$ respectively, and multiplying $(1.4)_5$ by

$|\mu|\,\varphi$, we obtain

$$\begin{cases} \rho_z z_{tt} z_t = \alpha_1 z_{xx} z_t + \alpha_2 u_{xx} z_t, \\ \rho_u u_{tt} u_t = \alpha_3 u_{xx} u_t + \alpha_2 z_{xx} u_t + \beta\theta_x u_t - \mu u_t \varphi\left(x,1,t\right), \\ c\theta_t \theta = -q_x \theta + \beta u_{tx} \theta, \\ \dfrac{\tau_0}{k} q_t q = -\dfrac{1}{k} q^2 - q\theta_x, \\ |\mu|\,\tau\varphi_t\varphi = -\,|\mu|\,\varphi_\rho\varphi. \end{cases}$$

Integrating over $(0,L)$ and integrating $(1.4)_5$ over $(0,L)\times(0,1)$, we get

$$\begin{cases} \rho_z \displaystyle\int_0^L z_{tt} z_t dx = \alpha_1 \int_0^L z_{xx} z_t dx + \alpha_2 \int_0^L u_{xx} z_t dx, \\ \rho_u \displaystyle\int_0^L u_{tt} u_t dx = \alpha_3 \int_0^L u_{xx} u_t dx + \alpha_2 \int_0^L z_{xx} u_t dx + \beta \int_0^L \theta_x u_t dx - \mu \int_0^L u_t \varphi\left(x,1,t\right) dx, \\ c \displaystyle\int_0^L \theta_t \theta dx = -\int_0^L q_x \theta dx + \beta \int_0^L u_{tx} \theta dx, \\ \dfrac{\tau_0}{k} \displaystyle\int_0^L q_t q dx = -\dfrac{1}{k} \int_0^L q^2 dx - \int_0^L q\theta_x dx, \\ |\mu|\,\tau \displaystyle\int_0^L \left(\int_0^1 \varphi_t\left(x,\rho,t\right) \varphi\left(x,\rho,t\right) d\rho \right) dx = -\,|\mu| \int_0^L \left(\int_0^1 \varphi_\rho\left(x,\rho,t\right) \varphi\left(x,\rho,t\right) d\rho \right) dx. \end{cases}$$

This last system is equivalent to

$$\begin{cases} \rho_z \dfrac{d}{2dt} \displaystyle\int_0^L z_t^2 dx = \alpha_1 \int_0^L z_{xx} z_t dx + \alpha_2 \int_0^L u_{xx} z_t dx, \\ \rho_u \dfrac{d}{2dt} \displaystyle\int_0^L u_t^2 dx = \alpha_3 \int_0^L u_{xx} u_t dx + \alpha_2 \int_0^L z_{xx} u_t dx + \beta \int_0^L \theta_x u_t dx, \\ -\mu \displaystyle\int_0^L u_t \varphi\left(x,1,t\right) dx, \\ c \dfrac{d}{2dt} \displaystyle\int_0^L \theta^2 dx = -\int_0^L q_x \theta dx + \beta \int_0^L u_{tx} \theta dx, \\ \dfrac{\tau_0}{k} \dfrac{d}{2dt} \displaystyle\int_0^L q^2 dx = -\dfrac{1}{k} \int_0^L q^2 dx - \int_0^L \theta_x q dx, \\ |\mu|\,\tau \dfrac{d}{2dt} \displaystyle\int_0^L \left(\int_0^1 \varphi^2\left(x,\rho,t\right) d\rho \right) dx = -\,|\mu| \int_0^L \left(\int_0^1 \varphi_\rho\left(x,\rho,t\right) \varphi\left(x,\rho,t\right) d\rho \right) dx. \end{cases}$$

Integrating by parts and taking into account the boundary conditions, we obtain

$$\begin{cases} \rho_z \dfrac{d}{2dt} \displaystyle\int_0^L z_t^2 dx = -\alpha_1 \int_0^L z_{tx} z_x dx - \alpha_2 \int_0^L z_{tx} u_x dx, \\ \rho_u \dfrac{d}{2dt} \displaystyle\int_0^L u_t^2 dx = -\alpha_3 \int_0^L u_{tx} u_x dx - \alpha_2 \int_0^L u_{tx} z_x dx - \beta \int_0^L u_{tx} \theta dx - \mu \int_0^L u_t \varphi(x,1,t)\, dx, \\ c \dfrac{d}{2dt} \displaystyle\int_0^L \theta^2 dx = -\int_0^L \theta q_x dx + \beta \int_0^L \theta u_{tx} dx, \\ \dfrac{\tau_0}{k} \dfrac{d}{2dt} \displaystyle\int_0^L q^2 dx = -\frac{1}{k} \int_0^L q^2 dx + \int_0^L q_x \theta dx, \\ |\mu|\, \tau \dfrac{d}{2dt} \displaystyle\int_0^L \left(\int_0^1 \varphi^2(x,\rho,t)\, d\rho \right) dx = -\frac{|\mu|}{2} \int_0^L \left(\varphi^2(x,1,t) - u_t^2 \right) dx. \end{cases}$$

Summing them up, we obtain

$$\begin{aligned} &\frac{d}{2dt} \int_0^L \Big(\rho_z z_t^2 + \rho_u u_t^2 + c\theta^2 + \frac{\tau_0}{k} q^2 + |\mu|\, \tau \int_0^1 \varphi^2(x,\rho,t)\, d\rho \\ &+\alpha_1 z_x^2 + \alpha_3 u_x^2 + 2\alpha_2 z_x u_x \Big)\, dx \\ &= -\frac{1}{k} \int_0^L q^2 dx - \frac{|\mu|}{2} \int_0^L \varphi^2(x,1,t)\, dx + \frac{|\mu|}{2} \int_0^L u_t^2 dx - \mu \int_0^L u_t \varphi(x,1,t)\, dx. \end{aligned} \tag{3.3}$$

Using the fact that

$$\begin{aligned} &\alpha_1 \int_0^L z_x^2 dx + \alpha_3 \int_0^L u_x^2 dx + 2\alpha_2 \int_0^L u_x z_x dx \\ &= \alpha_1 \int_0^L z_x^2 dx + \alpha_3 \int_0^L u_x^2 dx + 2\alpha_2 \int_0^L u_x z_x dx + \frac{\alpha_2^2}{\alpha_1} \int_0^L u_x^2 dx - \frac{\alpha_2^2}{\alpha_1} \int_0^L u_x^2 dx \\ &= \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L u_x^2 dx + \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 dx. \end{aligned} \tag{3.4}$$

Inserting (3.4) into (3.3), we get

$$\begin{aligned} &\frac{d}{2dt} \int_0^L \Bigg(\rho_z z_t^2 + \rho_u u_t^2 + c\theta^2 + \frac{\tau_0}{k} q^2 + |\mu|\, \tau \int_0^1 \varphi^2(x,\rho,t)\, d\rho \\ &+ \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 \Bigg)\, dx \\ &= -\frac{1}{k} \int_0^L q^2 dx - \frac{|\mu|}{2} \int_0^L \varphi^2(x,1,t)\, dx + \frac{|\mu|}{2} \int_0^L u_t^2 dx - \mu \int_0^L u_t \varphi(x,1,t)\, dx. \end{aligned}$$

Then, using Young's inequality to estimate the last term in (3.3), we have

$$-\mu \int_0^L u_t \varphi(x,1,t)\,dx \le \frac{|\mu|}{2}\int_0^L u_t^2 dx + \frac{|\mu|}{2}\int_0^L \varphi^2(x,1,t)\,dx, \tag{3.5}$$

which leads to (3.2). □

3.2 Lyapunov functional

To construct the Lyapunov functional, we consider the following lemmas

Lemma 3.2.1. *Let (z,u,θ,q,φ) be the solution of (1.4)-(1.5). Then, the functional*

$$I_1(t) = \rho_u \int_0^L u_t u dx - \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u dx,\ t \ge 0, \tag{3.6}$$

satisfies, for any $\varepsilon_1 > 0$,

$$I_1'(t) \le -\frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \left(\rho_u + \frac{\alpha_2^2\rho_z^2}{4\alpha_1^2\varepsilon_1}\right)\int_0^L u_t^2 dx$$
$$+\varepsilon_1 \int_0^L z_t^2 dx + C_0' \int_0^L \theta^2 dx + C_0 \int_0^L \varphi^2(x,1,t)dx,\ \forall t \ge 0. \tag{3.7}$$

Proof. Multiplying $(1.4)_2$ by u and integrating over $(0,L)$, we obtain

$$\rho_u \int_0^L u_{tt} u dx$$
$$= \alpha_3 \int_0^L u_{xx} u dx + \alpha_2 \int_0^L z_{xx} u dx + \beta \int_0^L \theta_x u dx - \mu \int_0^L \varphi(x,1,t) u dx,$$

integrating by parts

$$\rho_u \int_0^L u_{tt} u\, dx$$
$$= -\alpha_3 \int_0^L u_x^2 dx - \alpha_2 \int_0^L z_x u_x dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L \varphi(x,1,t) u dx,$$

thus

$$\rho_u \int_0^L u_{tt} u dx + \rho_u \int_0^L u_t^2 dx - \rho_u \int_0^L u_t^2 dx$$
$$= -\alpha_3 \int_0^L u_x^2 dx - \alpha_2 \int_0^L z_x u_x dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L \varphi(x,1,t) u dx,$$

which is equivalent to

$$\rho_u \frac{d}{dt}\int_0^L u_t u dx = \rho_u \int_0^L u_t^2 dx - \alpha_3 \int_0^L u_x^2 dx - \alpha_2 \int_0^L z_x u_x dx \\ - \beta \int_0^L \theta u_x dx - \mu \int_0^L \varphi(x,1,t) u dx. \tag{3.8}$$

Multiplying $(1.4)_1$ by $\frac{-\alpha_2}{\alpha_1}u$ and integrating over $(0, L)$, we get

$$\frac{-\alpha_2}{\alpha_1}\rho_z \int_0^L z_{tt} u dx = -\alpha_2 \int_0^L z_{xx} u dx - \frac{\alpha_2^2}{\alpha_1}\int_0^L u_{xx} u dx,$$

which is equivalent to

$$- \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_{tt} u dx - \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u_t dx + \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u_t dx \\ = -\alpha_2 \int_0^L z_{xx} u dx - \frac{\alpha_2^2}{\alpha_1}\int_0^L u_{xx} u dx,$$

therefore

$$-\frac{\alpha_2}{\alpha_1}\rho_z \frac{d}{dt}\int_0^L z_t u dx = -\frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u_t dx - \alpha_2 \int_0^L z_{xx} u dx - \frac{\alpha_2^2}{\alpha_1}\int_0^L u_{xx} u dx,$$

integrating by parts, we arrive at

$$-\frac{\alpha_2}{\alpha_1}\rho_z \frac{d}{dt}\int_0^L z_t u dx = -\frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u_t dx + \alpha_2 \int_0^L z_x u_x dx + \frac{\alpha_2^2}{\alpha_1}\int_0^L u_x^2 dx, \tag{3.9}$$

summing (3.8) and (3.9), we obtain

$$\frac{d}{dt}\left[\rho_u \int_0^L u u_t dx - \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u dx\right] \\ = -\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2\, dx + \rho_u \int_0^L u_t^2 dx - \beta \int_0^L \theta u_x dx \\ - \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u_t\, dx - \mu \int_0^L u \varphi(x,1,t) dx. \tag{3.10}$$

Using Young's and Poincaré's inequalities

$$\begin{aligned}
-\mu \int_0^L u\varphi(x,1,t)dx &= -\int_0^L (\sqrt{\chi}u)\left(\frac{\mu}{\sqrt{\chi}}\varphi(x,1,t)\right)dx \\
&\leq \frac{1}{4}\chi \int_0^L u_x^2\,dx + C_0 \int_0^L \varphi^2(x,1,t)dx,
\end{aligned} \tag{3.11}$$

where $\chi = \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) > 0$ and $C_0 = \frac{\mu^2}{\chi}$.

Young's inequality leads to

$$\begin{aligned}
-\beta \int_0^L \theta u_x dx &= -\int_0^L \left(\frac{\beta}{\sqrt{\chi}}\theta\right)(\sqrt{\chi}u_x)dx \\
&\leq C_0' \int_0^L \theta^2 dx + \frac{1}{4}\chi \int_0^L u_x^2\,dx,
\end{aligned} \tag{3.12}$$

where $C_0' = \frac{\beta^2}{\chi}$.

$$\begin{aligned}
\frac{-\alpha_2}{\alpha_1}\rho_z \int_0^L z_t u_t\,dx &= -\int_0^L z_t \left(\left(\frac{\alpha_2}{\alpha_1}\rho_z\right)u_t\right)dx \\
&\leq \varepsilon_1 \int_0^L z_t^2 dx + \frac{\alpha_2^2\rho_z^2}{4\alpha_1^2\varepsilon_1}\int_0^L u_t^2 dx.
\end{aligned} \tag{3.13}$$

Substituting (3.11),(3.12) and (3.13) into (3.10), we get (3.7). □

Lemma 3.2.2. *Let (z,u,θ,q,φ) be the solution of (1.4)-(1.5). Then, the functional*

$$I_2(t) = \rho_u\alpha_2 \int_0^L u_t \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx - \frac{\alpha_2^2}{\alpha_1}\rho_z \int_0^L z_t \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx,\ t \geq 0, \tag{3.14}$$

satisfies, for any $\varepsilon_2 > 0$,

$$\begin{aligned}
I_2'(t) \leq & -\frac{\alpha_2^2\rho_z}{2\sqrt{\alpha_1}}\int_0^L z_t^2 dx + \left(C_1 + \frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_u\right)\int_0^L u_t^2 dx + C_{\varepsilon_2}\int_0^L u_x^2 dx + C_{\varepsilon_2}'\int_0^L \theta^2 dx + \\
&+ C_{\varepsilon_2}''\int_0^L \varphi^2(x,1,t)dx + \varepsilon_2 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx.
\end{aligned} \tag{3.15}$$

Proof. Multiplying $(1.4)_2$ by $\dfrac{\alpha_2^2}{\sqrt{\alpha_1}}u$ and integrating over $(0, L)$, we obtain

$$\rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_{tt} u dx = \alpha_3 \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_{xx} u dx + \frac{\alpha_2^3}{\sqrt{\alpha_1}} \int_0^L z_{xx} u dx + \beta \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L \theta_x u dx - \mu \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u \varphi(x, 1, t) dx,$$

integrating by parts with the boundary conditions

$$\begin{aligned} &\rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_{tt} u dx \\ &= -\alpha_3 \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_x^2 dx - \frac{\alpha_2^3}{\sqrt{\alpha_1}} \int_0^L z_x u_x dx - \beta \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L \theta u_x dx \\ &- \mu \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u \varphi(x, 1, t) dx, \end{aligned}$$

thus

$$\begin{aligned} &\rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_{tt} u dx + \rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_t^2 dx - \rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_t^2 dx \\ &= -\alpha_3 \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_x^2 dx - \frac{\alpha_2^3}{\sqrt{\alpha_1}} \int_0^L z_x u_x dx - \beta \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L \theta u_x dx \\ &- \mu \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u \varphi(x, 1, t) dx, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{d}{dt}\left[\rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_t u dx\right] &= \rho_u \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_t^2 dx - \alpha_3 \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u_x^2 dx \\ &- \frac{\alpha_2^3}{\sqrt{\alpha_1}} \int_0^L z_x u_x dx - \beta \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L \theta u_x dx \\ &- \mu \frac{\alpha_2^2}{\sqrt{\alpha_1}} \int_0^L u \varphi(x, 1, t) dx. \end{aligned} \tag{3.16}$$

Multiplying $(1.4)_2$ by $\sqrt{\alpha_1}\alpha_2 z$ and integrating over $(0, L)$, we arrive at

$$\begin{aligned}
&\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L u_{tt}zdx\\
&=\alpha_3\sqrt{\alpha_1}\alpha_2\int_0^L u_{xx}zdx+\sqrt{\alpha_1}\alpha_2^2\int_0^L z_{xx}zdx+\beta\sqrt{\alpha_1}\alpha_2\int_0^L\theta_x zdx\\
&-\mu\sqrt{\alpha_1}\alpha_2\int_0^L z\varphi(x,1,t)dx,
\end{aligned}$$

integrating by parts

$$\begin{aligned}
&\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L u_{tt}zdx\\
&=-\alpha_3\sqrt{\alpha_1}\alpha_2\int_0^L u_x z_x dx-\sqrt{\alpha_1}\alpha_2^2\int_0^L z_x^2dx-\beta\sqrt{\alpha_1}\alpha_2\int_0^L\theta z_x dx\\
&-\mu\sqrt{\alpha_1}\alpha_2\int_0^L z\varphi(x,1,t)dx,
\end{aligned}$$

therefore

$$\begin{aligned}
&\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L u_{tt}zdx+\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L z_t u_t dx-\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L z_t u_t dx\\
&=-\alpha_3\sqrt{\alpha_1}\alpha_2\int_0^L u_x z_x dx-\sqrt{\alpha_1}\alpha_2^2\int_0^L z_x^2dx-\beta\sqrt{\alpha_1}\alpha_2\int_0^L\theta z_x dx\\
&-\mu\sqrt{\alpha_1}\alpha_2\int_0^L z\varphi(x,1,t)dx,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\frac{d}{dt}\left[\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L u_{tt}zdx\right]&=\rho_u\sqrt{\alpha_1}\alpha_2\int_0^L z_t u_t dx-\alpha_3\sqrt{\alpha_1}\alpha_2\int_0^L u_x z_x dx\\
&-\sqrt{\alpha_1}\alpha_2^2\int_0^L z_x^2dx-\beta\sqrt{\alpha_1}\alpha_2\int_0^L\theta z_x dx\\
&-\mu\sqrt{\alpha_1}\alpha_2\int_0^L z\varphi(x,1,t)dx. \qquad (3.17)
\end{aligned}$$

Multiplying $(1.4)_1$ by $-\dfrac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}u$ and integrating over $(0, L)$, we obtain

$$-\frac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}\rho_z\int_0^L z_{tt}udx=-\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L z_{xx}udx-\frac{\alpha_2^4}{\alpha_1\sqrt{\alpha_1}}\int_0^L u_{xx}udx,$$

integrating by parts

$$-\frac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}\rho_z\int_0^L z_{tt}udx=\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L z_xu_xdx+\frac{\alpha_2^4}{\alpha_1\sqrt{\alpha_1}}\int_0^L u_x^2dx,$$

thus

$$\begin{aligned}&-\frac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}\rho_z\int_0^L z_{tt}udx-\frac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}\rho_z\int_0^L z_tu_tdx+\frac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}\rho_z\int_0^L z_tu_tdx\\&=\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L z_xu_xdx+\frac{\alpha_2^4}{\alpha_1\sqrt{\alpha_1}}\int_0^L u_x^2dx,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{d}{dt}\left[-\frac{\alpha_2^2}{\alpha_1}\rho_z\int_0^L z_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u\right)dx\right]\\&=-\frac{\alpha_2^3}{\alpha_1\sqrt{\alpha_1}}\rho_z\int_0^L z_tu_tdx+\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L z_xu_xdx+\frac{\alpha_2^4}{\alpha_1\sqrt{\alpha_1}}\int_0^L u_x^2dx.\end{aligned}\tag{3.18}$$

Multiplying $(1.4)_1$ by $-\frac{\alpha_2^2}{\sqrt{\alpha_1}}z$ and integrating over $(0,L)$, we obtain

$$-\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_{tt}zdx=-\alpha_2^2\sqrt{\alpha_1}\int_0^L z_{xx}zdx-\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L u_{xx}zdx,$$

integrating by parts

$$-\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_{tt}zdx=\alpha_2^2\sqrt{\alpha_1}\int_0^L z_x^2dx+\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L u_xz_xdx,$$

thus

$$\begin{aligned}&-\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_{tt}zdx-\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_t^2dx+\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_t^2dx\\&=\alpha_2^2\sqrt{\alpha_1}\int_0^L z_x^2dx+\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L u_xz_xdx,\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{d}{dt}\left[-\frac{\alpha_2^2}{\alpha_1}\rho_z\int_0^L z_t(\sqrt{\alpha_1}z)dx\right]\\
&= -\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_t^2dx+\alpha_2^2\sqrt{\alpha_1}\int_0^L z_x^2dx+\frac{\alpha_2^3}{\sqrt{\alpha_1}}\int_0^L u_xz_xdx. \qquad (3.19)
\end{aligned}$$

Summing (3.16)-(3.19), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\rho_u\alpha_2\int_0^L u_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u+\sqrt{\alpha_1}z\right)dx-\frac{\alpha_2^2}{\alpha_1}\rho_z\int_0^L z_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u+\sqrt{\alpha_1}z\right)dx\right]\\
&= -\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_z\int_0^L z_t^2dx+\alpha_2(\rho_u\sqrt{\alpha_1}-\frac{\alpha_2^2}{\alpha_1\sqrt{\alpha_1}}\rho_z)\int_0^L u_tz_tdx+\frac{\alpha_2^2}{\sqrt{\alpha_1}}\rho_u\int_0^L u_t^2dx\\
&-\alpha_2\chi\int_0^L u_x\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)dx-\alpha_2\beta\int_0^L\theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)dx\\
&-\mu\alpha_2\int_0^L\varphi(x,1,t)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u+\sqrt{\alpha_1}z\right)dx. \qquad (3.20)
\end{aligned}$$

Using Young's inequality, with $\xi=\alpha_2(\rho_u\sqrt{\alpha_1}-\dfrac{\alpha_2^2\rho_z}{\alpha_1\sqrt{\alpha_1}})$, we get

$$\xi\int_0^L u_tz_tdx=\int_0^L(\xi u_t)z_tdx\le\int_0^L|\xi u_t||z_t|dx,$$

$$\begin{aligned}
\xi\int_0^L u_tz_tdx &= \int_0^L\frac{\sqrt[4]{\alpha_1}}{\alpha_2\sqrt{\rho_z}}|\xi|\,|u_t|\,\frac{\alpha_2\sqrt{\rho_z}}{\sqrt[4]{\alpha_1}}\,|z_t|\,dx\\
&\le\frac{\alpha_2^2\rho_z}{2\sqrt{\alpha_1}}\int_0^L z_t^2dx+C_1\int_0^L u_t^2dx, \qquad (3.21)
\end{aligned}$$

where $C_1=\dfrac{\sqrt{\alpha_1}}{2\alpha_2^2\rho_z}\xi^2$.

$$\begin{aligned}
-\alpha_2\chi\int_0^L u_x\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)dx &\le \alpha_2\chi\int_0^L|u_x|\left|\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right|dx\\
&=\int_0^L\alpha_2\chi|u_x|\left|\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right|dx\\
&\le\frac{\varepsilon_2}{3}\int_0^L\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)^2dx+C_{\varepsilon_2}\int_0^L u_x^2dx, \qquad (3.22)
\end{aligned}$$

where $C_{\varepsilon_2} = \dfrac{3\alpha_2^2\chi^2}{4\varepsilon_2}$.

$$
\begin{aligned}
-\alpha_2\beta\int_0^L \theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx &\le \alpha_2\beta\int_0^L |\theta|\left|\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right|dx \\
&\le \frac{\varepsilon_2}{3}\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx + C'_{\varepsilon_2}\int_0^L \theta^2 dx, \qquad (3.23)
\end{aligned}
$$

where $C'_{\varepsilon_2} = \dfrac{3\alpha_2^2\beta^2}{4\varepsilon_2}$.
Using Young's and Poincaré's inequalities, leads to

$$
\begin{aligned}
-\mu\alpha_2\int_0^L \varphi(x,1,t)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx &\le \mu\alpha_2\int_0^L |\varphi(x,1,t)|\left|\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right|dx \\
&\le \frac{\varepsilon_2}{3}\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)^2 dx + C''_{\varepsilon_2}\int_0^L \varphi^2(x,1,t)dx \\
&\le \frac{\varepsilon_2}{3}\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx + C''_{\varepsilon_2}\int_0^L \varphi^2(x,1,t)dx,
\end{aligned}
\tag{3.24}
$$

where $C''_{\varepsilon_2} = \dfrac{3\alpha_2^2\mu^2}{4\varepsilon_2}$.
Substituting (3.21)-(3.24) into (3.20), we obtain (3.15). □

Lemma 3.2.3. *Let $(z, u, \theta, q, \varphi)$ be the solution of (1.4)-(1.5). Then, the functional*

$$
I_3(t) = \rho_u\int_0^L u_t u dx + \rho_z\int_0^L z z_t dx,\ t \ge 0, \tag{3.25}
$$

satisfies

$$
\begin{aligned}
I_3'(t) \le &-\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx - \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx \\
&+ \rho_z\int_0^L z_t^2 dx + \rho_u\int_0^L u_t^2 dx + C_2\int_0^L \theta^2 + C_2'\int_0^L \varphi^2(x,1,t)\,dx.
\end{aligned}
\tag{3.26}
$$

Proof. Multiplying $(1.4)_2$ by u and integrating over $(0, L)$, we arrive at

$$
\rho_u\int_0^L u_{tt}u dx = \alpha_3\int_0^L u_{xx}u dx + \alpha_2\int_0^L z_{xx}u dx + \beta\int_0^L \theta_x u dx - \mu\int_0^L u\varphi(x,1,t)\,dx,
$$

integrating by parts, we obtain

$$\rho_u \int_0^L u_{tt} u dx = -\alpha_3 \int_0^L u_x^2 dx - \alpha_2 \int_0^L z_x u_x dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L u\varphi\left(x,1,t\right) dx,$$

thus

$$\begin{aligned} &\rho_u \int_0^L u_{tt} u dx + \rho_u \int_0^L u_t^2 dx - \rho_u \int_0^L u_t^2 dx \\ &= -\alpha_3 \int_0^L u_x^2 dx - \alpha_2 \int_0^L z_x u_x dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L u\varphi\left(x,1,t\right) dx, \end{aligned}$$

which implies that

$$\begin{aligned} \rho_u \frac{d}{dt} \int_0^L u_t u dx = \rho_u \int_0^L u_t^2 dx - \alpha_3 \int_0^L u_x^2 dx - \alpha_2 \int_0^L z_x u_x dx \\ - \beta \int_0^L \theta u_x dx - \mu \int_0^L u\varphi\left(x,1,t\right) dx. \end{aligned} \tag{3.27}$$

Multiplying $(1.4)_1$ by z and integrating over $(0,L)$, we obtain

$$\rho_z \int_0^L z z_{tt} dx = \alpha_1 \int_0^L z z_{xx} dx + \alpha_2 \int_0^L z u_{xx} dx,$$

integrating by parts

$$\rho_z \int_0^L z z_{tt} dx + \rho_z \int_0^L z_t^2 dx - \rho_z \int_0^L z_t^2 dx = -\alpha_1 \int_0^L z_x^2 dx - \alpha_2 \int_0^L z_x u_x dx,$$

which is equivalent to

$$\rho_z \frac{d}{dt} \int_0^L z z_t dx = \rho_z \int_0^L z_t^2 dx - \alpha_1 \int_0^L z_x^2 dx - \alpha_2 \int_0^L z_x u_x dx, \tag{3.28}$$

summing (3.27)-(3.23) and using (3.4), we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\rho_u \int_0^L u_t u dx + \rho_z \int_0^L z z_t dx\right] \\ &= -\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2 dx - \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx - \beta \int_0^L \theta u_x dx \\ &- \mu \int_0^L u\varphi\left(x,1,t\right) dx + \rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx. \end{aligned} \tag{3.29}$$

Using Young's inequality, we get

$$\begin{aligned} -\beta\int_0^L \theta u_x dx &= -\int_0^L \frac{\beta}{\sqrt{\chi}}\theta\sqrt{\chi}u_x dx \leq \left| -\int_0^L \frac{\beta}{\sqrt{\chi}}\theta\sqrt{\chi}u_x dx \right| \\ &\leq C_2\int_0^L \theta^2 dx + \frac{1}{4}\chi\int_0^L u_x^2 dx, \end{aligned} \tag{3.30}$$

where $C_2 = \dfrac{\beta^2}{\chi}$.

Using Young's and Poincaré's inequalities, we find

$$\begin{aligned} -\mu\int_0^L u\varphi(x,1,t)\,dx &= -\int_0^L \left(\frac{1}{\sqrt{\chi}}\mu\varphi(x,1,t)\right)(\sqrt{\chi}u)\,dx \\ &\leq \frac{1}{\chi}\mu^2\int_0^L \varphi^2(x,1,t)\,dx + \frac{1}{4}\chi\int_0^L u^2 dx \\ &\leq C_2'\int_0^L \varphi^2(x,1,t)\,dx + \frac{1}{4}\chi\int_0^L u_x^2 dx, \end{aligned} \tag{3.31}$$

where $C_2' = \dfrac{\mu^2}{\chi}$. Substituting (3.30) and (3.31) into (3.29), we get (3.26). □

Lemma 3.2.4. *Let (z,u,θ,q,φ) be the solution of (1.4)-(1.5). Then, the functional*

$$I_4(t) = -c\rho_u\int_0^L \theta\left(\int_x^L u_t(y)\,dy\right)dx,\ \forall t \geq 0, \tag{3.32}$$

satisfies, for any $\varepsilon_3,\varepsilon_4,\varepsilon_5 > 0$, the following estimate

$$\begin{aligned} I_4'(t) \leq& -\frac{\beta\rho_u}{2}\int_0^L u_t^2 dx + \varepsilon_3\int_0^L u_x^2 dx + \varepsilon_4\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx \\ &+ \frac{\rho_u}{2\beta}\int_0^L q^2 dx + \left(c\beta + \frac{C_3}{\varepsilon_3} + \frac{C_3'}{\varepsilon_4} + \frac{C_3''}{\varepsilon_5}\right)\int_0^L \theta^2 dx \\ &+ L^2\varepsilon_5\int_0^L \varphi^2(x,1,t)\,dx. \end{aligned} \tag{3.33}$$

Proof. Multiplying $(1.4)_3$ by $-\rho_u\int_x^L u_t(y)dy$ and integrating over $(0,L)$, we have

$$\begin{aligned} &-c\rho_u\int_0^L \theta_t\left(\int_x^L u_t(y)\,dy\right)dx \\ &= \rho_u\int_0^L q_x\left(\int_x^L u_t(y)\,dy\right)dx - \beta\rho_u\int_0^L u_{tx}\left(\int_x^L u_t(y)\,dy\right)dx, \end{aligned}$$

integrating by parts

$$\begin{aligned}
&- c\rho_u \int_0^L \theta_t \left(\int_x^L u_t\left(y\right) dy \right) dx - c\rho_u \int_0^L \theta \left(\int_x^L u_{tt}\left(y\right) dy \right) dx \\
&+ c\rho_u \int_0^L \theta \left(\int_x^L u_{tt}\left(y\right) dy \right) dx \\
&= \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&- c\rho_u \frac{d}{dt} \int_0^L \theta \left(\int_x^L u_t\left(y\right) dy \right) dx \\
&= \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx \\
&- c \int_0^L \theta \int_x^L \left[\alpha_3 u_{yy} + \alpha_2 z_{yy} + \beta\theta_y - \mu\varphi(y,1,t)\right] dy dx,
\end{aligned}$$

therefore

$$\begin{aligned}
I_4'(t) &= \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx \\
&- c \int_0^L \theta \left[-\alpha_3 u_x - \alpha_2 z_x - \beta\theta - \mu \int_x^L \varphi(y,1,t) dy \right] dx \\
&= \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx + c\alpha_3 \int_0^L \theta u_x dx + c\alpha_2 \int_0^L \theta z_x dx \\
&+ c\beta \int_0^L \theta^2 dx + c\mu \int_0^L \theta \int_x^L \varphi(y,1,t) dy dx,
\end{aligned} \tag{3.34}$$

using the fact that

$$\begin{aligned}
&c\alpha_3 \int_0^L \theta u_x dx + c\alpha_2 \int_0^L \theta z_x dx \\
&= c\chi \int_0^L \theta u_x dx + c\frac{\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right) dx,
\end{aligned}$$

then, (3.29) can be rewritten as follows

$$\begin{aligned}
I_4'(t) &= \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx + c\chi \int_0^L \theta u_x dx \\
&+ c\frac{\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right) dx
\end{aligned} \tag{3.35}$$

$$+c\beta\int_0^L \theta^2 dx + c\mu\int_0^L \theta\int_x^L \varphi(y,1,t)dydx.$$

Using Young's inequality, we obtain

$$\begin{aligned} c\chi\int_0^L \theta u_x dx &= \int_0^L (c\chi\theta)u_x dx \\ &\leq \varepsilon_3\int_0^L u_x^2 dx + \frac{C_3}{\varepsilon_3}\int_0^L \theta^2 dx, \end{aligned} \tag{3.36}$$

with $C_3 = \dfrac{c^2\chi^2}{4}$.

$$\begin{aligned} & c\frac{\alpha_2}{\sqrt{\alpha_1}}\int_0^L \theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx \\ &= \int_0^L \left(c\frac{\alpha_2}{\sqrt{\alpha_1}}\theta\right)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx \\ &\leq \varepsilon_4\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx + \frac{C_3'}{\varepsilon_4}\int_0^L \theta^2 dx, \end{aligned} \tag{3.37}$$

where $C_3' = \dfrac{c^2\alpha_2^2}{4\alpha_1}$.
and

$$\rho_u\int_0^L q u_t dx \leq \frac{\beta\rho_u}{2}\int_0^L u_t^2 dx + \frac{\rho_u}{2\beta}\int_0^L q^2 dx. \tag{3.38}$$

Young's inequality, leads to

$$c\mu\int_0^L \theta\int_x^L \varphi(y,1,t)dydx \leq \varepsilon_5\int_0^L\left(\int_x^L \varphi(y,1,t)dy\right)^2 dx + \frac{c^2\mu^2}{4\varepsilon_5}\int_0^L \theta^2 dx,$$

it follows from Cauchy Schwarz's inequality that

$$\begin{aligned} \int_0^L\left(\int_x^L \varphi(y,1,t)dy\right)^2 dx &\leq \int_0^L\left(\left(\int_x^L 1^2 dy\right)^{\frac{1}{2}}\left(\int_x^L \varphi^2(y,1,t)dy\right)^{\frac{1}{2}}\right)^2 dx \\ &\leq \int_0^L\left(\int_x^L dy\right)\left(\int_x^L \varphi^2(y,1,t)dy\right)dx \\ &\leq \int_0^L L\left(\int_0^L \varphi^2(y,1,t)dy\right)dx \\ &\leq L^2\int_0^L \varphi^2(x,1,t)dx, \end{aligned}$$

then

$$c\mu \int_0^L \theta \int_x^L \varphi(y,1,t)dydx \leq L^2\varepsilon_5 \int_0^L \varphi^2(x,1,t)dx + \frac{C_3''}{\varepsilon_5}\int_0^L \theta^2 dx, \tag{3.39}$$

where $C_3'' = \dfrac{c^2\mu^2}{4}$.
Inserting (**3.36**)-(3.39) into (3.35), we get (3.33). □

Lemma 3.2.5. *Let (z,u,θ,q,φ) be the solution of (1.4)-(1.5). Then, the functional*

$$I_5(t) = -c\tau_0 \int_0^L \theta \left(\int_x^L q(y)dy\right) dx,\ \forall t \geq 0, \tag{3.40}$$

satisfies, for any $\varepsilon_6 > 0$, the following estimate

$$I_5'(t) \leq -\frac{ck}{2}\int_0^L \theta^2 dx + \varepsilon_6 \int_0^L u_t^2 dx + \left(C_4' + \frac{C_4}{\varepsilon_6}\right)\int_0^L q^2 dx. \tag{3.41}$$

Proof. Multiplying $(1.4)_3$ by $-\tau_0 \displaystyle\int_x^L q(y)dy$ and integrating by parts, we obtain

$$-c\tau_0 \int_0^L \theta_t \left(\int_x^L q(y)dy\right) dx = \tau_0 \int_0^L q^2 dx - \tau_0\beta \int_0^L u_t q dx,$$

which implies that

$$\begin{aligned}
&- c\tau_0 \int_0^L \theta_t \left(\int_x^L q(y)dy\right) dx - c\tau_0 \int_0^L \theta \left(\int_x^L q_t(y)dy\right) dx \\
&+ c\tau_0 \int_0^L \theta \left(\int_x^L q_t(y)dy\right) dx \\
&= \tau_0 \int_0^L q^2 dx - \tau_0\beta \int_0^L u_t q dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{dt}\left[-c\tau_0 \int_0^L \theta \left(\int_x^L q(y)dy\right) dx\right] \\
&= \tau_0 \int_0^L q^2 dx - \tau_0\beta \int_0^L u_t q dx - c\tau_0 \int_0^L \theta \left(\int_x^L q_t(y)dy\right) dx,
\end{aligned}$$

using $(1.4)_4$, we get

$$\begin{aligned}&\frac{d}{dt}\left[-c\tau_0\int_0^L\theta\left(\int_x^L q(y)dy\right)dx\right]\\&=\tau_0\int_0^L q^2dx-\tau_0\beta\int_0^L u_tqdx-c\int_0^L\theta\left(\int_x^L(-q(y)-k\theta_y)dy\right)dx,\end{aligned}$$

thus

$$\begin{aligned}&\frac{d}{dt}\left[-c\tau_0\int_0^L\theta\left(\int_x^L q(y)dy\right)dx\right]\\&=-ck\int_0^L\theta^2dx+\tau_0\int_0^L q^2dx-\tau_0\beta\int_0^L u_tqdx+c\int_0^L\theta\left(\int_x^L q(y)dy\right)dx.\end{aligned}\tag{3.42}$$

Young's inequality, leads to

$$\begin{aligned}-\tau_0\beta\int_0^L u_tqdx&\le\varepsilon_6\int_0^L u_t^2dx+\frac{\tau_0^2\beta^2}{4\varepsilon_6}\int_0^L q^2dx\\&=\varepsilon_6\int_0^L u_t^2dx+\frac{C_4}{\varepsilon_6}\int_0^L q^2dx.\end{aligned}\tag{3.43}$$

where $C_4=\dfrac{\tau_0^2\beta^2}{4}$.
Applying Young's and Cauchy's inequalities yields

$$\begin{aligned}c\int_0^L\theta\left(\int_x^L q(y)dy\right)dx&\le\frac{ck}{2}\int_0^L\theta^2dx+\frac{cL^2}{2k}\int_0^L q^2dx.\\&=\frac{ck}{2}\int_0^L\theta^2dx+C_4'\int_0^L q^2dx.\end{aligned}\tag{3.44}$$

where $C_4'=\dfrac{cL^2}{2k}$. Inserting (3.43) and (3.44) in (3.42), we get (3.41). □

Lemma 3.2.6. *Let (z,u,θ,q,φ) be the solution of (1.4) - (1.5). Then, the functional*

$$I_6(t)=\tau\int_0^L\int_0^1\exp(-\tau\rho)\,\varphi^2(x,\rho,t)\,d\rho dx,\tag{3.45}$$

satisfies

$$I_6'(t)\le\int_0^L u_t^2dx-\exp(-\tau)\left(\int_0^L\varphi^2(x,1,t)\,dx+\tau\int_0^L\int_0^1\varphi^2(x,\rho,t)\,d\rho dx\right).\tag{3.46}$$

Proof. Multiplying $(1.4)_5$ by $2\exp(-\tau\rho)\,\varphi(x,\rho,t)$ and integrating over $(0,L)\times(0,1)$, we get

$$\int_0^L\int_0^1 \tau\exp(-\tau\rho)\frac{d}{dt}\Big(\varphi^2(x,\rho,t)\Big)\,d\rho dx = -\int_0^L\int_0^1 \exp(-\tau\rho)\frac{d}{d\rho}\Big(\varphi^2(x,\rho,t)\Big)\,d\rho dx,$$

which implies that

$$\tau\frac{d}{dt}\int_0^L\int_0^1 \exp(-\tau\rho)\,\varphi^2(x,\rho,t)\,d\rho dx = -\int_0^L\int_0^1 \exp(-\tau\rho)\frac{d}{d\rho}\Big(\varphi^2(x,\rho,t)\Big)\,d\rho dx.$$

The right-hand side of the last formula represents the time derivative of the functional $I_6(t)$

$$I_6'(t) = -\int_0^L\int_0^1 \exp(-\tau\rho)\frac{d}{d\rho}\Big(\varphi^2(x,\rho,t)\Big)\,d\rho dx. \tag{3.47}$$

integrating by parts to the second integral in (3.47) yields

$$\begin{aligned}\int_0^1 \exp(-\tau\rho)\frac{d}{d\rho}\Big(\varphi^2(x,\rho,t)\Big)\,d\rho &= \exp(-\tau)\,\varphi^2(x,1,t)-\varphi^2(x,0,t)\\ &\quad+\int_0^1 \tau\exp(-\tau\rho)\,\varphi^2(x,\rho,t)\,d\rho,\end{aligned} \tag{3.48}$$

substituting (3.48) into (3.47), we obtain

$$I_6'(t) = \int_0^L \varphi^2(x,0,t)\,dx - \exp(-\tau)\int_0^L \varphi^2(x,1,t)\,dx - \tau\int_0^L\int_0^1 \exp(-\tau\rho)\,\varphi^2(x,\rho,t)\,d\rho dx,$$

where $\varphi(x,0,t)=\psi(x,t)=u_t$, thus

$$I_6'(t) = \int_0^L u_t^2 dx - \exp(-\tau)\int_0^L \varphi^2(x,1,t)\,dx - \tau\int_0^L\int_0^1 \exp(-\tau\rho)\,\varphi^2(x,\rho,t)\,d\rho dx.$$

Note that

$$\exp(-\tau)\le\exp(-\tau\rho)\le 1 \text{ for all } \rho\in[0,1],$$

hence

$$I_6'(t)\le\int_0^L u_t^2 dx - \exp(-\tau)\left(\int_0^L \varphi^2(x,1,t)\,dx + \tau\int_0^L\int_0^1 \varphi^2(x,\rho,t)\,d\rho dx\right).$$

□

3.3 Lyapunov functional properties

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + 2(I_3(t) + I_4(t)) + N_3 I_5(t) + N_4 I_6(t), \tag{3.49}$$

where N, N_1, N_2, N_3, N_4 are positive constants.

Theorem 3.3.1. *Let $(z, u, \theta, q, \varphi)$ be the solution of (1.4)-(1.5). Then, there exist two positive constants $\mathcal{K}_1$ and $\mathcal{K}_2$ such that the Lyapunov functional (3.49) satisfies*

$$\mathcal{K}_1 E(t) \leq \mathcal{L}(t) \leq \mathcal{K}_2 E(t), \ \forall t \geq 0, \tag{3.50}$$

and

$$\mathcal{L}'(t) \leq -\beta_1 E(t), \ \beta_1 > 0. \tag{3.51}$$

Proof. From (3.49), we have

$$\begin{aligned}
|\mathcal{L}(t) - NE(t)| \leq{} & N_1 \rho_u \int_0^L |u_t u|\, dx + N_1 \frac{\alpha_2}{\alpha_1} \rho_z \int_0^L |z_t u|\, dx \\
& + N_2 \rho_u \alpha_2 \int_0^L \left| u_t \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u + \sqrt{\alpha_1} z \right) \right| dx \\
& + N_2 \frac{\alpha_2^2}{\alpha_1} \rho_z \int_0^L \left| z_t \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u + \sqrt{\alpha_1} z \right) \right| dx \\
& + 2\rho_u \int_0^L |u u_t|\, dx + 2\rho_z \int_0^L |z z_t|\, dx \\
& + 2c\rho_u \int_0^L \left| \theta \left(\int_x^L u_t(y) dy \right) \right| dx \\
& + N_3 c \tau_0 \int_0^L \left| \theta \left(\int_x^L q(y) dy \right) \right| dx \\
& + N_4 \tau \int_0^L \int_0^1 e^{-\tau\rho} \varphi^2(x, \rho, t) d\rho dx.
\end{aligned}$$

By using the Young's, Poincaré's and Cauchy-Schwarz's inequalities, we obtain

$$\begin{aligned}
|\mathcal{L}(t) - NE(t)| \leq{} & \delta_1 \int_0^L z_t^2 dx + \delta_2 \int_0^L u_t^2 dx + \delta_3 \int_0^L u_x^2 dx \\
& + \delta_4 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 dx + \delta_5 \int_0^L \theta^2 dx \\
& + \delta_6 \int_0^L q^2 dx + \delta_7 \int_0^L \int_0^1 \varphi^2 (x, \rho, t)\, d\rho dx,
\end{aligned}$$

then

$$|\mathcal{L}(t) - NE(t)| \leq \gamma_1 \int_0^L (z_t^2 + u_t^2 + u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t)\, d\rho) dx,$$

where

$$\gamma_1 = \max\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\} > 0.$$

Let

$$\gamma_2 = \min\left\{\frac{\rho_z}{2}, \frac{\rho_u}{2}, \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_3}\right), \frac{1}{2}, \frac{c}{2}, \frac{\tau_0}{2k}, \frac{|\mu|\,\tau}{2}\right\} > 0.$$

Then

$$\int_0^L \left(z_t^2 + u_t^2 + u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t)\, d\rho\right) dx \leq \frac{E(t)}{\gamma_2}.$$

Consequently

$$\gamma_1 \int_0^L \left(z_t^2 + u_t^2 + u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t)\, d\rho dx\right) \leq \frac{\gamma_1}{\gamma_2} E(t),$$

It means that

$$|\mathcal{L}(t) - NE(t)| \leq \gamma E(t), \text{ with } \gamma = \frac{\gamma_1}{\gamma_2} > 0,$$

which yields

$$(N - \gamma) E(t) \leq \mathcal{L}(t) \leq (N + \gamma) E(t).$$

By choosing N (depending on N_1, N_2, N_3 and N_4) sufficiently large we obtain (3.49).

Now, By differentiating $\mathcal{L}(t)$, exploiting (3.2), (3.7), (3.15), (3.26), (3.31), (3.41), (3.46) and setting $\varepsilon_1 = \frac{1}{N_1}$, $\varepsilon_2 = \frac{1}{N_2}$, $\varepsilon_3 = \varepsilon_5 = \frac{1}{2}$, $\varepsilon_4 = \frac{1}{4}$, $\varepsilon_6 = \frac{1}{N_3}$, we get

$$\mathcal{L}'(t) \leq -\left[\rho_u \beta - |\mu| N - N_1\left(\rho_u + \frac{\alpha_2^2 \rho_z^2}{4\alpha_1^2} N_1\right) - N_2\left(C_1 + \frac{\alpha_2^2}{\sqrt{\alpha_1}} \rho_u\right) - 2\rho_u - 1 - N_4\right] \int_0^L u_t^2 dx$$

$$
\begin{aligned}
&- \left[\frac{N}{k} - \frac{\rho_u}{\beta} - N_3(C_4' + C_4 N_3)\right] \int_0^L q^2 dx \\
&- \left[\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\left(\frac{N_1}{2} + 1\right) - C_{\varepsilon_2} N_2 - 1\right] \int_0^L u_x^2 dx \\
&- \left[\frac{\alpha_2^2 \rho_z N_2}{2\sqrt{\alpha_1}} - 1 - 2\rho_z\right] \int_0^L z_t^2 dx \\
&- \left[\frac{ckN_3}{2} - 8C_3' - 2(C_2 + c\beta) - 4(C_3 + C_3'') - C_{\varepsilon_2}' N_2 - C_0' N_1\right] \int_0^L \theta^2 dx \\
&- \frac{1}{2}\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx \\
&- \left[e^{-\tau} N_4 - L^2 - 2C_2' - N_2 C_{\varepsilon_2}'' - N_1 C_0\right] \int_0^L \varphi^2(x,1,t) dx \\
&- \tau e^{-\tau} N_4 \int_0^L \int_0^1 \varphi^2(x,\rho,t) d\rho dx.
\end{aligned}
$$

Now, we select our parameters appropriately as follows: First, we choose N_2 large enough so that

$$\xi_1 = \frac{\alpha_2^2 \rho_z}{2\sqrt{\alpha_1}} N_2 - 1 - 2\rho_z > 0.$$

Next, we select N_1 large enough so that

$$\xi_2 = \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{2\alpha_1}\right)(N_1 + 2) - C_{\varepsilon_2} N_2 - 1 > 0.$$

We take N_3 large such that

$$\xi_3 = \frac{ckN_3}{2} - 8C_3' - 2(C_2 + c\beta) - 4(C_3 + C_3'') - C_{\varepsilon_2}' N_2 - C_0' N_1 > 0.$$

We pick N_4 large so that

$$\xi_4 = e^{-\tau} N_4 - L^2 - 2C_2' - N_2 C_{\varepsilon_2}'' - N_1 C_0 > 0.$$

We choose N large enough so that (3.50) remains valid, further

$$\xi_5 = \frac{N}{k} - \frac{\rho_u}{\beta} - N_3(C_4' + C_4 N_3) > 0.$$

Finally, by taking $|\mu|$ small enough so that

$$\xi_6 = \rho_u \beta - |\mu| N - N_1 \left(\rho_u + \frac{\alpha_2^2 \rho_z^2}{4\alpha_1^2} N_1 \right) - N_2 \left(C_1 + \frac{\alpha_2^2}{\sqrt{\alpha_1}} \rho_u \right) - 2\rho_u - 1 - N_4 > 0$$

All these choices, leads to

$$\begin{aligned} \mathcal{L}'(t) \leq -\zeta_1 \int_0^L (z_t^2 + u_t^2 + u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 \\ + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t) \, d\rho) dx, \end{aligned} \tag{3.52}$$

we set

$$\zeta_1 = \min\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\} > 0,$$

and

$$\zeta_2 = \max \left\{ \frac{\rho_z}{2}, \frac{\rho_u}{2}, \frac{1}{2} \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_3} \right), \frac{1}{2}, \frac{c}{2}, \frac{\tau_0}{2k}, \frac{|\mu| \, \tau}{2} \right\} > 0.$$

Then

$$\begin{aligned} &\int_0^L \left(z_t^2 + u_t^2 + u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t) \, d\rho \right) dx \\ &\geq \frac{1}{\zeta_2} E(t), \end{aligned}$$

consequently

$$\begin{aligned} &-\zeta_1 \int_0^L \left(z_t^2 + u_t^2 + u_x^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t) \, d\rho \right) dx \\ &\leq -\frac{\zeta_1}{\zeta_2} E(t), \end{aligned} \tag{3.53}$$

The combination of (3.52) and (3.53) gives (3.51) where $\beta_1 = \dfrac{\zeta_1}{\zeta_2} > 0$. □

3.4 Main exponential stability result

Now, we can state and prove the following stability result

Lemma 3.4.1. *Let $(z, u, \theta, q, \varphi)$ be the solution of (1.4)-(1.5). Then, for any $U_0 \in D(\mathcal{A})$, there exist two* *positive constants λ_1 and λ_2 such that*

$$E(t) \leq \lambda_2 \exp\left(-\lambda_1 t\right), \ \forall t \geq 0. \tag{3.54}$$

Proof. By using the estimation (3.51), we get

$$\mathcal{L}'(t) \leq -\beta_1 E(t), \ t \geq 0,$$

having in mind the equivalence of $E(t)$ and $\mathcal{L}(t)$, we infer that

$$\mathcal{L}'(t) \leq -\frac{\beta_1}{\mathcal{K}_2}\mathcal{L}(t), \ t \geq 0,$$

we set $\lambda_1 = \dfrac{\beta_1}{\mathcal{K}_2}$,

$$\frac{\mathcal{L}'(t)}{\mathcal{L}(t)} \leq -\lambda_1, \tag{3.55}$$

integrating (3.55) yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) \exp\left(-\lambda_1 t\right), \ t \geq 0,$$

using the left side of the equivalence relation (3.50), we get

$$\mathcal{K}_1 E(t) \leq \mathcal{L}(0) \exp\left(-\lambda_1 t\right).$$

Thus

$$E(t) \leq \lambda_2 \exp\left(-\lambda_1 t\right), \ \forall t \geq 0,$$

where $\lambda_2 = \dfrac{\mathcal{L}(0)}{\mathcal{K}_1}$. □

CHAPTER 4

Numerical simulation

In the second section, we established the existence and uniqueness of the solution to system(1.4)-(1.5) , but it is challenging to determine the exact value of the solution. To address this, in the following, we will numerically solve the system in the one-dimension domain Ω of length L., allowing us to determine approximations of the solution. To this end, we will employ the Euler scheme for discretizing temporal variable and the classical finite differences method for discretizing space. To solve the discretized problem, we use a fixed point algorithm with study of their convergence ([1],[3], [4]). In addition, we provide an example where the numerical experiment demonstrates that the discrete energy E^n decays exponentially for different choices of the system parameters, supporting the asymptotic behavior of the discretized issue solution.

4.1 Discretization of the continuous problem

Let us introduce the functions $\widehat{z} = z_t$, $\widehat{u} = u_t$ and for any $N, m, M \in \mathbb{N}$, we introduce the nets

$$\begin{aligned}
&\Omega_N = \{x_i = \rho_i = ih,\ i = 0,1,...,N+1 \text{ , where } h = \frac{L}{N+1}\},\\
&\Gamma_M = \{t_n = n\Delta t,\ n = 0,1,...,M+1 \text{ , where } \Delta t = \frac{T}{M+1}\},\\
&\Upsilon_M = \{t_n = n\Delta t,\ n = -M', -M'+1, ..., 0 \text{ , with } 0 < M' < M\}.
\end{aligned}$$

such that the width of delay mesh is $\tau = M'\Delta t$. Now, using a backward Euler scheme in time and finite differences in space, we define the following approximation of the derivatives:

$$\phi_{xx}\left(x_i,t_n\right)=\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{h^2},\ \phi_x\left(x_i,t_n\right)=\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2h},\ \phi_t\left(x_i,t_n\right)=\frac{\phi_i^n-\phi_i^{n-1}}{\Delta t},$$

$$\phi_t\left(x_i,t_{n-M'}\right)=\frac{\phi_i^{n-M'}-\phi_i^{n-M'-1}}{\Delta t},\ 1\leq i\leq N,\ 1\leq n\leq M.$$

where $\phi=\phi\left(x,t\right)$ *be a function with second order partial derivatives. As a result, our problem consists to find* $\left(\hat{z},\hat{u},\theta,q,\varphi\right)$ *satisfying the discrete formulation of the system(1.4)-(1.5) presented by the following numerical scheme*

$$\begin{cases}
\frac{\rho_z}{\Delta t}(\widehat{z}_i^n-\widehat{z}_i^{n-1})=\frac{\alpha_1}{h^2}(z_{i+1}^n-2z_i^n+z_{i-1}^n)+\frac{\alpha_2}{h^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n),\\
\\
\frac{\rho_u}{\Delta t}(\widehat{u}_i^n-\widehat{u}_i^{n-1})=\frac{\alpha_3}{h^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n)+\frac{\alpha_2}{h^2}(z_{i+1}^n-2z_i^n+z_{i-1}^n)\\
+\frac{\beta}{2h}\left(\theta_{i+1}^n-\theta_{i-1}^n\right)-\frac{\mu}{\Delta t}\left(u_i^{n-M'}-u_i^{n-M'-1}\right),\\
\\
\frac{c}{\Delta t}(\theta_i^n-\theta_i^{n-1})=\frac{-1}{2h}\left(q_{i+1}^n-q_{i-1}^n\right)+\frac{\beta}{2h}(\widehat{u}_{i+1}^n-\widehat{u}_{i-1}^n),\\
\\
\frac{\tau_0}{\Delta t}(q_i^n-q_i^{n-1})=-q_i^n-\frac{k}{2h}\left(\theta_{i+1}^n-\theta_{i-1}^n\right),\\
\\
\frac{\tau}{\Delta t}\left(\varphi_{i,j}^n-\varphi_{i,j}^{n-1}\right)=-\frac{1}{2h}\left(\varphi_{i,j+1}^n-\varphi_{i,j-1}^n\right),
\end{cases}\tag{4.1.1}$$

where $z_i^n=z\left(x_i,t_n\right),\ \hat{z}_i^n=z_t\left(x_i,t_n\right),\ u_i^n=u\left(x_i,t_n\right),\ \hat{u}_i^n=u_t\left(x_i,t_n\right),\ \theta_i^n=\theta\left(x_i,t_n\right),\ q_i^n=q\left(x_i,t_n\right),$ $u_i^{n-M'}=u\left(x_i,t_{n-M'}\right)$, $\varphi_{i,j}^n=\varphi\left(x_i,\rho_j,t_n\right)$ *for all* $i=1,2,...,N$, $j=1,2,...,m$ *and* $n=1,2,...,M$, *with* $\varphi\left(x_i,\rho_{m+1},t_n\right)=u_i^{n-M'}$ *according to* (1.3). *To simplicity our numerical calculations in our scheme, we consider the discrete boundary conditions given by*

$$z_0^n=u_0^n=\theta_{N+1}^n=q_0^n=0,\ z_{N+1}^n=z_N^n,\ u_{N+1}^n=u_N^n,\ \theta_1^n=\theta_0^n,\tag{4.1.2}$$

and initial conditions

$$\begin{cases} z_i^0 = z_0\left(x_i\right),\ u_i^0 = u_0\left(x_i\right),\ \theta_i^0 = \theta_0\left(x_i\right),\ q_i^0 = q_0\left(x_i\right) \\ \widehat{z}_i^0 = z_1\left(x_i\right) \text{ and } \widehat{u}_i^0 = u_1\left(x_i\right) \\ \varphi_{i,0}^n = \widehat{u}_i^n,\ u_i^{n-M'} = f_0\left(x_i, t_{n-M'}\right),\ \varphi_{i,j}^0 = f_0\left(x_i, \rho_j.t_{-M'}\right),\ t_{n-M'} \in \Upsilon_M, \\ \varphi_{N+1,N+1}^n = \hat{u}_{N+1}^{n-M'} \end{cases} \tag{4.1.3}$$

where

$$z_i^n = z_i^{n-1} + \Delta t \widehat{z}_i^n \text{ and } u_i^n = u_i^{n-1} + \Delta t \widehat{u}_i^n$$

for all $i = 1, 2, ..., N$ and $n = 1, 2, ..., M$.
Note that to find $\{\widehat{z}^n, \widehat{u}^n, \theta^n, q^n, \varphi^n\}$, we need to solve five coupled systems of algebraic equations. So, to solve the problem (4.1.1)-(4.1.3) at each time step we propose to consider the following fixed-point algorithm:

$$\begin{cases} \widehat{z}_i^{n,l} = \dfrac{\alpha_1 \Delta t}{\rho_z h^2}(z_{i+1}^{n,l-1} - 2z_i^{n,l-1} + z_{i-1}^{n,l-1}) + \dfrac{\alpha_2 \Delta t}{\rho_z h^2}(u_{i+1}^{n,l-1} - 2u_i^{n,l-1} + u_{i-1}^{n,l-1}) + \widehat{z}_i^{n-1}, \\ \\ \widehat{u}_i^{n,l} = \dfrac{\alpha_3 \Delta t}{\rho_u h^2}(u_{i+1}^{n,l-1} - 2u_i^{n,l-1} + u_{i-1}^{n,l-1}) + \dfrac{\alpha_2 \Delta t}{\rho_u h^2}(z_{i+1}^{n,l-1} - 2z_i^{n,l-1} + z_{i-1}^{n,l-1}) \\ \\ + \dfrac{\beta \Delta t}{2\rho_u h}\left(\theta_{i+1}^{n,l-1} - \theta_{i-1}^{n,l-1}\right) + \widehat{u}_i^{n-1} - \dfrac{\mu}{\rho_u}\left(u_i^{n-M'} - u_i^{n-M'-1}\right), \\ \\ \theta_i^{n,l} = \theta_i^{n-1} - \dfrac{\Delta t}{2ch}\left(q_{i+1}^{n,l-1} - q_{i-1}^{n,l-1}\right) + \dfrac{\beta \Delta t}{2ch}(\widehat{u}_{i+1}^{n,l} - \widehat{u}_{i-1}^{n,l}), \\ \\ (1 + \dfrac{\Delta t}{\tau_0})q_i^{n,l} = -\dfrac{k \Delta t}{2h\tau_0}\left(\theta_{i+1}^{n,l} - \theta_{i-1}^{n,l}\right) + q_i^{n-1}, \\ \\ \varphi_{i,j}^{n,l} = \varphi_{i,j}^{n-1,l-1} - \dfrac{\Delta t}{2h\tau}\left(\varphi_{i,j+1}^{n,l-1} - \varphi_{i,j-1}^{n,l-1}\right) \end{cases} \tag{4.1.4}$$

with

$$\begin{cases} z_i^{n,0} = z_i^{n-1},\ u_i^{n,0} = u_i^{n-1},\ \theta_i^{n,0} = \theta_i^{n-1},\ q_i^{n,0} = q_i^{n-1},\ z_i^{n,l} = z_i^{n-1,l} + \Delta t \widehat{z}_i^{n,l}, \\ u_i^{n,l} = u_i^{n-1,l} + \Delta t \widehat{u}_i^{n,l},\ \varphi_{i,j}^{n,0} = \hat{u}_{i,j}^{n-M'}, \end{cases} \tag{4.1.5}$$

for all $i, j = 1, 2, ..., N$, $n = 1, 2, ..., M$ and $l = 1, 2, ...$
At each time step, we solve the scheme (4.1.4) by an iterative procedure that was stopped when the

difference between two successive iterations became smaller than a given tolerance ε.

4.2 Matrix form of the discretized problem

Let $\widehat{Z}^{n,l} = \left(\widehat{z}_i^{n,l}\right)_{1\le i\le N}$, $\widehat{U}^{n,l} = \left(\widehat{u}_i^{n,l}\right)_{1\le i\le N}$, $\widehat{\Theta}^{n,l} = \left(\widehat{\theta}_i^{n,l}\right)_{1\le i\le N}$, $\widehat{Q}^{n,l} = \left(\widehat{q}_i^{n,l}\right)_{1\le i\le N}$, $\vartheta^{n,l} = \left(\varphi_{i,j}^{n,l}\right)_{1\le i,j\le N}$.
Then, the system (4.1.4)-(4.1.5) can be rewritten as follows

$$\begin{cases} \widehat{Z}^{n,l} = \dfrac{\alpha_1\,\Delta t}{\rho_z h^2} A\, Z^{n,l-1} + \dfrac{\alpha_2\,\Delta t}{\rho_z h^2} A\, U^{n,l-1} + \widehat{Z}^{n-1} \\ \\ \widehat{U}^{n,l} = \dfrac{\alpha_3\,\Delta t}{\rho_u h^2} A\, U^{n,l-1} + \dfrac{\alpha_2\,\Delta t}{\rho_u h^2} A\, Z^{n,l} + \dfrac{\beta\,\Delta t}{2\rho_u h} B\,\Theta^{n,l-1} + \widehat{U}^{n-1} \\ -\dfrac{\mu}{\rho_u}\left(U^{n-M'} - U^{n-M'-1}\right) \\ \\ \Theta^{n,l} = \Theta^{n-1} - \dfrac{\Delta t}{2c\,h} C\, Q^{n,l-1} + \dfrac{\beta\,\Delta t}{2c\,h} D\,\widehat{U}^{n,l} \\ \\ \left(1 + \dfrac{\Delta t}{\tau_0}\right) Q^{n,l} = -\dfrac{k\,\Delta t}{2h\,\tau_0} B\,\Theta^{n,l} + Q^{n-1} \\ \\ \vartheta^{n,l} = \vartheta^{n-1,l-1} - \dfrac{\Delta t}{2h\,\tau} E\,\vartheta^{n,l-1} \end{cases} \tag{4.2.6}$$

with

$$\begin{cases} Z^{n,0} = Z^{n-1},\ U^{n,0} = U^{n-1},\ \Theta^{n,0} = \Theta^{n-1},\ Q^{n,0} = Q^{n-1},\ Z^{n,l} = Z^{n-1,l} + \Delta t \widehat{Z}^{n,l}, \\ U^{n,l} = U^{n-1,l} + \Delta t \widehat{U}_i^{n,l},\ \vartheta^{n,0} = \widehat{U}^{n-M'} \end{cases} \tag{4.2.7}$$

where A, B, C, D *are real matrices of dimensions* $(n \times n)$ *and* E *is a real matrix of dimension* $\left(n^2 \times n^2\right)$ *defined as follows*

$$D = diag(-1, 0, 1), C = diag(-1, 1, 0),$$

$$E = \begin{pmatrix} D & E_1 & \cdots & E_1 \\ E_1 & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & E_1 \\ E_1 & \cdots & E_1 & D \end{pmatrix},\ E_1 = 0_{\mathbb{R}^n \times \mathbb{R}^n}$$

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}, A = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

Now, we are ready to state and prove the following convergence result

4.3 Approximation of the discrete energy

To approximate the continuous energy (3.1), we use the trapezoidal quadrature formula to compute the integral $I = \int_0^L f(x)dx$

$$I_N = \sum_{i=0}^{N+1} a_i f(x_i) \simeq I,$$

where the weights $\{a_i\}_{i=1}^N$ *are given by* $a_0 = a_{N+1} = \dfrac{h}{2}$ *and for* $i = 1, 2, ..., N$, $a_i = h$. *Concerning the trapezoidal quadrature formula in tow dimensional case to compute the last part of* (3.1), *we use the following approximation*

$$\begin{aligned} \int_0^L \int_0^1 f(x,y)dydx \;\; \simeq \;\; & \frac{h^2}{4}\left(f(x_0,y_0) + f(x_0,y_{N+1}) + f(x_{N+1},y_0) + f(x_{N+1},y_{N+1})\right) \\ & + \frac{h^2}{2}\sum_{i=1}^{N}\left(f(x_i,y_0) + f(x_i,y_{N+1})\right) \\ & + \frac{h^2}{2}\sum_{j=1}^{N}\left(f(x_0,y_j) + f(x_{N+1},y_j)\right) \\ & + h^2\sum_{i=1}^{N}\sum_{j=1}^{N} f(x_i,y_j). \end{aligned}$$

Therefore, the discrete energy at the time step t_n *of system (4.1.1)-(4.1.3) is given by*

$$\begin{aligned} E^n = & \frac{1}{2}\sum_{i=0}^{N+1} a_i[\rho_z\,(\hat{z}_i^n)^2 + \rho_u\,(\hat{u}_i^n)^2 + c\,(\theta_i^n)^2 + \frac{\tau_0}{k}\,(q_i^n)^2 + \alpha_1\left((z_x)_i^n\right)^2 \qquad (4.3.8) \\ & + \alpha_3\left((u_x)_i^n\right)^2 + 2b\,(u_x)_i^n\,(z_x)_i^n] + \frac{|\mu|\,\tau}{2}[\frac{h^2}{4}\left(\left(\varphi_{0,0}^n\right)^2 + \left(\varphi_{0,N+1}^n\right)^2 + \left(\varphi_{N+1,0}^n\right)^2 + \left(\varphi_{N+1,N+1}^n\right)^2\right) \end{aligned}$$

$$+\frac{h^2}{2}\sum_{j=1}^{N}\left(\varphi_{0,j}^{n}\right)^2+\frac{h^2}{2}\sum_{j=1}^{N}\left(\varphi_{N+1,j}^{n}\right)^2+\frac{h^2}{2}\sum_{i=1}^{N}\left(\varphi_{i,0}^{n}\right)^2+\frac{h^2}{2}\sum_{i=1}^{N}\left(\varphi_{i,N+1}^{n}\right)^2+h^2\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\varphi_{i,j}^{n}\right)^2],$$

with

$$\begin{aligned}\hat{u}_i^n &= u_t\left(x_i,t_n\right),\ \left(u_x\right)_i^n=\frac{u_{i+1}^n-u_{i-1}^n}{2h},\\ \hat{z}_i^n &= z_t\left(x_i,t_n\right),\ \left(z_x\right)_i^n=\frac{z_{i+1}^n-z_{i-1}^n}{2h}.\end{aligned}$$

which is a discretization of the continuous energy (3.1).

4.4 Numerical illustration

In the table below, we consider five different choices of delay's weight and we note that the asymptotic behavior was reached in the last case even if in the first four cases the weights of delay are not considerable. This shows the great influence of the delay on the stability of this type of system throughout time. In the

Case	Weight of delay	Iterations time	Asymptotic behavior
1	$\mu = 0.9$	200	Unreached
2	$\mu = 0.03$	200	Unreached
3	$\mu = 0.005$	200	Unreached
4	$\mu = 0.0004$	200	Unreached
5	$\mu = 0.00008$	200	Reached

Table 4.1: Asymptotic behavior for different cases of delay's weight.

next, we describe the following numerical example where the asymptotic behavior was reached, that is the case when $\mu_0 = 0.00008$ *and for different choices of the system parameters with the condition* (1.3) *holds.*

Test 1 *For this numerical test, we choose the following different values for the coefficients of the system*

$$\begin{gathered}\rho_z = 3.5,\ \alpha_3 = 0.2,\ \mu = 0.8,\ \alpha_2 = 0.1,\ \beta = 0.05,\ \tau = 0.1\\ \rho_u = 1.5,\ \alpha_1 = 0.1,\ c = 4,\ \tau_0 = 0.06,\ c = 1.5,\ k = 10^{-5}\end{gathered}$$

We run our code for the following discretization parameters: $N = 100$, $m = 150$, $M = 200$, $L = 1$,

$T = 1$ and take $\varepsilon = 10^{-5}$. With the following initial conditions

$$z_0(x) = 10^{-2}\left(x^3 - \frac{3}{2}x^2\right),\ z_1(x) = \frac{1}{8}\left(2x^2 - 4x\right),\ u_0(x) = \frac{1}{8}\left(2x^2 - 2x\right),$$
$$u_1(x) = 0,\ \theta_0(x) = 0,\ q_0(x) = \frac{1}{5}x^3 e^{-\frac{3}{2}x^2},$$
$$f_0(x, t-\tau) = \frac{1}{2}.10^{-3}\cos(x).\cos\left(\frac{1}{10\pi}(t-\tau)\right)$$

Here are the evolution in time of the solutions $z, u, \theta, q, \varphi(x,1,t)$ and of the discrete energy.

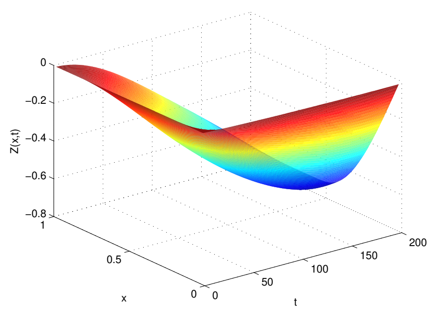

Figure 4.1: Evolution in time of the function z

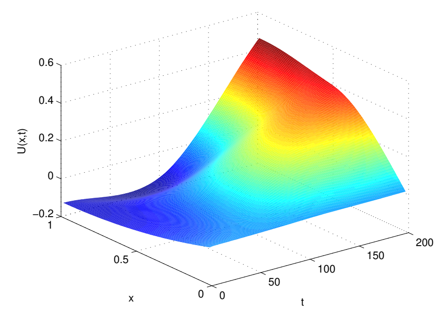

Figure 4.2: Evolution in time of the function u

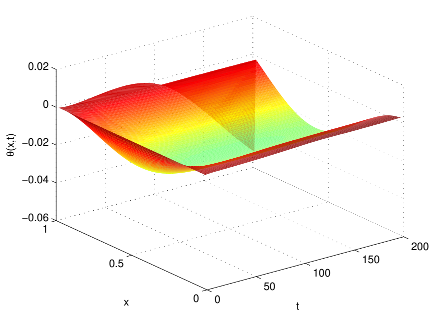

Figure 4.3: Evolution in time of the function θ

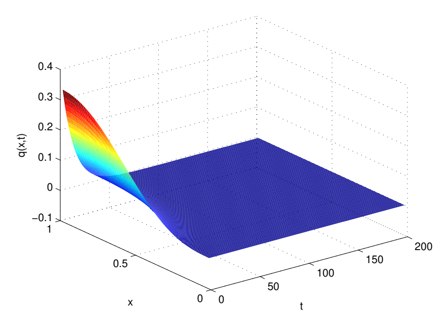

Figure 4.4: Evolution in time of the function q

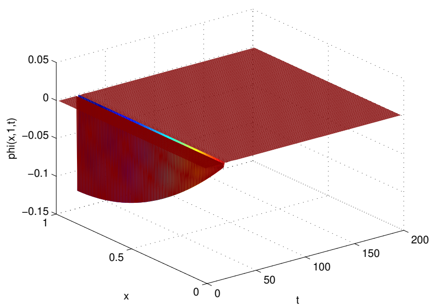

Figure 4.5: Evolution in time of the function $\varphi(x,1,t)$

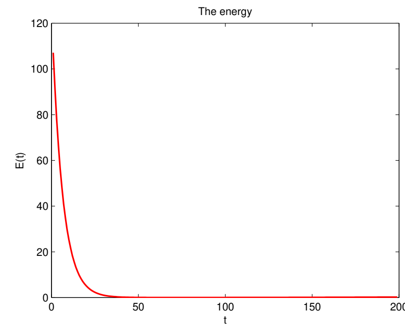

Figure 4.6: Evolution in time of the discrete energy

In above numerical test, the condition (1.3) *holds and graphics presented in the Figures 4.1–4.5 show the evolution in time of the approximations solutions* z, u, θ, q *and* $\varphi(x,1,t)$ *on the interval* $[0,T]$, *for different choices of the system parameters and of the initial data. Furthermore, the Figure 4.6 shows that the approximate energy (4.3.8) decays in an exponential manner which confirms the main theoretical result obtained.*

CONCLUSION

In this manuscript, we have conducted a theoretical and numerical investigation of a one-dimensional model describing the behavior of swelling porous thermoelastic soils, incorporating both thermal effects via Cattaneo's law and a constant internal time delay. We showed that the system is well-posed using semigroup theory, which allowed us to prove the existence and uniqueness of the solution. Furthermore, by constructing an appropriate Lyapunov functional, we demonstrated that the dissipation induced by the second sound effect is sufficient to ensure exponential stability of the system, despite the presence of time delay. Finally, we validate the theoritical results by carrying out some numerical experiments.

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