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Theoretical and numerical study of a porous thermoelastic system

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Dedication

In the name of **Allah**, the Merciful, the Compassionate

With pride and gratitude, I dedicate this work to:

My dear parents, **Mohamed** and **Samira**, who instilled in me the values of diligence and perseverance, and always supported me in every step.

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HAMDI MOHAMED EL AMINE

Abstract

In the present work, we consider a one dimensional porous-thermoelastic system with dissipation only due to microtemperatures effect where the heat conduction is given by Cattaneo's law. First, we give an existence and uniqueness of the solution using semiroup theory. Then, by constructing a suitable Lyapunov functional using the multipliers method and by introducing a stability number that obtained at the first in [20], we prove that the dissipation given only by the microtemperature is strong enough to give an exponential stability of the energy. Also, by constructing a suitable Lyapunov functional using the multipliers method, we establish a polynomial decay result of the solution in the case when the stability number not holds. Then, we give some numerical tests to illustrate the theoretical results by carrying out an Euler scheme for time discretization and the classical finite difference method for the spatial discretization.

Key words

Porous-elastic system, microtemperature effect, semigroups,
Lyapunov functional, energy method.

الملخص

في هذا العمل، تم التطرق الى دراسة نظام مسامي مرن حراري أحادي البعد علما أن تخامد الطاقة يكون فقط بتأثير الحرارة الدقيقة حيث تنتقل الحرارة حسب قانون كاتانيو. في البداية، أثبتنا وجود ووحدانية الحل باستخدام نظرية أنصاف الزمر. بعد ذلك، من خلال إنشاء دالة Lyapunov مناسبة باستخدام الطريقة الضربية و بإدخال رقم الاستقرار الذي تم الحصول عليه في [20]، أثبتنا أن التخامد الناتج عن درجة الحرارة الدقيقة وحده قوي بما يكفي من أجل استقرارالطاقة بشكل أسي. بالإظافة الى ذلك، من خلال إنشاء دالة Lyapunov أخرى مناسبة، تحصلنا على نتيجة استقرار جبري للطاقة في حالة عدم تحقق رقم الاستقرار المتحصل عليه في الحالة الأولى. بعد ذلك، تم إجراء بعض الاختبارات العددية لتأكيد النتائج النظرية من خلال تنفيذ مخطط أولر لتقديرالزمن وطريقة الفروق المنتهية الكلاسيكية للتقدير المكاني.

الكلمات المفتاحية

نظام مسامي مرن، تأثير الحرارة الدقيقة، أنصاف الزمر، دالة ليابونوف، طريقة الطاقة

Contents

Introduction

In 1972, Goodman and Cowin [10] have given an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin [16] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. Furthermore, this representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rocks and soils and to manufactured porous materials. In [11], Grot has developed a theory of thermodynamics of elastic materials with inner structure whose microelements, in addition to microdeformations of the string, possess microtemperatures which represent the variation of the temperature within a microvolume. Many investigations were realized concerning the study of the asymptotic behavior of different type of models and between them, we cited the following works ([1, 2, 3, 4, 5, 7, 8, 12]) and the references therein.

In the present work, we consider the following porous thermoelastic system that has been considered at the first in ([20])

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, & \text{in } (0,1)\times(0,+\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - d\omega_x + m\theta, & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1\omega_x, & \text{in } (0,1)\times(0,+\infty), \\ \alpha\omega_t = k_2\omega_{xx} - k_3\omega - k_1\theta_x - d\varphi_{tx}, & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{1}$$

subject to the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \omega(x,0) = \omega_0(x),\ \theta(x,0) = \theta_0(x), & x \in (0,1), \\ u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,+\infty), \\ \omega_x(0,t) = \omega_x(1,t) = \theta(0,t) = \theta(1,t) = 0, & t \in (0,+\infty), \end{cases} \tag{2}$$

where the initial data u_0, u_1, φ_0, φ_1, ω_0, θ_0 belongs to the suitable functional spaces. The functions u, φ, θ, ω represent, respectively, the displacement of the solid elastic material, the volume fraction, the temperature difference and the microtemperature vector. The parameters ρ and J which are assumed to be strictly positive constants, represent, respectively, the mass density and product of the mass density by the equilibrated inertia. The coefficients c, μ, δ, γ, ξ, m, d, k_1, k_2, k_3, α are positive constants represent the constitutive parameters defining the coupling among the different components of the materials such that

$$\mu\xi > b^2.$$

First, we give the existence and uniqueness of solutions using semigroup theory and based on the energy method, we prove that a unique dissipation given by microtemperatures is sufficiently strong enough to produce an exponential stability in the case when

$$\chi_1 = \frac{\mu}{\rho} - \frac{\delta}{J} - \frac{\gamma^2}{c\rho} = 0. \tag{3}$$

Later, we present the result that find in ([22]), where the authors showed a polynomial decay in the case where $\chi_1 \neq 0$ by constructing an appropriate Lyapunov functional. Also, we present some numerical tests using MATLAB software to illustrate the decay rate ($\chi_1 = 0$ and $\chi_1 \neq 0$).
In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. To avoid such case and also to be able to use Poincaré's inequality, we use the following transformation as in ([20]): By using $(1)_1$ and $(1)_4$, we arrive at

$$\begin{aligned} \int_0^1 u\,dx &= t\int_0^1 u_1 dx + \int_0^1 u_0 dx, \\ \int_0^1 \omega\,dx &= \left(\int_0^1 \omega_0 dx\right) e^{-\frac{t}{\alpha}k_3}. \end{aligned}$$

If we take

$$\bar{u}(x,t) = u(x,t) - \int_0^1 u dx,$$
$$\bar{\omega}(x,t) = \omega(x,t) - \int_0^1 \omega dx,$$

we get

$$\int_0^1 \bar{u} dx = \int_0^1 \bar{\omega} dx = 0.$$

This work is organized as follows: In chapter 1, we introduce some assumptions and transformations needed in the next chapters to prove the main results of this work. In chapter 2, we give the existence and uniqueness result of the solution. In chapter 3, we use the energy method to prove the decay rate presented in this work. Finally, some numerical simulations are obtained using MATLAB software.

Chapter 1

Preliminaries

In this chapter, we introduce all the mathematical background needed to achieve the results presented in this manuscript

1.1 Functional spaces

1.1.1 Banach space

Definition 1.1.1 *A Banach space is a complete normed vector space.*

1.1.2 Hilbert space

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. We will suffice to mention its definition.

Definition 1.1.2 *[6] A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

1.1.3 The $L^p(\Omega)$ spaces

Definition 1.1.3 *[6] Let $1 \leq p \leq \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_{\Omega} |f(x)|^p \, dx < \infty \right\}.$$

Notation 1.1.1 *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, denote by*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there exists a constant } C \text{ such that, } |f(x)| \leq C \text{ a.e in } \Omega \}.$$

Also, we denote by

$$\|f\|_\infty = \inf \{ C, \ |f(x)| \leq C \text{ a.e in } \Omega \}.$$

Notation 1.1.2 *Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark 1.1.1 *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) \, g(x) \, dx,$$

is a Hilbert space.

1.1.4 The Sobolev spaces $\mathbf{W}^{m,p}(\Omega)$:

Definition 1.1.4 *Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \left\{ f \in L^p(\Omega), \text{ such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that } |\alpha| = \sum_{j=1}^n \alpha_j \leq m, \text{ where } \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} \right\}.$$

Theorem 1.1.1 *([9]) $W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{1 \leq |\alpha| \leq m} \|\partial^\alpha f\|_{L^p}, \ 1 \leq p \leq \infty, \text{ for all } f \in L^p(\Omega).$$

Definition 1.1.5 *When* $p=2$, *we prefer to denote by* $W^{m,2}(\Omega)=H^m(\Omega)$ *and* $W_0^{m,p}(\Omega)=H_0^m(\Omega)$ *for* $p\in[1,\infty[$ *supplied with the norm*

$$\|f\|_{H^m(\Omega)}=\left(\sum_{|\alpha|\leq m}\left(\|\partial^\alpha f\|_{L^2}\right)^2\right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ *a real Hilbert space with their usual scalar product*

$$\langle u,v\rangle_{H^m(\Omega)}=\sum_{|\alpha|\leq m}\int_\Omega \partial^\alpha u\partial^\alpha v dx.$$

The next result provides a basic characterization of functions in $W_0^{1,p}(\Omega)$.

Theorem 1.1.2 *[6] Let* $u\in W^{1,p}(\Omega)$. *Then* $u\in W_0^{1,p}(\Omega)$ *if and only if* $u=0$ *on* $\partial\Omega$.

Remark 1.1.2

1. *Theorem 1.1.2 explains the central role played by the space* $W_0^{1,p}(\Omega)$. *Differential equations (or partial differential equations) are often coupled with boundary conditions, i.e., the value of* u *is prescribed on* $\partial\Omega$.

2. *We have the following characterization of* $H_0^m(\Omega)$

$$H_0^m(\Omega)=\{u\in H^m(\Omega),\ u=u'=...=u^{(m-1)}=0 \text{ on } \partial\Omega\}$$

It is essential to notice the distinction between

$$H_0^2(\Omega)=\{u\in H^2(\Omega),\ u=u'=0 \text{ on } \partial\Omega\},$$

and

$$H^2(\Omega)\cap H_0^1(\Omega)=\{u\in H^2(\Omega),\ u=0 \text{ on } \partial\Omega\}.$$

1.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Lemma 1.2.1 *([6], Hölder's Inequality) Let* $1 \leq p \leq \infty$*, assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *then,* $fg \in L^1(\Omega)$ *and*

$$\int_{\Omega} |fg|\, dx \leq \|f\|_p \|g\|_q . \tag{1.1}$$

The next result is an important prototype of a Sobolev inequality (also called a Sobolev embedding).

Lemma 1.2.2 *([6]) There exists a constant* C *(depending only on* $|I| \leq \infty$*) such that*

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}, \ \forall\ u \in W^{1,p}(I), \ \forall\ 1 \leq p \leq \infty. \tag{1.2}$$

Lemma 1.2.3 *[6] (Poincaré's inequality) Suppose* I *is a bounded interval. Then there exists a constant* C *(depending on* $|I| < \infty$*) such that*

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^P(I)}, \ \textit{for all}\ u \in W_0^{1,p}(I). \tag{1.3}$$

Lemma 1.2.4 *[14] (Poincaré type Scheeffer's inequality): Let* $h \in H_0^1(0, L)$*. Then it holds*

$$\int_0^L |h|^2\, dx \leq \ l \int_0^L |h_x|^2\, dx, \quad l = \frac{L^2}{\pi^2}. \tag{1.4}$$

1.3 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.3.1 *([6], Cauchy-Schwarz Inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\| . \tag{1.5}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Lemma 1.3.2 *[6](Young's Inequality) For all* $a, b \in \mathbb{R}^+$*, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{1.6}$$

where ϵ *is any positive constant.*

Lax-Milgram Lemma

The existence and uniqueness of a solution to the weak formulation of the problem can be proved using the Lax-Milgram Lemma. This states that the weak formulation admits a unique solution.

Lemma 1.3.3 *[6] (Lax-Milgram lemma). Let $a(\cdot,\cdot)$ be a bilinear form on a Hilbert space $\mathcal{H}$ equipped with norm $\|\cdot\|_{\mathcal{H}}$ and the following properties:*
i) $a(\cdot,\cdot)$ is continuous, that is

$$\exists \gamma_1 > 0 \text{ such that } |a(w,v)| \leq \gamma_1 \|w\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \ \forall w, v \in \mathcal{H},$$

ii) $a(\cdot,\cdot)$ coercive (or $\mathcal{H}$-elliptic), that is

$$\exists \alpha > 0 \text{ such that } |a(v,v)| \leq \alpha \|v\|^2_{\mathcal{H}}, \ \forall v \in \mathcal{H},$$

iii) L is a linear mapping on $\mathcal{H}$ (thus L is continuous), that is

$$\exists \gamma_2 > 0 \text{ such that } |L(w)| \leq \gamma_2 \|w\|_{\mathcal{H}}, \ \forall w \in \mathcal{H},$$

Then there exists a unique $u \in \mathcal{H}$ such that

$$a(w,u) = L(w), \ \forall w \in \mathcal{H}.$$

1.4 Definitions and notions

Definition 1.4.1 *([6]) An unbounded linear operator in Y is a pair $(A, D(A))$ where $D(A)$ is a vector subspace of Y that represents the domain of A and A is a linear map of $D(A)$ in Y.*

Definition 1.4.2 *([6]) An unbounded linear operator $(A; D(A))$ in Y, is closed if its graph*

$$G(A) = \{(x, Ax) \mid x \in D(A)\} \text{ is closed in } Y \times Y$$

Definition 1.4.3 *([17]) Let $(A, D(A))$ an unbounded linear operator in Y where $D(A)$ is dense in Y, it's said that $(A, D(A))$ is of dense domain in Y.*

C_0-Semigroup of bounded linear operators

Let Y be a Banach space on the field $\mathbb{C}$ or $\mathbb{R}$, and let $B(Y)$ be the Banach algebra of unbounded linear operators of Y in Y.

Definition 1.4.4 *[17] Let Y be a Banach space. A one parameter family $(S(t))_{t\geq 0}$ of bounded linear operators defined from Y into Y is a strongly continuous semigroup of bounded linear operators on Y if*

- *$S(0) = I$ (I identity operator on Y).*
- *$S(t+s) = S(t)S(s)$ for every $t, s \geq 0$.*
- *$S(t)x \to x$, as $t \to 0$, $\forall x \in Y$.*

A strongly continuous semigroup is called a C_0-semigroup.

Definition 1.4.5 *[17] The infinitesimal generator $\mathcal{A}$ of the semigroup $(S(t))_{t\geq 0}$ is defined by:*

$$D(\mathcal{A}) = \left\{ x \in Y : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

and

$$\mathcal{A}x = \lim_{t\to 0} \frac{S(t)x - x}{t}, \ x \in D(\mathcal{A}).$$

1.5 The m-dissipatives operators

Definition 1.5.1 *([17]) An unbounded linear operator $(A; D(A))$ in Y, is dissipative if*

$$\forall x \in D(A), \forall \lambda > 0, \ \|\lambda x - Ax\| \ \geq \lambda \|x\|.$$

Definition 1.5.2 *([17]) An unbounded linear operator $(A; D(A))$ in Y, is m-dissipative if A is dissipative and*

$$\forall f \in Y, \forall \lambda > 0, \quad \exists x \in D(A) \quad \text{as that } \lambda x - Ax = f.$$

Theorem 1.5.1 *([17]) If A is m-dissipative, then, for all $\lambda > 0$, the operator $(\lambda I - A)$ admits an inverse, $(\lambda I - A)^{-1} f$ belongs to $D(A)$ for all $f \in Y$, and $(\lambda I - A)^{-1}$is a bounded verifying linear operator*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}.$$

Theorem 1.5.2 *([17])Let $(A, D(A))$ an unbounded dissipative linear operator in Y. The operator A is m-dissipative if and only if*

$$\exists\ \lambda_0 \text{ such that } \forall\ f \in Y,\ \ \exists\ x \in D(A) \text{ verifies } \lambda_0 x - Ax = f.$$

Theorem 1.5.3 *([17]) Let $(A, D(A))$ an unbounded linear operator in Y. If it exists $\lambda_0 > 0$ for which the operator $\lambda_0 I - A$ is a bijection of $D(A)$ o, Y, and if $(\lambda_0 I - A)^{-1}$ is a bounded operator on Y then A is closed.*

Particularly, if A is m-dissipative then A is closed.

Theorem 1.5.4 *([17]) Let A a dissipative operator and $R(I - A) = Y$, if Y is reflexive, then*

$$\overline{D(A)} = Y.$$

Theorem 1.5.5 *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$, there exists a unique function*

$$u \in C([0, \infty[, D(\mathcal{A})) \cap C^1([0, \infty[, Y)$$

satisfying

$$\begin{cases} u' + \mathcal{A}(t)u = 0 & \text{on } [0, \infty[\\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0|,\ \forall t \geq 0 \text{ and } |\frac{du}{dt}(t)| = |Au(t)| \leq |Au_0|,\ \forall t > 0.$$

Remark 1.5.1

1. *The main interest of Theorem 1.5.5 lies in the fact that we reduce the study of an "evolution problem" to the study of the "stationary equation" $u' + Au = f$.*

2. *The space $D(A)$ is equipped with the graph norm $|u| + |Au|$ or with the equivalent Hilbert norm $\sqrt{|u|^2 + |Au|^2}$.*

3. *We refer the interested readers to [15, 21] and references therein for details discussion on existence and uniqueness of local or global solutions of nonlinear evolution equations.*

Theorem 1.5.6 *([17]) Let $(S(t)_{t\geq 0})$ a strongly continued semigroup on Y and $(A, D(A))$ are infinitesimal generator, the following properties are verified:*
i) For all $x \in X$, we have

$$\lim_{t \longrightarrow 0} \frac{1}{h} \int_t^{t+h} S(s)x ds = S(t)x.$$

ii) For all $x \in X$, and all $t > 0$,

$$\int_0^t S(s)x ds,$$

belongs to $D(A)$ and

$$A(\int_0^t S(s)x ds) = S(t)x - x.$$

iii) If $x \in D(A)$ then $S(t)x \in D(A)$ and

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax.$$

iv) If $x \in D(A)$ so

$$S(t)x - S(s)x = \int_s^t S(\tau)Ax d\tau = \int_s^t As(\tau)x d\tau.$$

1.5.1 The m-dissipative (m-monotone) operators in a Hilbert space

Definition 1.5.3 *[6] An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \ \rightarrow \mathcal{H}$ is said to be monotone if it satisfies*

$$(\mathcal{A}u, u) \geq 0, \ \forall u \in D(\mathcal{A}),$$

It is called maximal monotone if, in addition

$$R(\mathcal{I} + \mathcal{A}) = \mathcal{H}, \ \text{i.m.} \ \forall f \in \mathcal{H}, \ \exists u \in D(\mathcal{A}) \ \text{such that} \ u + Au = f.,$$

Proposition 1.5.1 *[6] Let $\mathcal{A}$ be a maximal monotone operator. Then $D(\mathcal{A})$ is dense in $\mathcal{H}$.*

Theorem 1.5.7 *([6]) An unbounded linear operator $(A, D(A))$ in $\mathcal{H}$, is dissipative if and only if*

$$\forall x \in D(A),\ (Ax, x) \leq 0.$$

In the case of a complex Hilbert space, the previous condition is replaced by

$$\forall x \in D(A),\ Re\,(Ax, x) \leq 0.$$

Theorem 1.5.8 *([6]) If A is m-dissipative then $D(A)$ dense in $\mathcal{H}$.*

1.6 Semi groups of unbounded linear operators

1.6.1 Hille–Yosida theorem

Theorem 1.6.1 *([17])(Hille-Yosida 1) An unbounded linear operator $(A, D(A))$ in $\mathcal{H}$ is the infinitesimal generator of a strongly continued semigroup of contraction on $\mathcal{H}$ if and only if the following conditions are satisfied*

i) A is closed.
ii) $D(A)$ dense in $\mathcal{H}$.
iii) For all $\lambda > 0$, $(\lambda I - A)$ is a bijective application of $D(A)$ on $\mathcal{H}$, $(\lambda I - A)^{-1}$ is a bounded operator on $\mathcal{H}$ verifies

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}.$$

Theorem 1.6.2 *([17]) (Hille-Yosida 2) An unbounded linear operator $(A, D(A))$ in $\mathcal{H}$ is the infinitesimal generator of a strongly continued semigroup of contraction on $\mathcal{H}$ if and only if A is m-dissipative and of dense domain in X.*

1.6.2 Lumer Phillips theorem

Theorem 1.6.3 *([17])Let A an unbounded linear operator of $D(A)$ in $\mathcal{H}$ of dense domain .*

a) *If A is dissipative and if there exists $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = \mathcal{H}$. Therefore A is an infinitesimal generator of a strongly continued C_0-semigroup of contraction in $\mathcal{H}$.*

b) *If A is an infinitesimal generator of a strongly continued contraction* C_0-semigroup*, then*

$$R(\lambda I - A) = \mathcal{H} \quad \forall \lambda > 0$$

and

$$A \text{ dissipative.}$$

1.7 Concept of stability

Definition 1.7.1 *([19])The semigroup $T(t) = e^{\mathcal{A}t}$ is said to be exponentially stable if there exists two constants $\alpha > 0$ and $M \geq 1$ as that*

$$\|T(t)\| \leq M e^{\alpha t} \quad \forall t \geq 0$$

1.7.1 Stability in the Lyapunov sense

Definition 1.7.2 (*Internal stability*) *An equilibrium point is stable if the state trajectories of the system converge to an initial state different from the equilibrium state.*

Definition 1.7.3 (*Balance state*) *x_e is a state of balance if $x(t_0) = x_e \Longleftrightarrow x(t) = x_e \quad t \geq t_0$ in the absence of control and disturbances.*

Definition 1.7.4 (*Asymptotic stability*) *The state of equilibrium point x_e is said to be stable if $\forall t \geq 0$*

$$\forall \epsilon > 0, \; \exists \alpha > 0 \mid \quad \|x(0) - x_e\| < \alpha \Longrightarrow \|x(t) - x_e\| < \epsilon$$

Otherwise, x_e is said to be unstable.

Definition 1.7.5 (*Lyapunov's stability*) *An equilibrium point is asymptotically stable if it is stable and if*

$$\exists \alpha > 0 \mid \quad \|x(0) - x_e\| < \alpha \Longrightarrow \lim_{t \to +\infty} x(t) = x_e$$

Chapter 2

Well-posedness

In this chapter, we prove the existence and uniqueness of solutions for (1)-(2) based on the semigroup theory and more precisely the Lumer-Phillips theorem. For this reason, we start to transform the system (1)-(2) as a Cauchy problem. Indeed, we introduce the vector function $U = (u, v, \varphi, \psi, \theta, \omega)^T$, where $v = u_t$,and $\psi = \varphi_t$. Then, the system (1)-(2) can be rewritten as follows

$$\begin{cases} U_t + \mathcal{A}U = 0,\ t > 0, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \omega_0)^T, \end{cases} \tag{2.1}$$

where

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\varphi_x + \frac{\gamma}{\rho}\theta_x \\ -\psi \\ -\frac{\delta}{J}\varphi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\varphi + \frac{d}{J}\omega_x - \frac{m}{J}\theta \\ \frac{\gamma}{c}v_x + \frac{m}{c}\psi + \frac{k_1}{c}\omega_x \\ -\frac{k_2}{\alpha}\omega_{xx} + \frac{k_3}{\alpha}\omega + \frac{k_1}{\alpha}\theta_x + \frac{d}{\alpha}\psi_x \end{pmatrix}, \tag{2.2}$$

with $\mathcal{A}$ is unbounded operator on $\mathcal{H}$ defined by

$$\mathcal{A} = \begin{pmatrix} 0 & -I & 0 & 0 & 0 & 0 \\ -\frac{\mu}{\rho}\partial_{xx}(.) & 0 & -\frac{b}{\rho}\partial_x(.) & 0 & \frac{\gamma}{\rho}\partial_x(.) & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ \frac{b}{J}\partial_x(.) & 0 & -\frac{\delta}{J}\partial_{xx}(.) + \frac{\xi}{J}I & 0 & -\frac{m}{J}I & \frac{d}{J}\partial_x(.) \\ 0 & \frac{\gamma}{c}\partial_x(.) & 0 & \frac{m}{c}I & 0 & \frac{k_1}{c}\partial_x(.) \\ 0 & 0 & 0 & \frac{d}{\alpha}\partial_x(.) & \frac{k_1}{\alpha}\partial_x(.) & -\frac{k_2}{\alpha}\partial_{xx}(.) + \frac{k_3}{\alpha}I \end{pmatrix}, \tag{2.3}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H}=H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times L_*^2(0,1) \times L^2(0,1),$$

where

$$\begin{aligned}
H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\
L_*^2(0,1) &= \left\{ \varphi \in L^2(0,1) : \int_0^1 \varphi(x)\, dx = 0 \right\}, \\
H_*^2(0,1) &= \left\{ \Psi \in H^2(0,1) : \Psi_x(0) = \Psi_x(1) = 0 \right\}.
\end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned}
D(\mathcal{A}) = \big\{ & U \in \mathcal{H} \mid u \in H^2(0,1) \cap H_0^1(0,1);\ v \in H_0^1(0,1); \\
& \varphi \in H_*^2(0,1) \cap H_*^1(0,1);\ \psi \in H_*^1(0,1);\ \theta \in H_*^1(\Omega); \\
& \omega \in H^2(0,1) \cap H_0^1(0,1) \big\}.
\end{aligned}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

2.1 Energy space

First, we define the energy of our system as follows

Lemma 2.1.1 *Let $(u, \varphi, \theta, \omega)$ be a solution of (1)-(2). Then, the energy functional $E(t)$, defined by*

$$\begin{aligned}
E(t) = \frac{1}{2} \int_0^1 & \left(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + \delta \varphi_x^2 + c\theta^2 + \xi \varphi^2 + \alpha \omega^2 \right. \\
& \left. + 2b\varphi u_x \right) dx, \qquad (2.4)
\end{aligned}$$

satisfies

$$E'(t) = -k_3 \int_0^1 \omega^2 dx - k_2 \int_0^1 \omega_x^2 dx \leq 0. \qquad (2.5)$$

Proof 2.1.1 *Multiplying $(1)_1$, $(1)_2$, $(1)_3$, $(1)_4$ by u_t, φ_t, θ, ω respectively,* *integrating by parts over* $(0,1)$, *we obtain*

$$\begin{cases} \rho \int_0^1 u_{tt}u_t dx = -\mu \int_0^1 u_{tx}u_x dx + b\int_0^1 \varphi_x u_t dx \\ -\gamma \int_0^1 \theta_x u_t dx, \ in\ (0,1)\times(0,+\infty), \\ \\ J\int_0^1 \varphi_{tt}\varphi_t dx = -\delta \int_0^1 \varphi_x \varphi_{tx} dx - b\int_0^1 u_x\varphi_t dx - \xi \int_0^1 \varphi_t \varphi dx \\ +d\int_0^1 \omega\varphi_{tx} dx + m\int_0^1 \theta\varphi_t dx, \ in\ (0,1)\times(0,+\infty), \\ \\ c\int_0^1 \theta\theta_t dx = \gamma \int_0^1 u_t\theta_x dx - m\int_0^1 \varphi_t \theta dx + k_1 \int_0^1 \omega\theta_x dx, \\ \ in\ (0,1)\times(0,+\infty), \\ \\ \alpha \int_0^1 \omega\omega_t dx = -k_2 \int_0^1 \omega_x^2 dx - k_3 \int_0^1 \omega^2 dx - k_1 \int_0^1 \theta_x \omega dx \\ -d\int_0^1 \varphi_{tx}\omega dx, \ in\ (0,1)\times(0,+\infty). \end{cases}$$

This last system is equivalent to

$$\begin{cases} \rho \frac{d}{2dt} \int_0^1 u_t^2 dx = -\mu \frac{d}{2dt} \int_0^1 u_x^2 dx - b\int_0^1 \varphi u_{tx} dx \\ -\gamma \int_0^1 \theta_x u_t dx, \ in\ (0,1)\times(0,+\infty), \\ \\ J\frac{d}{2dt} \int_0^1 \varphi_t^2 dx = -\delta \frac{d}{2dt} \int_0^1 \varphi_x^2 dx - b\int_0^1 u_x\varphi_t dx - \xi \frac{d}{2dt} \int_0^1 \varphi^2 dx \\ +d\int_0^1 \omega\varphi_{tx} dx + m\int_0^1 \theta\varphi_t dx, \ in\ (0,1)\times(0,+\infty), \\ \\ c\frac{d}{2dt} \int_0^1 \theta^2 dx = \gamma \int_0^1 u_t\theta_x dx - m\int_0^1 \varphi_t \theta dx + k_1 \int_0^1 \omega\theta_x dx, \\ \ in\ (0,1)\times(0,+\infty), \\ \\ \alpha \frac{d}{2dt} \int_0^1 \omega^2 dx = -k_2 \int_0^1 \omega_x^2 dx - k_3 \int_0^1 \omega^2 dx - k_1 \int_0^1 \theta_x \omega dx \\ -d\int_0^1 \varphi_{tx}\omega dx, \ in\ (0,1)\times(0,+\infty). \end{cases}$$

Summing them up, we obtain

$$\begin{aligned} &\frac{d}{2dt} \int_0^1 (\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + \delta\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha\omega^2 + 2b\varphi u_x) \\ &= -k_2 \int_0^1 \omega_x^2 dx - k_3 \int_0^1 \omega^2 dx, \end{aligned} \tag{2.6}$$

and this is give us (2.4) *and* (2.5).

Remark 2.1.1 *The energy $E(t)$ defined by (2.4) is non-negative. In fact,*

$$\mu u_x^2 + 2bu_x\varphi + \xi\varphi^2 = \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + \xi\left(\varphi + \frac{b}{\xi}u_x\right)^2 + \left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right],$$

since $\mu\xi > b^2$, we deduce that

$$\mu u_x^2 + 2bu_x\varphi + \xi\varphi^2 > \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right].$$

Consequently,

$$E(t) > \frac{1}{2}\int_0^1 \{\rho u_t^2 + J\varphi_t^2 + \mu_1 u_x^2 + \delta\varphi_x^2 + c\theta^2 + \xi_1\varphi^2 + \alpha w^2\ ,$$

where

$$\mu_1 = \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right) > 0,\ \xi_1 = \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right) > 0,$$

then $E(t)$ is non-negative.

.

For any $U = (u, v, \varphi, \psi, \theta, \omega)^T \in \mathcal{H}$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \bar{\omega})^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product

$$\left\langle U, \tilde{U}\right\rangle_{\mathcal{H}} = \rho\int_0^1 v\tilde{v}dx + \mu\int_0^1 u_x\tilde{u}_x dx + J\int_0^1 \psi\tilde{\psi}dx + b\int_0^1 (u_x\tilde{\varphi} + \tilde{u}_x\varphi)\, dx + \xi\int_0^1 \varphi\tilde{\varphi}dx + \delta\int_0^1 \varphi_x\tilde{\varphi}_x dx + \alpha\int_0^1 \omega\bar{\omega}dx + c\int_0^1 \theta\tilde{\theta}dx. \tag{2.7}$$

To define the space $\mathcal{H}$, we would follow the following steps:

$$\rho\int_0^1 u_t^2 dx < \infty,$$

implies that $u_t \in L^2(0,1)$. Also we

$$\mu \int_0^1 u_x^2 dx < \infty,$$

and by Poincaré inequality, we deduce that

$$\int_0^1 u^2 dx < \infty.$$

Then, by taking into account the boundary conditions, we conclude that $u \in H_0^1(0,1)$.

$$\begin{aligned} \delta \int_0^1 \varphi_x^2 dx &< \infty, \\ \xi \int_0^1 \varphi^2 dx &< \infty. \end{aligned}$$

Then, $\varphi \in H^1(0,1)$ and thanks to the Neumann boundary conditions, we have $\varphi \in H_*^1(0,1)$. On the other hand

$$J \int_0^1 \varphi_t^2 dx < \infty.$$

So $\varphi_t \in L^2(0,1)$ and because $\int_0^1 \varphi dx = 0$. Therefore, $\varphi_t \in L_*^2(0,1)$. Then,

$$c \int_0^1 \theta^2 dx < \infty,$$

and because $\int_0^1 \theta dx = 0$. Therefore, $\theta \in L_*^2(0,1)$. Finally

$$\alpha \int_0^1 \omega^2 dx < \infty,$$

this last, confirm that $\omega \in L^2(0,1)$.

2.2 Domain of the operator $\mathcal{A}$

Since $D(\mathcal{A})$ has a value in $\mathcal{H}$, then, we can confirm that

$$v \in H_0^1(0,1).$$

- The second components of $\mathcal{A}$ in $L^2(0,1)$. Then,

$$u_{xx} \in L^2(0,1).$$

So

$$u \in H^2(0,1) \cap H_0^1(0,1).$$

- The third components of $\mathcal{A}$ in $H_*^1(0,1)$. Then

$$\psi \in H_*^1(0,1).$$

- The fourth components of $\mathcal{A}$ is in $L_*^2(0,1)$. Then,

$$\varphi_{xx} \in L_*^2(0,1),$$

and because $L_*^2(0,1) \subset L^2(0,1)$, then

$$\varphi_{xx} \in L^2(0,1),$$

and since $\varphi_x(0) = \varphi_x(1) = 0$, we deduce that

$$\varphi \in H_*^2(0,1).$$

- We have $\theta \in L^2(0,1)$ because the second components of $\mathcal{A}$ is in $L^2(0,1)$. Therefore

$$\theta \in H_*^1(0,1).$$

- The last components of $\mathcal{A}$ is in $L^2(0,1)$ and by Poincaré's inequality, we get

$$\omega_x, \omega_{xx} \in L^2(0,1).$$

Then

$$\omega \in H^2(0,1) \cap H_0^1(0,1).$$

Now, we can give and prove the following existence and uniqueness result.

2.3 Existence and uniqueness result

Theorem 2.3.1 *Let* $U_0 \in \mathcal{H}$*, then there exists a unique solution* $U \in C(\mathbb{R}_+, \mathcal{H})$ *of problem* (2.1)*. Moreover, if* $U_0 \in D(\mathcal{A})$*, then*

$$U \in C(\mathbb{R}_+,\, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof 2.3.1 *The result follows from Lumer-Phillips theorem provided, we prove that* $\mathcal{A}$ *is a maximal monotone operator. In what follows, we prove that* $\mathcal{A}$ *is a maximal monotone operator. For any* $U \in D(\mathcal{A})$*, by using* (2.7) *and the boundary conditions, we obtain*

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} -v \\ -\frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\varphi_x + \frac{\gamma}{\rho}\theta_x \\ -\psi \\ -\frac{\delta}{J}\varphi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\varphi + \frac{d}{J}\omega_x - \frac{m}{J}\theta \\ \frac{\gamma}{c}v_x + \frac{m}{c}\psi + \frac{k_1}{c}\omega_x \\ -\frac{k_2}{\alpha}\omega_{xx} + \frac{k_3}{\alpha}\omega + \frac{k_1}{\alpha}\theta_x + \frac{d}{\alpha}\psi_x \end{pmatrix}, \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ \theta \\ \omega \end{pmatrix} \right\rangle_{\mathcal{H}}, \quad (2.8)$$

which leads to with integration by parts

$$\begin{aligned}
&\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \\
&= -\mu \int_0^1 u_{xx} v dx - b\int_0^1 \varphi_x v dx + \gamma \int_0^1 \theta_x v dx - \mu \int_0^1 u_x v_x dx \\
&- \delta \int_0^1 \psi\varphi_{xx} dx + b\int_0^1 \psi u_x dx + \xi \int_0^1 \psi\varphi dx + d\int_0^1 \psi\omega_x dx - m\int_0^1 \psi\theta dx \\
&- b\int_0^1 \varphi v_x dx - b\int_0^1 \psi u_x dx - \xi\int_0^1 \psi\varphi dx - \delta\int_0^1 \psi_x\varphi_x dx \\
&- k_2\int_0^1 \omega_{xx}\omega dx + k_3\int_0^1 \omega^2 dx + k_1\int_0^1 \theta_x\omega dx + d\int_0^1 \psi_x\omega dx
\end{aligned}$$

$$+\gamma\int_0^1 v_x\theta dx+m\int_0^1\psi\theta dx+k_1\int_0^1\omega_x\theta dx$$

$$=\mu\int_0^1 u_xv_xdx+b\int_0^1\varphi v_xdx-\gamma\int_0^1\theta v_xdx-\mu\int_0^1 u_xv_xdx$$

$$+\delta\int_0^1\psi_x\varphi_xdx+b\int_0^1\psi u_xdx+\xi\int_0^1\psi\varphi dx-d\int_0^1\psi_x\omega dx-m\int_0^1\psi\theta dx$$

$$-b\int_0^1\varphi v_xdx-b\int_0^1\psi u_xdx-\xi\int_0^1\psi\varphi dx-\delta\int_0^1\psi_x\varphi_xdx$$

$$+k_2\int_0^1\omega_x^2dx+k_3\int_0^1\omega^2dx-k_1\int_0^1\theta\omega_xdx+d\int_0^1\psi_x\omega dx$$

$$+\gamma\int_0^1 v_x\theta dx+m\int_0^1\psi\theta dx+k_1\int_0^1\omega_x\theta dx.$$

Finally, we obtain

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=k_2\int_0^1\omega_x^2dx+k_3\int_0^1\omega^2dx\geq 0$$

Therefore, the operator $\mathcal{A}$ *is monotone. Next, we prove that the operator* $(I+\mathcal{A})$ *is surjective. For any* $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in\mathcal{H}$, *we prove that there exists a unique* $U\in D(\mathcal{A})$ *such that*

$$(I+\mathcal{A})\,U=F. \tag{2.9}$$

The problem (2.9), *leads to solve the following system*

$$\begin{cases} u-v=f_1\in H_0^1(0,1)\,,\\ \rho v-\mu u_{xx}-b\varphi_x+\gamma\theta_x=\rho f_2\in L^2(0,1)\,,\\ \varphi-\psi=f_3\in H_*^1(0,1)\,,\\ J\psi-\delta\varphi_{xx}+bu_x+\xi\varphi+d\omega_x-m\theta=Jf_4\in L_*^2(0,1)\,,\\ c\theta+\gamma v_x+m\psi+k_1\omega_x=cf_5\in L_*^2(0,1)\,,\\ \alpha_1\omega-k_2\omega_{xx}+k_1\theta_x+d\psi_x=\alpha f_6\in L^2(0,1)\,,\ \ \alpha_1=\alpha+f_3. \end{cases} \tag{2.10}$$

Inserting $v = u - f_1$, $\psi = \varphi - f_3$ *in* $(2.10)_2$, $(2.10)_4$, $(2.10)_5$ *and* $(2.10)_6$, *we get*

$$\begin{cases} \rho u - \mu u_{xx} - b\varphi_x + \gamma\theta_x = h_1 \in L^2(0,1), \\ \mu_3\varphi - \delta\varphi_{xx} + bu_x + d\omega_x - m\theta = h_2 \in L^2_*(0,1), \\ c\theta + \gamma u_x + m\varphi + k_1\omega_x = h_3 \in L^2_*(0,1), \\ \alpha_1\omega - k_2\omega_{xx} + k_1\theta_x + d\varphi_x = h_4 \in L^2(0,1), \end{cases} \tag{2.11}$$

where

$$\begin{aligned} h_1 &= \rho(f_2 + f_1), \\ h_2 &= J(f_3 + f_4), \\ h_3 &= cf_5 + \gamma f_{1x} + mf_3, \\ h_4 &= \alpha f_6 + df_{3x}, \\ \mu_3 &= J + \xi. \end{aligned}$$

To solve (2.11), *we consider*

$$B((u,\varphi,\theta,\omega),(u_1,\varphi_1,\theta_1,\omega_1)) = \mathcal{G}(u_1,\varphi_1,\theta_1,\omega_1), \tag{2.12}$$

where $B : \left[H^1_0(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times L^2(0,1)\right]^2 \longrightarrow \mathbb{R}$ *is the bilinear form defined by*

$$\begin{aligned} &B((u,\varphi,\theta,\omega),(u_1,\varphi_1,\theta_1,\omega_1)) \\ &= \rho\int_0^1 uu_1dx + \mu\int_0^1 u_xu_{1x}dx + \mu_3\int_0^1 \varphi\varphi_1dx + \delta\int_0^1 \varphi_x\varphi_{1x}dx \\ &+ c\int_0^1 \theta\theta_1dx + \alpha_1\int_0^1 \omega\omega_1dx + k_2\int_0^1 \omega_x\omega_{1x}dx + \gamma\int_0^1 (u_x\theta_1 + u_1\theta_x)\,dx \\ &+ b\int_0^1 (u_x\varphi_1 + \varphi u_{1x})\,dx + d\int_0^1 (\omega_x\varphi_1 + \varphi_x\omega_1)\,dx \\ &+ k_1\int_0^1 (\omega_x\theta_1 + \omega_1\theta_x)\,dx + m\int_0^1 (\varphi\theta_1 - \varphi_1\theta)\,dx, \end{aligned}$$

and $\mathcal{G} : \left[H^1_0(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times L^2(0,1)\right] \longrightarrow \mathbb{R}$ *is the linear form given by*

$$\mathcal{G}(u_1,\varphi_1,\theta_1,\omega_1) = \int_0^1 h_1u_1dx + \int_0^1 h_2\varphi_1dx + \int_0^1 h_3\theta_1dx + \int_0^1 h_4\omega_1dx.$$

Let $V = H_0^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times L^2(0,1)$ *equipped with the norm*

$$\|(u,\varphi,\theta,\omega)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\theta\|_2^2 + \|\omega\|_2^2 + \|\omega_x\|_2^2,$$

then, we can easily prove that

$$\begin{aligned}|B((u,\varphi,\theta,\omega),(u,\varphi,\theta,\omega))| &= \rho\int_0^1 u^2dx + \mu\int_0^1 u_x^2dx + \mu_3\int_0^1 \varphi^2dx + \delta\int_0^1 \varphi_x^2dx \\ &\quad + c\int_0^1 \theta^2dx + \alpha_1\int_0^1 \omega^2dx + k_2\int_0^1 \omega_x^2dx \\ &\geq M_0\|(u,\varphi,\theta,\omega)\|_V^2,\end{aligned}$$

where $M_0 = \min\{\rho,\mu,\mu_3,\delta,c,\alpha_1,k_2\}$. *Thus,* B *is coercive. Moreover, we can easily see that* B *and* $\mathcal{G}$ *are bounded. Consequently, by Lax-Milgram Lemma, system* (2.12) *has a unique solution* $(u,\varphi,\theta,\omega) \in V$ *satisfying* (2.11).
Substituting u *and* φ *in* $(2.10)_1$ *and* $(2.10)_2$*, respectively, we obtain*

$$v \in H_0^1(0,1), \ \psi \in H_*^1(0,1).$$

Moreover, if we take $(\varphi_1,\theta_1,\omega_1) \equiv (0,0,0) \in H_*^1(0,1) \times L_*^2(0,1) \times L^2(0,1)$, *then* (2.12) *reduces to*

$$\begin{aligned}&\rho\int_0^1 uu_1dx + \mu\int_0^1 u_xu_{1x}dx + \gamma\int_0^1 u_1\theta_xdx + b\int_0^1 \varphi u_{1x}dx \\ &= \int_0^1 h_1u_1dx, \ \forall u_1 \in H_0^1(0,1).\end{aligned} \tag{2.13}$$

That is

$$\mu\int_0^1 u_xu_{1x}dx = \int_0^1 (h_1 - \rho u - \gamma\theta_x + b\varphi_x)\,u_1dx, \ \forall u_1 \in H_0^1(0,1),$$

which implies

$$\mu u_{xx} = -h_1 + \rho u - b\varphi_x + \gamma\theta_x \in L^2(0,1).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$u \in H^2(0,1) \cap H_0^1(0,1).$$

If we choose $(u_1, \theta_1, \omega_1) \equiv (0,0,0) \in H_0^1(0,1) \times L_*^2(0,1) \times L^2(0,1)$ *in* (2.12), *we have*

$$\mu_3 \int_0^1 \varphi\varphi_1 dx + \delta \int_0^1 \varphi_x \varphi_{1x} dx + b \int_0^1 u_x \varphi_1 dx + d \int_0^1 \omega_x \varphi_1 dx$$
$$- m \int_0^1 \theta\varphi_1 dx = \int_0^1 h_2 \varphi_1 dx, \ \forall \varphi_1 \in H_*^1(0,1).$$

Furthermore, (2.13) *is also true for any* $\Psi \in C^1([0,1]) \subset H_*^1(0,1)$. *Hence, we have*

$$\delta \int_0^1 \varphi_x \Psi_x dx = \int_0^1 (h_2 - bu_x - d\omega_x - \mu_3\varphi + m\theta) \Psi dx, \ \forall \Psi \in C^1([0,1]). \tag{2.14}$$

Thus, integrating by parts the left side of (2.14) *and taking into account* $(2.11)_2$, *we get*

$$\varphi_x(1)\Psi(1) - \varphi_x(0)\Psi(0) = 0, \ \forall \Psi \in C^1([0,1]).$$

Therefore,

$$\varphi_x(1) = \varphi_x(0) = 0.$$

Consequently, we obtain

$$\varphi \in H_*^2(0,1) \cap H_*^1(0,1).$$

Similarly, we obtain

$$\omega \in H_0^2(0,1) \cap H_0^1(0,1), \ \theta \in H_*^1(\Omega).$$

According to the classical elliptic regularity, it follows from (2.12) *that there exists a unique* $U \in D(\mathcal{A})$ *satisfying* (2.9). *Therefore,* $\mathcal{A}$ *is a maximal operator. Hence the result of Theorem (1) follows from Lumer-Phillips theorem. (see [17]).*

Chapter 3

Stability

3.1 Exponential stability

In this section, we use the energy method to establish the exponential stability of the system (1)-(2). To achieve our goal we state and prove the following lemmas.

Lemma 3.1.1 *Let* $(u, \varphi, \theta, \omega)$ *be a solution of* (1)-(2). *Then, the functional*

$$I_1(t) = -\rho \int_0^1 u_t u dx, \ t \geq 0,$$

satisfies, $\forall t \geq 0$

$$I_1'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx + \frac{b^2}{\mu} \int_0^1 \varphi^2 dx + \frac{\gamma^2}{\mu} \int_0^1 \theta^2 dx. \quad (3.1)$$

Proof 3.1.1 *Direct computation, we get*

$$I_1'(t) = -\rho \int_0^1 u_{tt} u dx - \rho \int_0^1 u_t^2 dx.$$

Using equation $(1)_1$*, we have*

$$\begin{aligned} I_1'(t) &= -\int_0^1 (\mu u_{xx} + b\varphi_x - \gamma\theta_x)\, u dx - \rho \int_0^1 u_t^2 dx \\ &= -\mu \int_0^1 u_{xx} u dx - b \int_0^1 \varphi_x u dx - \gamma \int_0^1 \theta_x u dx - \rho \int_0^1 u_t^2 dx. \end{aligned}$$

Integrating by parts

$$I_1'(t) = \mu \int_0^1 u_x^2 dx + b \int_0^1 \varphi u_x dx + \gamma \int_0^1 \theta u_x dx - \rho \int_0^1 u_t^2 dx. \tag{3.2}$$

Using Young's inequality

$$\begin{aligned} b \int_0^1 \varphi u_x dx &\leq \frac{b^2}{\mu} \int_0^1 \varphi^2 dx + \frac{\mu}{4} \int_0^1 u_x^2 dx, & (3.3) \\ \gamma \int_0^1 \theta u_x dx &\leq \frac{\gamma^2}{\mu} \int_0^1 \theta^2 dx + \frac{\mu}{4} \int_0^1 u_x^2 dx. & (3.4) \end{aligned}$$

Inserting (3.3) *and* (3.4) *in* (3.2), *we get (3.1).*

Lemma 3.1.2 *Let* $(u, \varphi, \theta, \omega)$ *be a solution of* (1)-(2). *Then, the functional*

$$I_2(t) = J \int_0^1 \varphi_t \varphi dx - \frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi(y)\, dy \right) dx,\ t \geq 0,$$

satisfies, for any $\varepsilon_1 > 0$,

$$\begin{aligned} I_2'(t) \leq & -\frac{\delta}{2} \int_0^1 \varphi_x^2 dx - 2\xi_1 \int_0^1 \varphi^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + \frac{1}{\delta} \left(m - \frac{b\gamma}{\mu} \right)^2 \int_0^1 \theta^2 dx \\ & + \frac{d^2}{\delta} \int_0^1 \omega^2 dx + \left(\frac{b^2 \rho^2}{4\varepsilon_1 \mu^2} + J \right) \int_0^1 \varphi_t^2 dx, \end{aligned} \tag{3.5}$$

where $\xi_1 = \frac{1}{2} \left(\xi - \frac{b^2}{\mu} \right)$.

Proof 3.1.2 *By differentiating* $I_2(t)$, *we obtain*

$$\begin{aligned} I_2'(t) = & J \int_0^1 \varphi_{tt} \varphi dx + J \int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi_t(y)\, dy \right) dx \\ & - \frac{b\rho}{\mu} \int_0^1 u_{tt} \left(\int_0^x \varphi(y)\, dy \right) dx. \end{aligned}$$

Using $(1)_1$ *and* $(1)_2$, *we get*

$$\begin{aligned} I_2'(t) = & \int_0^1 (\delta \varphi_{xx} - b u_x - \xi \varphi - d\omega_x + m\theta)\, \varphi dx + J \int_0^1 \varphi_t^2 dx \\ & - \frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi_t(y)\, dy \right) dx \\ & - \frac{b}{\mu} \int_0^1 (\mu u_{xx} + b\varphi_x - \gamma \theta_x) \left(\int_0^x \varphi(y)\, dy \right) dx. \end{aligned}$$

So

$$I_2'(t) = \delta \int_0^1 \varphi_{xx}\varphi dx - b\int_0^1 u_x\varphi dx - \xi \int_0^1 \varphi^2 dx - d\int_0^1 \omega_x\varphi dx$$
$$+ m\int_0^1 \theta\varphi dx + J\int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx$$
$$- b\int_0^1 u_{xx}\left(\int_0^x \varphi(y)\,dy\right)dx + \frac{b^2}{\mu}\int_0^1 \varphi_x\left(\int_0^x \varphi(y)\,dy\right)dx$$
$$+ \frac{b\gamma}{\mu}\int_0^1 \theta_x\left(\int_0^x \varphi(y)\,dy\right)dx.$$

Now, by using integration by parts together with the boundary conditions, we get

$$I_2'(t) = -\delta \int_0^1 \varphi_x^2 dx - b\int_0^1 u_x\varphi dx - \xi \int_0^1 \varphi^2 dx + d\int_0^1 \omega\varphi_x dx$$
$$+ m\int_0^1 \theta\varphi dx + J\int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx$$
$$+ b\int_0^1 u_x\varphi dx + \frac{b^2}{\mu}\int_0^1 \varphi^2 dx$$
$$- \frac{b\gamma}{\mu}\int_0^1 \theta\varphi dx.$$

After simplification, we arrive at

$$I_2'(t) = -\delta \int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx + J\int_0^1 \varphi_t^2 dx$$
$$+ d\int_0^1 \omega\varphi_x dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx$$
$$+ \left(m - \frac{b\gamma}{\mu}\right)\int_0^1 \theta\varphi dx. \tag{3.6}$$

Using Young's and Cauchy Schwarz inequalities, we get

$$-\frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)\,dy\right)dx \le \varepsilon_1\int_0^1 u_t^2 dx + \frac{b^2\rho^2}{4\varepsilon_1\mu^2}\int_0^1 \varphi_t^2 dx. \tag{3.7}$$

Young's inequality leads to

$$d\int_0^1 \omega\varphi_x dx \le \frac{\delta}{4}\int_0^1 \varphi_x^2 dx + \frac{d^2}{\delta}\int_0^1 \omega^2 dx. \tag{3.8}$$

Young's and Poincaré inequalities give us

$$\left(m-\frac{b\gamma}{\mu}\right)\int_0^1 \theta\varphi dx \le \frac{\delta}{4}\int_0^1 \varphi_x^2 dx + \frac{1}{\delta}\left(m-\frac{b\gamma}{\mu}\right)^2\int_0^1 \theta^2 dx. \tag{3.9}$$

Inserting (3.7)-(3.9) *in* (3.6), *we obtain* (3.5).

Lemma 3.1.3 *Let* $(u,\varphi,\theta,\omega)$ *be a solution of* (1)-(2). *Then, the functional*

$$I_3(t)=\frac{J\mu}{b}\int_0^1 \varphi_t u_x dx - \frac{J\gamma}{b}\int_0^1 \varphi_t\theta dx + \frac{\delta\rho}{b}\int_0^1 u_t\varphi_x dx,\ t\ \ge\ 0,$$

satisfies, for any $\varepsilon_2>0$, *the following estimate*

$$\begin{aligned} I_3'(t) \le & -\frac{\mu}{4}\int_0^1 u_x^2 dx + \varepsilon_2\int_0^1 \varphi_x^2 dx + \frac{\gamma J}{bc}\left(\frac{k_1}{2}+m\right)\int_0^1 \varphi_t^2 dx \\ & + \left(\frac{\gamma k_1 J}{2bc}+\frac{\mu d^2}{b^2}+\frac{\gamma d}{2b}\right)\int_0^1 \omega_x^2 dx \\ & + \left(\frac{\gamma^2\xi^2}{4\varepsilon_2 b^2}+\frac{\gamma d}{2b}+\frac{1}{\mu}\left(\gamma+\frac{\mu m}{b}\right)^2\right)\int_0^1 \theta^2 dx \\ & + \frac{J\rho}{b}\left(\chi-\frac{\gamma^2}{\rho c}\right)\int_0^1 \varphi_{tx}u_t dx. \end{aligned} \tag{3.10}$$

Proof 3.1.3 *By exploiting the functional* $I_3(t)$

$$\begin{aligned} I_3'(t) = & \frac{J\mu}{b}\int_0^1 \varphi_{tt}u_x dx + \frac{J\mu}{b}\int_0^1 \varphi_t u_{tx} dx - \frac{J\gamma}{b}\int_0^1 \varphi_{tt}\theta dx - \frac{J\gamma}{b}\int_0^1 \varphi_t\theta_t dx \\ & + \frac{\delta\rho}{b}\int_0^1 u_{tt}\varphi_x dx + \frac{\delta\rho}{b}\int_0^1 u_t\varphi_{tx} dx. \end{aligned}$$

Using $(1)_1$, $(1)_2, (1)_3$ *and integrating by parts, we obtain*

$$
\begin{aligned}
I_3'(t) &= \frac{\mu}{b}\int_0^1 (\delta\varphi_{xx} - bu_x - \xi\varphi - d\omega_x + m\theta)\, u_x dx - \frac{J\mu}{b}\int_0^1 \varphi_{tx} u_t dx \\
&\quad -\frac{\gamma}{b}\int_0^1 (\delta\varphi_{xx} - bu_x - \xi\varphi - d\omega_x + m\theta)\, \theta dx \\
&\quad -\frac{J\gamma}{cb}\int_0^1 (-\gamma u_{tx} - m\varphi_t - k_1\omega_x)\, \varphi_t dx \\
&\quad +\frac{\delta}{b}\int_0^1 (\mu u_{xx} + b\varphi_x - \gamma\theta_x)\, \varphi_x dx + \frac{\delta\rho}{b}\int_0^1 u_t\varphi_{tx} dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I_3'(t) &= -\mu\int_0^1 u_x^2 dx - \frac{\mu\delta}{b}\int_0^1 \varphi_x u_{xx} dx - \frac{\mu\xi}{b}\int_0^1 u_x\varphi dx - \frac{\mu d}{b}\int_0^1 \omega_x u_x dx \\
&\quad + \left(\gamma + \frac{\mu m}{b}\right)\int_0^1 \theta u_x dx + \frac{\gamma\delta}{b}\int_0^1 \varphi_x\theta_x dx + \frac{\gamma\xi}{b}\int_0^1 \varphi\theta dx \\
&\quad +\frac{\gamma d}{b}\int_0^1 \omega_x\theta dx - \frac{\gamma m}{b}\int_0^1 \theta^2 dx + \frac{\gamma m J}{bc}\int_0^1 \varphi_t^2 dx + \frac{\gamma k_1 J}{bc}\int_0^1 \omega_x\varphi_t dx \\
&\quad +\frac{\delta\mu}{b}\int_0^1 u_{xx}\varphi_x dx + \frac{b\delta}{b}\int_0^1 \varphi_x^2 dx - \frac{\gamma\delta}{b}\int_0^1 \theta_x\varphi_x dx \\
&\quad +\frac{J\rho}{b}\left(\chi - \frac{\gamma^2}{\rho c}\right)\int_0^1 \varphi_{tx} u_t dx.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
I_3'(t) = -\mu\int_0^1 u_x^2 dx - \frac{\mu\xi}{b}\int_0^1 u_x\varphi dx - \frac{\mu d}{b}\int_0^1 \omega_x u_x dx& \\
+ \left(\gamma + \frac{\mu m}{b}\right)\int_0^1 \theta u_x dx + \frac{\gamma\xi}{b}\int_0^1 \varphi\theta dx + \frac{\gamma d}{b}\int_0^1 \omega_x\theta dx& \\
- \frac{\gamma m}{b}\int_0^1 \theta^2 dx + \frac{\gamma m J}{bc}\int_0^1 \varphi_t^2 dx + \frac{\gamma k_1 J}{bc}\int_0^1 \omega_x\varphi_t dx& \\
+ \delta\int_0^1 \varphi_x^2 dx + \frac{J\rho}{b}\left(\chi - \frac{\gamma^2}{\rho c}\right)\int_0^1 \varphi_{tx} u_t dx.& \qquad (3.11)
\end{aligned}
$$

Using Young's and Poincaré's inequalities,

$$-\frac{\mu\xi}{b}\int_0^1 u_x\varphi dx \le \frac{\mu}{4}\int_0^1 u_x^2 dx + \frac{\mu\xi^2}{b^2}\int_0^1 \varphi_x^2 dx, \tag{3.12}$$

$$-\frac{\mu d}{b}\int_0^1 \omega_x u_x dx \le \frac{\mu}{4}\int_0^1 u_x^2 dx + \frac{\mu d^2}{b^2}\int_0^1 \omega_x^2 dx, \tag{3.13}$$

$$\left(\gamma+\frac{\mu m}{b}\right)\int_0^1 \theta u_x dx \le \frac{\mu}{4}\int_0^1 u_x^2 dx + \frac{1}{\mu}\left(\gamma+\frac{\mu m}{b}\right)^2\int_0^1 \theta^2 dx, \tag{3.14}$$

$$\frac{\gamma d}{b}\int_0^1 \omega_x\theta dx \le \frac{\gamma d}{2b}\int_0^1 \left(\omega_x^2+\theta^2\right)dx, \tag{3.15}$$

$$\frac{\gamma\xi}{b}\int_0^1 \varphi\theta dx \le \varepsilon_2\int_0^1 \varphi_x^2 dx + \frac{\gamma^2\xi^2}{4\varepsilon_2 b^2}\int_0^1 \theta^2 dx, \tag{3.16}$$

$$\frac{\gamma k_1 J}{bc}\int_0^1 \omega_x\varphi_t dx \le \frac{\gamma k_1 J}{2bc}\int_0^1 \left(\varphi_t^2+\omega_x^2\right)dx. \tag{3.17}$$

Substituting (3.12)-(3.17) into (3.11), we get (3.1).

Lemma 3.1.4 *Let $(u,\varphi,\theta,\omega)$ be a solution of (1)-(2). Then, the functional*

$$I_4(t) = c\alpha\int_0^1 \theta\left(\int_0^x \omega(y)\,dy\right)dx,\ t\ge 0.$$

satisfies, for any $\varepsilon_3>0$, the following estimate

$$I_4'(t) \le -\frac{k_1 c}{2}\int_0^1 \theta^2 dx + \varepsilon_3\int_0^1 u_t^2 dx + ck_2^2\int_0^1 \omega_x^2 dx + \varepsilon_3\int_0^1 \varphi_t^2 dx$$
$$+\left(k_1\alpha + ck_3^2 + \frac{\alpha^2\gamma^2}{4\varepsilon_3} + \frac{m^2\alpha^2}{4\varepsilon_3}\right)\int_0^1 \omega^2 dx - dc\int_0^1 \theta\varphi_t dx,\ t\ge 0. \tag{3.18}$$

Proof 3.1.4 *By differentiating $I_4(t)$, we obtain*

$$I_4'(t) = c\alpha\int_0^1 \theta_t\left(\int_0^x \omega(y)\,dy\right)dx + c\alpha\int_0^1 \theta\left(\int_0^x \omega_t(y)\,dy\right)dx,$$

using $(1)_3$ and $(1)_4$, we get

$$I_4'(t) = \alpha\int_0^1 \left(-\gamma u_{tx} - m\varphi_t - k_1\omega_x\right)\left(\int_0^x \omega(y)\,dy\right)dx$$
$$+ c\int_0^1 \theta\left(\int_0^x \left(k_2\omega_{yy} - k_3\omega - k_1\theta_y - d\varphi_{ty}\right)dy\right)dx.$$

So

$$\begin{aligned} I_4'(t) &= -\alpha\gamma\int_0^1 u_{tx}\left(\int_0^x \omega(y)\,dy\right)dx - m\alpha\int_0^1 \varphi_t\left(\int_0^x \omega(y)\,dy\right)dx \\ &\quad -k_1\alpha\int_0^1 \omega_x\left(\int_0^x \omega(y)\,dy\right)dx + ck_2\int_0^1 \theta\omega_x dx \\ &\quad -k_3c\int_0^1 \theta\left(\int_0^x \omega(y)\,dy\right)dx - k_1c\int_0^1 \theta^2 dx - dc\int_0^1 \theta\varphi_t dx \end{aligned}$$

integrating by parts

$$\begin{aligned} I_4'(t) &= -k_1c\int_0^1 \theta^2 dx + k_1\alpha\int_0^1 \omega^2 dx + ck_2\int_0^1 \theta\omega_x dx + \alpha\gamma\int_0^1 u_t\omega dx \\ &\quad - m\alpha\int_0^1 \varphi_t\left(\int_0^x \omega(y)\,dy\right)dx - dc\int_0^1 \theta\varphi_t dx - k_3c\int_0^1 \theta\left(\int_0^x \omega(y)\,dy\right)dx. \end{aligned} \tag{3.19}$$

Thanks to, the inequalities of Young and Cauchy Schwarz, we obtain

$$ck_2\int_0^1 \theta\omega_x dx \le ck_2^2\int_0^1 \omega_x^2 dx + \frac{ck_1}{4}\int_0^1 \theta^2 dx. \tag{3.20}$$

$$-k_3c\int_0^1 \theta\left(\int_0^x \omega(y)\,dy\right)dx \le ck_3^2\int_0^1 \omega^2 dx + \frac{ck_1}{4}\int_0^1 \theta^2 dx. \tag{3.21}$$

$$\alpha\gamma\int_0^1 u_t\omega dx \le \varepsilon_3\int_0^1 u_t^2 dx + \frac{\alpha^2\gamma^2}{4\varepsilon_3}\int_0^1 \omega^2 dx \tag{3.22}$$

$$-m\alpha\int_0^1 \varphi_t\left(\int_0^x \omega(y)\,dy\right)dx \le \varepsilon_3\int_0^1 \varphi_t^2 dx + \frac{m^2\alpha^2}{4\varepsilon_3}\int_0^1 \omega^2 dx \tag{3.23}$$

Substituting (3.20)-(3.23) *in* (3.19), *we obtain* (3.18).

Lemma 3.1.5 *Let* $(u, \varphi, \theta, \omega)$ *be a solution of* (1)-(2). *Then, the functional*

$$\begin{aligned} I_5(t) &= J\alpha\int_0^1 \varphi_t\left(\int_0^x \omega(y)\,dy\right)dx + Jk_1\int_0^1 \theta\varphi dx \\ &\quad +\frac{mJk_1}{2c}\int_0^1 \varphi^2 dx - \frac{\gamma Jk_1}{c}\int_0^1 u\varphi_x dx, \end{aligned}$$

satisfies, for any $\varepsilon_4 > 0$, the following estimate

$$I_5'(t) \leq -\frac{dJ}{2}\int_0^1 \varphi_t^2 dx + \varepsilon_4 \int_0^1 u_x^2 dx + \varepsilon_4 \int_0^1 \theta^2 dx + \varepsilon_4 \int_0^1 \varphi_x^2 dx$$
$$\left(\frac{m^2\alpha^2}{4\varepsilon_4} + \frac{Jk_3^2}{d} + \frac{\alpha^2\delta^2}{4\varepsilon_4} + \frac{b^2\alpha^2}{4\varepsilon_4} + \frac{\xi^2\alpha^2}{2\varepsilon_4}\right)\int_0^1 \omega^2 dx$$
$$+ \varepsilon_4 \int_0^1 \varphi^2 dx + \left(\frac{Jk_2^2}{d} + \frac{J^2k_1^4}{2\varepsilon_4 c^2}\right)\int_0^1 \omega_x^2 dx + \frac{\gamma J k_1}{c}\int_0^1 u_x\varphi_t dx, \quad (3.24)$$

Proof 3.1.5 *Differentiating $I_5(t)$, we obtain*

$$\begin{aligned} I_5'(t) &= J\alpha \int_0^1 \varphi_{tt}\left(\int_0^x \omega(y)\, dy\right) dx + J\alpha \int_0^1 \varphi_t \left(\int_0^x \omega_t(y)\, dy\right) dx \\ &\quad + Jk_1 \int_0^1 \theta_t \varphi dx + Jk_1 \int_0^1 \theta\varphi_t dx \\ &\quad + \frac{mJk_1}{c}\int_0^1 \varphi\varphi_t dx - \frac{\gamma J k_1}{c}\int_0^1 u_t\varphi_x dx - \frac{\gamma J k_1}{c}\int_0^1 u\varphi_{tx} dx, \end{aligned}$$

Using $(1)_2$, $(1)_3$ and integrating by parts, we arrive at

$$I_5'(t) = \alpha \int_0^1 \left(\delta\varphi_{xx} - bu_x - \xi\varphi - d\omega_x + m\theta\right)\left(\int_0^x \omega(y)\, dy\right) dx$$
$$+ J\int_0^1 \varphi_t \left(\int_0^x \left(k_2\omega_{yy} - k_3\omega - k_1\theta_y - d\varphi_{ty}\right) dy\right) dx + Jk_1 \int_0^1 \theta\varphi_t dx$$
$$+ Jk_1 \int_0^1 \left(-\frac{\gamma}{c}u_{tx} - \frac{m}{c}\varphi_t - \frac{k_1}{c}\omega_x\right)\varphi dx$$
$$+ \frac{mJk_1}{c}\int_0^1 \varphi\varphi_t dx - \frac{\gamma J k_1}{c}\int_0^1 u_t\varphi_x dx - \frac{\gamma J k_1}{c}\int_0^1 u\varphi_{tx} dx$$

$$= -\alpha\delta \int_0^1 \varphi_x \omega dx - b\alpha \int_0^1 u_x \left(\int_0^x \omega(y)\, dy \right) dx - \xi\alpha \int_0^1 \varphi \left(\int_0^x \omega(y)\, dy \right) dx \tag{3.25}$$

$$+ d\alpha \int_0^1 \omega^2 dx + m\alpha \int_0^1 \theta \left(\int_0^x \omega(y)\, dy \right) dx + Jk_2 \int_0^1 \varphi_t \omega_x dx \tag{3.26}$$

$$- k_3 J \int_0^1 \varphi_t \left(\int_0^x \omega(y) dy \right) dx - dJ \int_0^1 \varphi_t^2 dx - \frac{Jk_1^2}{c} \int_0^1 \omega_x \varphi dx$$

$$+ \frac{\gamma J k_1}{c} \int_0^1 u_x \varphi_t dx. \tag{3.27}$$

Using Young's and Cauchy Schwarz's inequalities for any $\varepsilon_4 > 0$, we obtain

$$Jk_2 \int_0^1 \varphi_t \omega_x dx \leq \frac{Jd}{4} \int_0^1 \varphi_t^2 dx + \frac{Jk_2^2}{d} \int_0^1 \omega_x^2 dx, \tag{3.28}$$

$$-k_3 J \int_0^1 \varphi_t \left(\int_0^x \omega(y) dy \right) dx \leq \frac{Jd}{4} \int_0^1 \varphi_t^2 dx + \frac{Jk_3^2}{d} \int_0^1 \omega^2 dx, \tag{3.29}$$

$$-\alpha\delta \int_0^1 \varphi_x \omega dx \leq \varepsilon_4 \int_0^1 \varphi_x^2 dx + \frac{\alpha^2\delta^2}{4\varepsilon_4} \int_0^1 \omega^2 dx, \tag{3.30}$$

$$-b\alpha \int_0^1 u_x \left(\int_0^x \omega(y)\, dy \right) dx \leq \varepsilon_4 \int_0^1 u_x^2 dx + \frac{b^2\alpha^2}{4\varepsilon_4} \int_0^1 \omega^2 dx, \tag{3.31}$$

$$-\xi\alpha \int_0^1 \varphi \left(\int_0^x \omega(y)\, dy \right) dx \leq \frac{\varepsilon_4}{2} \int_0^1 \varphi^2 dx + \frac{\xi^2\alpha^2}{2\varepsilon_4} \int_0^1 \omega^2 dx, \tag{3.32}$$

$$-\frac{Jk_1^2}{c} \int_0^1 \omega_x \varphi dx \leq \frac{\varepsilon_4}{2} \int_0^1 \varphi^2 dx + \frac{J^2 k_1^4}{2\varepsilon_4 c^2} \int_0^1 \omega_x^2 dx, \tag{3.33}$$

$$m\alpha \int_0^1 \theta \left(\int_0^x \omega(y)\, dy \right) dx \leq \varepsilon_4 \int_0^1 \theta^2 dx + \frac{m^2\alpha^2}{4\varepsilon_4} \int_0^1 \omega^2 dx \tag{3.34}$$

Inserting (3.28)-(3.34) *into* (3.27), *we obtain* (3.24).

Lemma 3.1.6 *Let* $(u, \varphi, \theta, \omega)$ *be a solution of* (1)-(2). *Then, the functional*

$$I_6(t) = \frac{J}{2}\int_0^1 \varphi_t^2 dx + \frac{\delta}{2}\int_0^1 \varphi_x^2 dx + \frac{\xi}{2}\int_0^1 \varphi^2 dx;\ t \geq 0,$$

satisfies the following estimate

$$I_6'(t) \leq \delta_1 \int_0^1 \varphi_t^2 dx + \frac{d^2}{4\delta_1}\int_0^1 \omega_x^2 dx + m\int_0^1 \theta\varphi_t dx - b\int_0^1 u_x\varphi_t dx,\ t \geq 0. \quad (3.35)$$

Proof 3.1.6 *Differentiating* $I_6(t)$ *with respect to* t, *we obtain*

$$I_6'(t) = J\int_0^1 \varphi_t\varphi_{tt} dx + \delta\int_0^1 \varphi_x\varphi_{tx} dx + \xi\int_0^1 \varphi_t\varphi dx.$$

Exploiting $(1)_2$, *we get*

$$\begin{aligned} I_6'(t) &= \int_0^1 \varphi_t\left(\delta\varphi_{xx} - bu_x - \xi\varphi - d\omega_x + m\theta\right) dx + \delta\int_0^1 \varphi_x\varphi_{tx} dx \\ &\quad + \xi\int_0^1 \varphi_t\varphi dx. \end{aligned}$$

Integrating by parts

$$I_6'(t) = -b\int_0^1 u_x\varphi_t dx - d\int_0^1 \omega_x\varphi_t dx + m\int_0^1 \theta\varphi_t dx. \quad (3.36)$$

Young's inequality leads to

$$-d\int_0^1 \omega_x\varphi_t dx \leq \delta_1\int_0^1 \varphi_t^2 dx + \frac{d^2}{4\delta_1}\int_0^1 \omega_x^2 dx. \quad (3.37)$$

Inserting (3.37) *into* (3.36), *we obtain* (3.35).

Now, we define the Lyapunov functional $L(t)$ by

$$\begin{aligned} L(t) &= NE_1(t) + I_1(t) + N_1 I_2(t) + N_2 I_3(t) \\ &+ N_3 \left(\frac{m}{dc} I_4(t) + \frac{bc}{\gamma J k_1} I_5(t) + I_6(t) \right). \end{aligned} \tag{3.38}$$

where N, N_1, N_2, N_3 are positive constants.

Theorem 3.1.1 *Let $(u, \varphi, \omega, \theta)$ be a solution of (1). Then, there exist two positive constants κ_1 and κ_2 such that the Lyapunov functional (3.38) satisfies*

$$\kappa_1 E(t) \leq L(t) \leq \kappa_2 E(t), \ \forall t \geq 0, \tag{3.39}$$

and

$$L'(t) \leq -\beta_1 E(t). \tag{3.40}$$

Proof 3.1.7 *From (3.38), we have*

$$\begin{aligned} |L(t) - NE(t)| \leq{}& \rho \int_0^1 |u_t u| \, dx + N_1 J \int_0^1 |\varphi_t \varphi| \, dx \\ &+ N_1 \frac{b\rho}{\mu} \int_0^1 \left| u_t \left(\int_0^x \varphi(y) \, dy \right) \right| dx + N_2 \frac{J\mu}{b} \int_0^1 |\varphi_t u_x| \, dx \\ &+ N_2 \frac{J\gamma}{b} \int_0^1 |\varphi_t \theta| \, dx + N_2 \frac{\delta\rho}{b} \int_0^1 |u_t \varphi_x| \, dx \\ &+ N_3 \frac{m\alpha}{d} \int_0^1 \left| \theta \left(\int_0^x \omega(y) \, dy \right) \right| dx \\ &+ N_3 \frac{bc\alpha}{\gamma k_1} \int_0^1 \left| \varphi_t \left(\int_0^x \omega(y) \, dy \right) \right| dx + N_3 \frac{bc}{\gamma} \int_0^1 |\theta \varphi| \, dx \\ &+ N_3 \frac{bm}{2\gamma} \int_0^1 \left| \varphi^2 \right| dx + N_3 b \int_0^1 |\varphi_x u| \, dx \\ &+ N_3 \frac{J}{2} \int_0^1 \left| \varphi_t^2 \right| dx + N_3 \frac{\delta}{2} \int_0^1 \left| \varphi_x^2 \right| dx + N_3 \frac{\xi}{2} \int_0^1 \left| \varphi^2 \right| dx. \end{aligned}$$

By using Young's, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$|L(t) - NE(t)| \leq \tau E(t),$$

which yields

$$(N-\tau)\,E(t) \leq L(t) \leq (N+\tau)\,E(t)\,,$$

by choosing N (depending on N_1, N_2, N_3) sufficiently large we obtain (3.39). *Now, by differentiating $L(t)$, exploiting* (3.1), (3.5), (3.10) *and setting* $\varepsilon_1 = \dfrac{\rho}{4N_1}$, $\varepsilon_2 = \dfrac{1}{N_2}$, $\varepsilon_3 = \dfrac{dc}{4mN_3}$, $\varepsilon_4 = \dfrac{1}{N_3}$, $\delta_1 = \dfrac{1}{N_3}$, *we get*

$$\begin{aligned} L'(t) \;\leq\; & -\left(\frac{\mu}{4}N_2 - \frac{3\mu}{2} - \frac{bc}{\gamma J k_1}\right)\int_0^1 u_x^2 dx \\ & -\frac{\rho}{2}\int_0^1 u_t^2 dx - \left(2\xi_1 N_1 - \frac{bc}{\gamma J k_1} - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx \\ & -\left(N_3\frac{k_1 m}{2d} - N_1\frac{1}{\delta}\left(m - \frac{b\gamma}{\mu}\right)^2 - \frac{\gamma^2}{\mu} - N_2 C_0 - \frac{bc}{\gamma J k_1}\right)\int_0^1 \theta^2 dx \end{aligned}$$

$$\begin{aligned} & -\left(\frac{N_1\delta}{2} - 1 - \frac{bc}{\gamma J k_1}\right)\int_0^1 \varphi_x^2 dx \\ & -\left(N_3\frac{dbc}{2\gamma k_1} - N_1\left(N_1\frac{b^2\rho}{\mu^2} + J\right) - N_1\frac{\gamma J}{bc}\left(\frac{k_1}{2} + m\right) - \frac{5}{4}\right)\int_0^1 \varphi_t^2 dx \\ & -\left(Nk_3 - \frac{d^2}{\delta}N_1 - C_1\frac{m}{dc}N_3 - C_2 N_3\frac{bc}{\gamma J k_1}\right)\int_0^1 \omega^2 dx \\ & -\left(Nk_2 - N_2 C_3 - k_2^2\frac{m}{d}N_3 - C_4\frac{bc}{\gamma J k_1}N_3 - \frac{d^2}{4\delta_1}N_3\right)\int_0^1 \omega_x^2 dx. \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} C_0 &= \left(\frac{\gamma^2\xi^2}{4\varepsilon_2 b^2} + \frac{\gamma d}{2b} + \frac{1}{\mu}\left(\gamma + \frac{\mu m}{b}\right)^2\right), \\ C_1 &= \left(k_1\alpha + ck_3^2 + \frac{\alpha^2\gamma^2}{4\varepsilon_3} + \frac{m^2\alpha^2}{4\varepsilon_3}\right), \\ C_2 &= \left(\frac{m^2\alpha^2}{4\varepsilon_4} + \frac{Jk_3^2}{d} + \frac{\alpha^2\delta^2}{4\varepsilon_4} + \frac{b^2\alpha^2}{4\varepsilon_4} + \frac{\xi^2\alpha^2}{2\varepsilon_4}\right), \\ C_3 &= \left(\frac{\gamma k_1 J}{2bc} + \frac{\mu d^2}{b^2} + \frac{\gamma d}{2b}\right), \\ C_4 &= \left(\frac{Jk_2^2}{d} + \frac{J^2 k_1^4}{2\varepsilon_4 c^2}\right) \end{aligned}$$

Now, we select our parameters appropriately as follows:

First, we choose N_1 *large enough so that*

$$2\xi_1 N_1 - \frac{bc}{\gamma J k_1} - \frac{b^2}{\mu} > 0,$$

and

$$\frac{N_1\delta}{2} - 1 - \frac{bc}{\gamma J k_1} > 0.$$

Next, we select N_2 *large enough so that*

$$\frac{\mu}{4}N_2 - \frac{3\mu}{2} - \frac{bc}{\gamma J k_1} > 0.$$

We select N_3 *large enough so that*

$$N_3\frac{k_1 m}{2d} - N_1\frac{1}{\delta}\left(m - \frac{b\gamma}{\mu}\right)^2 - \frac{\gamma^2}{\mu} - N_2 C_0 - \frac{bc}{\gamma J k_1} > 0,$$

and

$$N_3\frac{dbc}{2\gamma k_1} - N_1\left(N_1\frac{b^2\rho}{\mu^2} + J\right) - N_1\frac{\gamma J}{bc}\left(\frac{k_1}{2} + m\right) - \frac{5}{4} > 0.$$

Finally, we choose N *large enough (even larger so that* (3.39) *remains valid) such that*

$$\begin{cases} Nk_3 - \frac{d^2}{\delta}N_1 - C_1\dfrac{m}{dc}N_3 - C_2 N_3\dfrac{bc}{\gamma J k_1} > 0, \\ and \\ Nk_2 - N_2C_3 - k_2^2\frac{m}{d}N_3 - C_4\frac{bc}{\gamma J k_1}N_3 - \frac{d^2}{4\delta_1}N_3 > 0. \end{cases}$$

All these choices with the relation (3.41) *lead to*

$$L'(t) \leq -\alpha_1 \int_0^1 \left(u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + \omega^2\right) dx. \tag{3.42}$$

On the other hand, from Eq. (2.4) *and by using Young's inequality, we obtain*

$$E(t) \leq \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + (\mu + |b|)\,u_x^2 + \delta\varphi_x^2 + c\theta^2 + (\xi + |b|)\,\varphi^2 + \alpha\omega^2\right) dx$$

$$\leq \varrho_1\left(\int_0^1 \left(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \theta^2 + \varphi^2 + \omega^2\right) dx\right),\ \ \varrho_1 > 0,$$

which implies that

$$-\int_0^1 \left(u_t^2+\varphi_t^2+u_x^2+\varphi_x^2+\theta^2+\varphi^2+\omega^2\right)dx \le -\varrho_2 E(t)\,,\ \varrho_2>0. \quad (3.43)$$

The combination of (3.42) *and* (3.43) *gives* (3.40).

We are now ready to state and prove the following exponential stability result.

Lemma 3.1.7 *Let* $(u,\varphi,\theta,\omega)$ *be a solution of* (1)-(2) *and assume that* (3) *holds. Then, for any* $U_0 \in D(\mathcal{A})$, *there exist two positive constants* λ_1 *and* λ_2 *such that*

$$E(t) \le \lambda_2 E(0)\, e^{-\lambda_1 t},\ \forall t \ge 0. \quad (3.44)$$

Proof 3.1.8 *By using the estimation* (3.40), *we get*

$$L'(t) \le -\beta_1 E(t),\ t \ge 0,$$

having in mind the equivalence of $E(t)$ *and L(t) we infer that*

$$L'(t) \le -\lambda_1 L(t),\ t \ge 0, \quad (3.45)$$

where $\lambda_1 = \dfrac{\beta_1}{\kappa_2}$. *A simple integration of* (3.45) *gives*

$$L'(t) \le -L(0)e^{-\lambda_2 t},\ t \ge 0,$$

which yields the serial result (3.44) *and by using the other side of the equivalence relation* (3.39) *again. The proof is complete.*

3.2 Polynomial decay

To state our decay result, we introduce the first and second-order energy functional.

Lemma 3.2.1 *Let* $(u,\varphi,\theta,\omega)$ *be the solution of system (1)-(2). Then, the second-order energy functional defined by*

$$E_2(t) = \frac{1}{2}\int_0^1 \left[\rho u_{tt}^2 + J\varphi_{tt}^2 + \mu u_{xt}^2 + c\theta_t^2 + \alpha\omega_t^2 + \delta\varphi_{xt}^2 + \xi\varphi_t^2 + 2b u_{xt}\varphi_t\right] dx, \quad (3.46)$$

satisfies

$$E_2'(t) = -k_2 \int_0^1 \omega_{tx}^2 dx - k_3 \int_0^1 \omega_t^2 dx. \tag{3.47}$$

Proof 3.2.1 *Differentiating* $(1)_1$, $(1)_2$, $(1)_3$, $(1)_4$, *with respect to* t, *we obtain*

$$\begin{cases} \rho u_{ttt} = \mu u_{txx} + b\varphi_{tx} - \gamma\theta_{tx}, & in\ (0,1)\times(0,+\infty), \\ J\varphi_{ttt} = \delta\varphi_{txx} - bu_{tx} - \xi\varphi_t - d\omega_{tx} + m\theta_t, & in\ (0,1)\times(0,+\infty), \\ c\theta_{tt} = -\gamma u_{ttx} - m\varphi_{tt} - k_1\omega_{tx}, & in\ (0,1)\times(0,+\infty), \\ \alpha\omega_{tt} = k_2\omega_{txx} - k_3\omega_t - k_1\theta_{tx} - d\varphi_{ttx}, & in\ (0,1)\times(0,+\infty). \end{cases}$$

Multiplying the resulting equations by u_{tt}, φ_{tt}, θ_t, ω_t *respectively, integrating by parts over* $(0,1)$, *we have*

$$\begin{cases} \rho \int_0^1 u_{ttt}u_{tt}dx = -\mu \int_0^1 u_{tx}u_{ttx}dx - b\int_0^1 \varphi_t u_{ttx}dx \\ +\gamma \int_0^1 \theta_t u_{ttx}dx,\ in\ (0,1)\times(0,+\infty), \\ \\ J\int_0^1 \varphi_{ttt}\varphi_{tt}dx = -\delta\int_0^1 \varphi_{tx}\varphi_{ttx}dx - b\int_0^1 u_{tx}\varphi_{tt}dx - \xi\int_0^1 \varphi_t\varphi_{tt}dx \\ +d\int_0^1 \omega_t\varphi_{ttx}dx + m\int_0^1 \theta_t\varphi_{tt}dx,\ in\ (0,1)\times(0,+\infty), \\ \\ c\int_0^1 \theta_{tt}\theta_t dx = -\gamma\int_0^1 u_{ttx}\theta_t dx - m\int_0^1 \varphi_{tt}\theta_t dx + k_1\int_0^1 \omega_t\theta_{tx}dx, \\ \ in\ (0,1)\times(0,+\infty), \\ \\ \alpha\int_0^1 \omega_{tt}\omega_t dx = -k_2\int_0^1 \omega_{tx}^2 dx - k_3\int_0^1 \omega_t^2 dx - k_1\int_0^1 \theta_{tx}\omega_t dx \\ -d\int_0^1 \varphi_{ttx}\omega_t dx,\ \ in\ (0,1)\times(0,+\infty). \end{cases}$$

This last system is equivalent to

$$\begin{cases} \rho\dfrac{d}{2dt}\int_0^1 u_{tt}^2dx = -\mu\dfrac{d}{2dt}\int_0^1 u_{tx}^2dx - b\int_0^1 \varphi_t u_{ttx}dx \\ +\gamma\int_0^1 \theta_t u_{ttx}dx,\ in\ (0,1)\times(0,+\infty), \\ \\ J\dfrac{d}{2dt}\int_0^1 \varphi_{tt}^2dx = -\delta\dfrac{d}{2dt}\int_0^1 \varphi_{tx}^2dx - b\int_0^1 u_{tx}\varphi_{tt}dx - \xi\dfrac{d}{2dt}\int_0^1 \varphi_t^2dx \\ +d\int_0^1 \omega_t\varphi_{ttx}dx + m\int_0^1 \theta_t\varphi_{tt}dx,\ in\ (0,1)\times(0,+\infty), \\ \\ c\dfrac{d}{2dt}\int_0^1 \theta_t^2dx = -\gamma\int_0^1 u_{ttx}\theta_tdx - m\int_0^1 \varphi_{tt}\theta_tdx + k_1\int_0^1 \omega_t\theta_{tx}dx, \\ in\ (0,1)\times(0,+\infty), \\ \\ \alpha\dfrac{d}{2dt}\int_0^1 \omega_t^2dx = -k_2\int_0^1 \omega_{tx}^2dx - k_3\int_0^1 \omega_t^2dx - k_1\int_0^1 \theta_{tx}\omega_tdx \\ -d\int_0^1 \varphi_{ttx}\omega_tdx,\ \ in\ (0,1)\times(0,+\infty). \end{cases}$$

Summing up, we obtain

$$\frac{d}{2dt}\int_0^1 \left(\rho u_{tt}^2 + J\varphi_{tt}^2 + c\theta_t^2 + \alpha\omega_t^2 + \mu u_{tx}^2 + \delta\varphi_{tx}^2 + \xi\varphi_t^2 + 2b\varphi_t u_{tx}\right)dx$$

$$= -k_2\int_0^1 \omega_{tx}^2dx - k_3\int_0^1 \omega_t^2dx.$$

Finally, we get (3.46) and (3.47).

Now, we are ready to state and prove the main result of this part.

Theorem 3.2.1 *Assume* $\chi_1 \neq 0$, *and let* $(u,\varphi,\theta,\omega)$ *be the solution of system (1)-(2). Then there exists a positive constant* λ *such that the energy functional* (2.4) *satisfies for all* $t>0$,

$$E_1(t) \leq \frac{\lambda}{t}. \tag{3.48}$$

The proof of the result will be established through several lemmas.

Lemma 3.2.2 *Let* $(u,\varphi,\theta,\omega)$ *be the solution of system (1)-(2). Then the functional*

$$F_7(t) = \frac{k_2\rho J}{db}\chi_1\int_0^1 u_x\omega_xdx + \frac{ck_1\rho J}{2\gamma db}\chi_1\int_0^1 \theta^2dx, \tag{3.49}$$

satisfies for any $\varepsilon_2' > 0$, the following estimate

$$F_7'(t) \leq -\frac{\rho J}{b}\chi_1 \int_0^1 u_t \varphi_{tx} dx + 2\varepsilon_2' \int_0^1 u_t^2 dx + \frac{\mu}{4} \int_0^1 u_x^2 dx \tag{3.50}$$

$$+ c_0 \int_0^1 \left(\theta^2 + \varphi_t^2 + \omega_{tx}^2\right) dx + \frac{c_0}{\varepsilon_2'} \int_0^1 \omega_t^2 dx$$

$$+ c_0 \left(1 + \frac{1}{\varepsilon_2'}\right) \int_0^1 \omega_x^2 dx. \tag{3.51}$$

Proof 3.2.2 *By differentiating $F_7(t)$ and integrating by parts, we obtain*

$$\begin{aligned} F_7'(t) &= -\frac{k_2 \rho J}{db}\chi_1 \int_0^1 u_t \omega_{xx} dx + \frac{k_2 \rho J}{db}\chi_1 \int_0^1 u_x \omega_{tx} dx \\ &\quad + \frac{c k_1 \rho J}{\gamma d b}\chi_1 \int_0^1 \theta \theta_t dx. \end{aligned} \tag{3.52}$$

From (1)$_4$, we have

$$\omega_{xx} = \frac{\alpha}{k_2}\omega_t + \frac{d}{k_2}\varphi_{tx} + \frac{k_3}{k_2}\omega + \frac{k_1}{k_2}\theta_x. \tag{3.53}$$

By substituting (1)$_3$ and (3.53) in (3.52), and integrating by parts, we obtain

$$\begin{aligned} F_7'(t) &= -\frac{\alpha \rho J}{db}\chi_1 \int_0^1 u_t \omega_t dx - \frac{\rho J}{b}\chi_1 \int_0^1 u_t \varphi_{tx} dx \\ &\quad - \frac{k_3 \rho J}{db}\chi_1 \int_0^1 u_t \omega dx - \frac{k_1 \rho J}{db}\chi_1 \int_0^1 u_t \theta_x dx \\ &\quad + \frac{k_2 \rho J}{db}\chi_1 \int_0^1 u_x \omega_{tx} dx + \frac{k_1 \rho J}{db}\chi_1 \int_0^1 u_t \theta_x dx \\ &\quad - \frac{m k_1 \rho J}{\gamma d b}\chi_1 \int_0^1 \varphi_t \theta dx - \frac{k_1^2 \rho J}{\gamma d b}\chi_1 \int_0^1 \omega_x \theta dx. \end{aligned}$$

Using Young's inequality

$$-\frac{\alpha\rho J}{db}\chi_1\int_0^1 u_t\omega_t dx \leq \varepsilon_2'\int_0^1 u_t^2 dx + \frac{c_0}{\varepsilon_2'}\int_0^1 \omega_t^2 dx.$$

Using Young's and Poincaré's inequalities:

$$-\frac{k_3\rho J}{db}\chi_1\int_0^1 u_t\omega dx \leq \varepsilon_2'\int_0^1 u_t^2 dx + \frac{c_0}{\varepsilon_2'}\int_0^1 \omega_x^2 dx.$$

Using Young's inequality

$$\begin{aligned}\frac{k_2\rho J}{db}\chi_1\int_0^1 u_x\omega_{tx} dx &\leq \frac{\mu}{4}\int_0^1 u_x^2 dx + c_0\int_0^1 \omega_{tx}^2 dx.\\ -\frac{mk_1\rho J}{\gamma db}\chi_1\int_0^1 \theta\varphi_t dx &\leq c_0\int_0^1 \left(\theta^2+\varphi_t^2\right) dx.\end{aligned}$$

$$-\frac{k_1^2\rho J}{\gamma db}\chi_1\int_0^1 \theta\omega_x dx \leq c_0\int_0^1 \left(\theta^2+\omega_x^2\right) dx.$$

Then, we get

$$\begin{aligned}F_7'(t) \leq& -\frac{\rho J}{b}\chi_1\int_0^1 u_t\varphi_{tx} dx + 2\varepsilon_2'\int_0^1 u_t^2 dx + \frac{\mu}{4}\int_0^1 u_x^2 dx\\ &+c_0\int_0^1 \left(\theta^2+\varphi_t^2+\omega_{tx}^2\right) dx + \frac{c_0}{\varepsilon_2'}\int_0^1 \omega_t^2 dx\\ &+c_0\left(1+\frac{1}{\varepsilon_2'}\right)\int_0^1 \omega_x^2 dx.\end{aligned}$$

Now, we define the following Lyapunov functional as follows:

$$\begin{aligned}\mathcal{L}(t) =& N\left(E_1(t)+E_2(t)\right) + F_1(t) + N_1F_2(t) + N_2\left(F_3(t)+F_7(t)\right)\\ &+ N_3\left(\frac{m}{dc}F_4(t) + \frac{bc}{\gamma Jk_1}F_5(t) + F_6(t)\right).\end{aligned} \tag{3.54}$$

Lemma 3.2.3 *The Lyapunov functional $\mathcal{L}$ defined by (3.54) is equivalent to* $(E_1 + E_2)$.

Proof 3.2.3 *Indeed, by using Young's, Poincaré's and Cauchy-Schwarz inequalities, we obtain*

$$|\mathcal{L}(t) - N(E_1(t) + E_2(t))| \leq c(E_1(t) + E_2(t)),$$

that is

$$(N - c)(E_1(t) + E_2(t)) \leq \mathcal{L}(t) \leq (N + c)(E_1(t) + E_2(t)).$$

Now by choosing N sufficiently large, we get

$$\tau_2(E_1(t) + E_2(t)) \leq \mathcal{L}(t) \leq \tau_1(E_1(t) + E_2(t)). \tag{3.55}$$

where $\tau_1, \tau_2 > 0$.

We are now ready to prove our main result (Theorem 3.2.1).

Proof 3.2.4 *Differentiating (3.54) and using (2.5), (3.47), (3.1), (3.5), (3.10), (3.18), (3.24), (3.35) and (3.51), we get*

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[k_2 N - N_2 c_0\left(2 + \frac{1}{\varepsilon_2'}\right) - \frac{m}{dc}N_3 c_0\right. \\
& \left. -\frac{bcc_0}{\gamma J k_1}N_3\left(1 + \frac{1}{\varepsilon_4}\right) - \frac{c_0}{\delta_1}N_3\right]\int_0^1 \omega_x^2 dx \\
& -\left[k_3 N - c_0 N_1 - \frac{mc_0}{dc}N_3\left(1 + \frac{1}{\varepsilon_3}\right) - \right. \\
& \left. \frac{bcc_0}{\gamma J k_1}N_3\left(1 + \frac{1}{\varepsilon_4}\right)\right]\int_0^1 \omega^2 dx \\
& -\left[\rho - \varepsilon_1 N_1 - \frac{m}{dc}N_3\varepsilon_3 - 2N_2\varepsilon_2'\right]\int_0^1 u_t^2 dx
\end{aligned}$$

$$
\begin{aligned}
&-\left[N_1\mu_1 - \frac{bc}{\gamma J k_1} N_3 \varepsilon_4 - c_0\right] \int_0^1 \varphi^2 dx \\
&-\left[\frac{\delta}{2} N_1 - \varepsilon_2 N_2 - \frac{bc}{\gamma J k_1} N_3 \varepsilon_4\right] \int_0^1 \varphi_x^2 dx \\
&-\left[\frac{\mu}{4} N_2 - 3\frac{\mu}{2} - \frac{bc}{\gamma J k_1} N_3 \varepsilon_4\right] \int_0^1 u_x^2 dx \\
&-\left[\frac{dbc}{2\gamma k_1} N_3 - N_1 c_0 \left(1 + \frac{1}{\varepsilon_1}\right) - 2c_0 N_2 - N_3 \delta_1\right] \int_0^1 \varphi_t^2 dx \\
&-\left[\frac{k_1 m}{2d} N_3 - c_0 N_1 - c_0 \left(1 + \frac{1}{\varepsilon_2}\right) N_2 - \frac{bc}{\gamma J k_1} N_3 \varepsilon_4 - c_0\right] \int_0^1 \theta^2 dx \\
&-\left[N k_3 - \frac{c_0}{\varepsilon_2'} N_2\right] \int_0^1 \omega_t^2 dx \\
&-\left[N k_2 - N_2 c_0\right] \int_0^1 \omega_{tx}^2 dx
\end{aligned}
$$

By setting $\varepsilon_1 = \dfrac{\rho}{4N_1}$, $\varepsilon_2 = \dfrac{\delta N_1}{4N_2}$, $\varepsilon_2' = \dfrac{\rho}{8N_2}$, $\varepsilon_3 = \dfrac{dc\rho}{4mN_3}$, $\varepsilon_4 = \delta_1 = \dfrac{1}{N_3}$, *we arrive at*

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\left[k_2 N - N_2 c_0 \left(2 + \frac{8N_2}{\rho}\right) - \frac{m}{dc} N_3 c_0 \right. \\
& \left. - \frac{bcc_0}{\gamma J k_1} N_3 (1 + N_3) - c_0 N_3^2\right] \int_0^1 \omega_x^2 dx \\
& - \left[k_3 N - c_0 N_1 - \frac{mc_0}{dc} N_3 \left(1 + \frac{4mN_3}{dc\rho}\right) \right. \\
& \left. - \frac{bcc_0}{\gamma J k_1} N_3 (1 + N_3)\right] \int_0^1 \omega^2 dx
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
&-\frac{\rho}{4}\int_0^1 u_t^2 dx - \left[N_1\mu_1 - \frac{bc}{\gamma J k_1} - c_0\right]\int_0^1 \varphi^2 dx \\
&-\left[\frac{\delta N_1}{4} - \frac{bc}{\gamma J k_1}\right]\int_0^1 \varphi_x^2 dx - \left[\frac{\mu}{4}N_2 - 3\frac{\mu}{2} - \frac{bc}{\gamma J k_1}\right]\int_0^1 u_x^2 dx \\
&-\left[\frac{dbc}{2\gamma k_1}N_3 - N_1 c_0\left(1 + \frac{4N_1}{\rho}\right) - 2c_0N_2 - 1\right]\int_0^1 \varphi_t^2 dx \\
&-\left[\frac{k_1 m}{2d}N_3 - c_0N_1 - c_0\left(1 + \frac{4N_2}{\delta N_1}\right)N_2 - \frac{bc}{\gamma J k_1} - c_0\right]\int_0^1 \theta^2 dx \\
&-\left[Nk_3 - \frac{8N_2^2}{\rho}c_0\right]\int_0^1 \omega_t^2 dx \\
&-\left[Nk_2 - N_2c_0\right]\int_0^1 \omega_{tx}^2 dx. \qquad (3.57)
\end{aligned}
$$

We choose N_1 large enough so that

$$N_1\mu_1 - \frac{bc}{\gamma J k_1} - c_0 > 0, \text{ and } \frac{\delta N_1}{4} - \frac{bc}{\gamma J k_1} > 0.$$

Now, we select N_2 large enough such that

$$\frac{\mu}{4}N_2 - 3\frac{\mu}{2} - \frac{bc}{\gamma J k_1} > 0,$$

for any N_1 and N_2 we take N_3 large so that

$$\frac{k_1 m}{2d}N_3 - c_0N_1 - c_0\left(1 + \frac{4N_2}{\delta N_1}\right)N_2 - \frac{bc}{\gamma J k_1} - c_0 > 0,$$

and

$$\frac{dbc}{2\gamma k_1}N_3 - N_1c_0\left(1 + \frac{4N_1}{\rho}\right) - 2c_0N_2 - 1 > 0.$$

Finally by choose N large enough (even larger so that (3.55) remains valid) such that :

$$k_2N - N_2c_0\left(2 + \frac{8N_2}{\rho}\right) - \frac{m}{dc}N_3c_0 - \frac{bcc_0}{\gamma J k_1}N_3\left(1 + N_3\right) - c_0N_3^2 > 0,$$

$$k_3N - c_0N_1 - \frac{mc_0}{dc}N_3\left(1 + \frac{4mN_3}{dc\rho}\right) - \frac{bcc_0}{\gamma Jk_1}N_3\left(1 + N_3\right) > 0,$$
$$Nk_3 - \frac{8N_2^2}{\rho}c_0 > 0,$$

and

$$Nk_2 - N_2c_0 > 0.$$

All these choices with the relation (3.57) lead to

$$\mathcal{L}'(t) \leq -\lambda_1 \int_0^1 \left(u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + \omega^2\right) dx, \ \ \lambda_1 > 0. \tag{3.58}$$

On the other hand, from (2.4) and using Young's inequality, we obtain

$$\begin{aligned} E_1(t) &\leq \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + (\mu + b)\, u_x^2 + c\theta^2 + \delta\varphi_x^2 + \alpha\omega^2 + (\xi + b)\,\varphi^2\right) dx \\ &\leq \lambda_2 \int_0^1 \left(u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + \omega^2\right) dx, \ \ \lambda_2 > 0, \end{aligned}$$

which implies that

$$-\int_0^1 \left(u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + \omega^2\right) dx \leq -\lambda_3 E_1(t), \ \ \lambda_3 > 0. \tag{3.59}$$

By combining (3.59) and (3.58), we obtain

$$\mathcal{L}'(t) \leq -\kappa E_1(t). \tag{3.60}$$

Integrating (3.60) over $(0,t)$ and using the fact that $E_1(t)$ is positive and non-increasing, we get

$$tE_1(t) \leq \int_0^t E_1(s)\, ds \leq \frac{1}{\kappa}\left(\mathcal{L}(0) - \mathcal{L}(t)\right) \leq \frac{\mathcal{L}(0)}{\kappa}.$$

Therefore

$$E_1(t) \leq \frac{\mathcal{L}(0)}{\kappa t}.$$

Finally, for $\lambda = \dfrac{\mathcal{L}(0)}{\kappa}$, we have (3.48). Which is the conclusion of Theorem 3.2.1.

Chapter 4

Numerical approximation

In this chapter, we will solve numerically the system (1) − (2) in the one-dimension domain. For that, we used the Euler scheme for discretization of temporal variable and the classic finite difference method for discretization of spatial variable. Furthermore, in order to verify the asymptotic behavior of the solution of discretize problem, we give some examples in which the numerical experiments show that the discrete energy E^n decays polinomially for different choices of the system parameters.

4.1 Discretization of the problem

Let us introduce the functions $\hat{u} = u_t$, $\hat{\varphi} = \varphi_t$, and for any $M, N \in \mathbb{N}$, we introduce the nets

$$\Omega_N = \left\{ x_i = ih,\ i = 0, ..., N+1 \text{ where } h = \frac{1}{N+1} \right\},$$

$$\Gamma_M = \left\{ t_n = n\Delta t,\ n = 0, ..., M+1 \text{ where } \Delta t = \frac{T}{M+1} \right\}.$$

Taking a backward Euler scheme in time and finite differences in space, our problem consists to find $(\hat{u}, \hat{\varphi}, \theta, \omega)$ satisfying the following numerical scheme

$$\begin{cases} \frac{\rho}{\Delta t}\left(\hat{u}_i^n - \hat{u}_i^{n-1}\right) = \frac{\mu}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \frac{b}{2h}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) \\ -\frac{\gamma}{2h}\left(\theta_{i+1}^n - \theta_{i-1}^n\right), \\ \frac{J}{\Delta t}\left(\hat{\varphi}_i^n - \hat{\varphi}_i^{n-1}\right) = \frac{\delta}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right) - \frac{b}{2h}\left(u_{i+1}^n - u_{i-1}^n\right) - \xi\varphi_i^n \\ -\frac{d}{2h}\left(\omega_{i+1}^n - \omega_{i-1}^n\right) + m\theta_i^n, \\ \frac{c}{\Delta t}\left(\theta_i^n - \theta_i^{n-1}\right) = -\frac{\gamma}{2h}\left(\hat{u}_{i+1}^n - \hat{u}_{i-1}^n\right) - m\hat{\varphi}_i^n - \frac{k}{2h}\left(\omega_{i+1}^n - \omega_{i-1}^n\right), \\ \frac{\alpha}{\Delta t}\left(\omega_i^n - \omega_i^{n-1}\right) = \frac{k_2}{h^2}\left(\omega_{i+1}^n - 2\omega_i^n + \omega_{i-1}^n\right) - k_3\omega_i^n - \frac{k_1}{2h}\left(\theta_{i+1}^n - \theta_{i-1}^n\right) \\ -\frac{d}{2h}\left(\hat{\varphi}_{i+1}^n - \hat{\varphi}_{i-1}^n\right), \end{cases} \tag{4.1}$$

where $\varphi_i^n = \varphi(x_i, t_n)$, $\hat{\varphi}_i^n = \varphi_t(x_i, t_n)$, $\psi_i^n = \psi(x_i, t_n)$, $\hat{\psi}_i^n = \psi_t(x_i, t_n)$, $\omega_i^n = \omega(x_i, t_n)$, for all $i = 1, ..., N$ and $n = 1, ..., M$. To simplicity our numerical calculations in our scheme, we consider the discrete boundary conditions given by

$$\begin{cases} \varphi_0^n = \varphi_{N+1}^n = \theta_0^n = \theta_{N+1}^n = 0, \\ u_{N+1}^n = u_N^n,\ u_1^n = u_0^n, \\ \omega_{N+1}^n = \omega_N^n,\ \omega_1^n = \omega_0^n, \end{cases} \tag{4.2}$$

and initial conditions

$$\begin{cases} u_i^0 = u_0(x_i),\ \hat{u}_i^0 = u_1(x_i),\ \varphi_i^0 = \varphi_0(x_i),\ \hat{\varphi}_i^0 = \varphi_1(x_i), \\ \theta_i^0 = \theta_0(x_i),\ \omega_i^0 = \omega_0(x_i), \end{cases} \tag{4.3}$$

where

$$u_i^n = u_i^{n-1} + \Delta t\hat{u}_i^n,\ \varphi_i^n = \varphi_i^{n-1} + \Delta t\hat{\varphi}_i^n,$$

for all $i = 1, ..., N$ and $n = 1, ..., M$.

Note that to find $(\hat{u}, \hat{\varphi}, \theta, \omega)$. we need to solve three coupled systems of algebraic equations. So, to solve the problem $(4.1) - (4.3)$ at each time step, we propose to consider the following fixed-point algorithm

$$\left\{\begin{array}{l}\hat{u}_i^{n,l}=\dfrac{\Delta t\mu}{\rho h^2}\left(u_{i+1}^{n,l-1}-2u_i^{n,l-1}+u_{i-1}^{n,l-1}\right)+\dfrac{b\Delta t}{2h\rho}\left(\varphi_{i+1}^{n,l-1}-\varphi_{i-1}^{n,l-1}\right)\\ -\dfrac{\gamma\Delta t}{2h\rho}\left(\theta_{i+1}^{n,l-1}-\theta_{i-1}^{n,l-1}\right)+\hat{u}_i^{n-1,l-1},\\ \hat{\varphi}_i^{n,l}=\dfrac{\delta\Delta t}{Jh^2}\left(\varphi_{i+1}^{n,l-1}-2\varphi_i^{n,l-1}+\varphi_{i-1}^{n,l-1}\right)-\dfrac{b\Delta t}{2hJ}\left(u_{i+1}^{n,l}-u_{i-1}^{n,l}\right)\\ -\dfrac{\xi\Delta t}{J}\varphi_i^{n,l-1}-\dfrac{d\Delta t}{2hJ}\left(\omega_{i+1}^{n,l-1}-\omega_{i-1}^{n,l-1}\right)+\dfrac{m\Delta t}{J}\theta_i^n+\hat{\varphi}_i^{n-1,l-1},\\ \theta_i^{n,l}=-\dfrac{\gamma\Delta t}{2hc}\left(\hat{u}_{i+1}^{n,l}-\hat{u}_{i-1}^{n,l}\right)-\dfrac{m\Delta t}{c}\hat{\varphi}_i^{n,l}-\dfrac{k\Delta t}{2hc}\left(\omega_{i+1}^{n,l-1}-\omega_{i-1}^{n,l-1}\right)+\theta_i^{n-1,l-1},\\ \dfrac{k_2}{h^2}\omega_{i+1}^n+\left(-\dfrac{\alpha}{\Delta t}-k_3-2\dfrac{k_2}{h^2}\right)\omega_i^n+\dfrac{k_2}{h^2}\omega_{i-1}^n=\dfrac{k_1}{2h}\left(\theta_{i+1}^{n,l}-\theta_{i-1}^{n,l}\right)\\ +\dfrac{d}{2h}\left(\hat{\varphi}_{i+1}^{n,l}-\hat{\varphi}_{i-1}^{n,l}\right)-\dfrac{\alpha}{\Delta t}\omega_i^{n-1,l-1},\end{array}\right. \tag{4.4}$$

with

$$\left\{\begin{array}{l}u_i^{n,0}=u_i^{n-1},\ \varphi_i^{n,0}=\varphi_i^{n-1},\ \theta_i^{n,0}=\theta_i^{n-1},\ \omega_i^{n,0}=\omega_i^{n-1},\ \varphi_i^{n,l}=\varphi_i^{n-1}+\Delta t\hat{\varphi}_i^{n,l},\\ u_i^{n,l}=u_i^{n-1}+\Delta t\hat{u}_i^{n,l},\end{array}\right.$$

for all $i=1,...,N$ and $n=1,...,M$, and $l=1,2...$
At each time step, we solve the scheme (4.4) by an iterative procedure that was stopped when the difference between two successive iterations becomes smaller than a given tolerance ε.

4.2 Energy estimation

To approximate the continuous energy (2.4), we use the trapezoidal quadrature formula to compute the integral $I=\int_0^1 f(x)dx$

$$I_N=\sum_{i=1}^{N}a_i f(x_i)\approx I,$$

where the weights $\{a_i\}_{i=1}^N$ are given by $a_1=a_N=\dfrac{h}{2}$ and for $i=2,3,...,N-1$, $a_i=h$. Therefore, the discrete energy formula is given by

$$\begin{aligned}E_1(t_n)\ \approx\ J^n=\frac{1}{2}\sum_{i=1}^{N}a_i[\rho\left(\hat{u}_i^n\right)^2+J\left(\hat{\varphi}_i^n\right)^2+\mu\left((u_x)_i^n\right)^2+c\left(\theta_i^n\right)^2 \quad (4.5)\\ +\delta\left((\varphi_x)_i^n\right)^2+\xi\left(\varphi_i^n\right)^2+\alpha\left(\omega_i^n\right)^2+2b\left(u_x\right)_i^n\left(\varphi_i^n\right)^2],\end{aligned}$$

with

$$\hat{u}_i^n = u_t(x_i, t_n), \quad (u_x)_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h},$$
$$\hat{\varphi}_i^n = \varphi_t(x_i, t_n), \quad (\varphi_x)_i^n = \frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2h}.$$

4.3 Numerical examples

In the next, we describe some numerical examples.

Example 4.3.1 *For this numerical test, we choose the following different values for the coefficients of the system*

$$\rho = 1.5, \ \delta = 0.1, \ \mu = 0.8, \ b = 0.2, \ m = 0.35$$
$$\gamma = \frac{3\sqrt{2}}{\sqrt{15}}, \ J = 0.3, \ \xi = 7, \ d = 0.5, \ \beta = 3$$
$$c = 4, \ k_1 = 5, \ \alpha = 0.8, \ k_2 = 8, \ k_3 = 4.5.$$

We run our code for the following discretization parameters: $N = 150$, $M = 300$, $T = 1$ *and take* $\varepsilon = 10^{-5}$. *With the following initial conditions*

$$u_0(x) = 10^{-2}\left(x^3 - \frac{3}{2}x^2\right), \ u_1(x) = \frac{1}{8}\left(2x^2 - 2x\right), \ \varphi_0(x) = \frac{1}{8}\left(2x^2 - 2x\right),$$
$$\varphi_1(x) = 10\sin(\pi(x+1)), \ \omega_0(x) = \frac{1}{5}x^3 e^{-\frac{3}{2}x^2}, \ \theta_0(x) = 0.$$

Here are the evolution in time of the solutions $u, \varphi, \omega, \theta$ *and of the discrete energy*

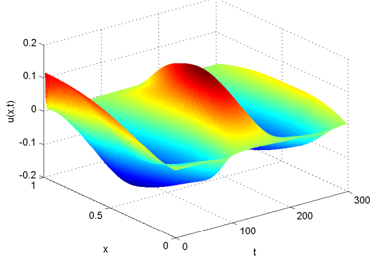

Figure 4.1:
The function u

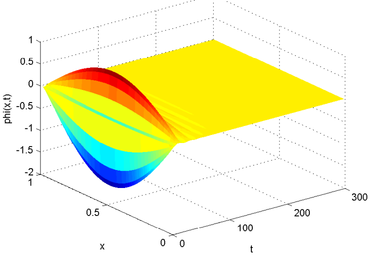

Figure 4.2:
The function φ

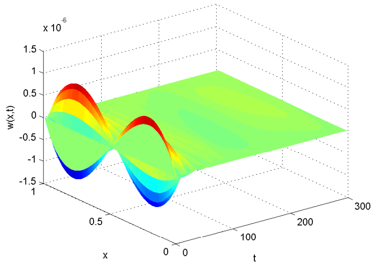

Figure 4.3:
The function θ

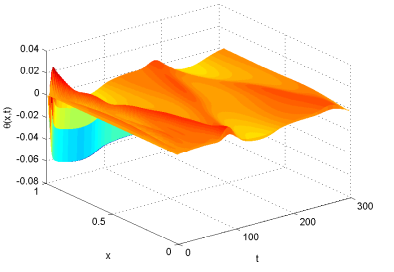

Figure 4.4:
The function ω

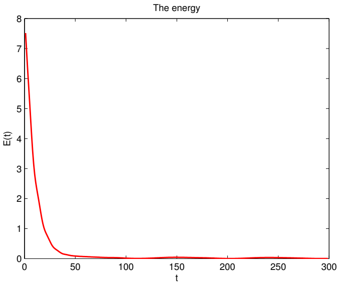

Figure 4.5:
Evolution in time of the function θ

Example 4.3.2 *For this numerical test, we choose the following different values for the coefficients of the system*

$$\begin{aligned}\rho &= 2.5,\ \delta = 0.1,\ \mu = 0.8,\ b = 1.2,\ m = 0.35\\ \gamma &= 0.05,\ J = 0.3,\ \xi = 2,\ d = 10.5,\ \beta = 1.5\\ c &= 1,\ k_1 = 0.5,\ \alpha = 0.8,\ k_2 = 0.01,\ k_3 = 4.5.\end{aligned}$$

We run our code for the following discretization parameters: $N = 150$, $M = 390$, $T = 1$ *and take* $\varepsilon = 10^{-5}$. *With the following initial conditions*

$$\begin{aligned}u_0(x) &= 10^{-2}\left(x^3 - \frac{3}{2}x^2\right),\ u_1(x) = \frac{1}{8}\left(2x^2 - 2x\right),\ \varphi_0(x) = \frac{1}{8}\left(2x^2 - 2x\right),\\ \varphi_1(x) &= 10\sin\left(\pi\left(x+1\right)\right),\ \omega_0(x) = \frac{1}{5}x^3 e^{-\frac{3}{2}x^2},\ \theta_0(x) = 0.\end{aligned}$$

Here are the evolution in time of the solutions $u, \varphi, \omega, \theta$ *and of the discrete energy*

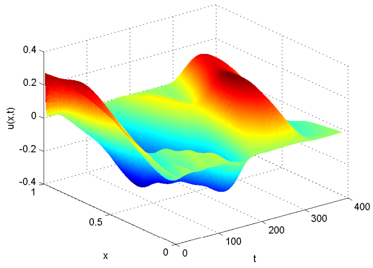

Figure 4.6:
Evolution in time of the function u (Exemple1)

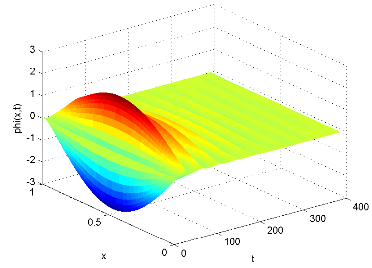

Figure 4.7:
Evolution in time of the function φ (Exemple1)

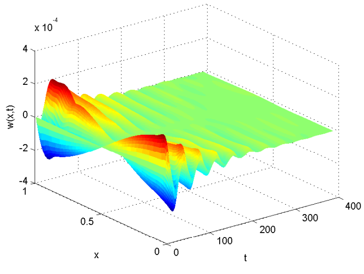

Figure 4.8:
Evolution in time of the function θ (Exemple1)

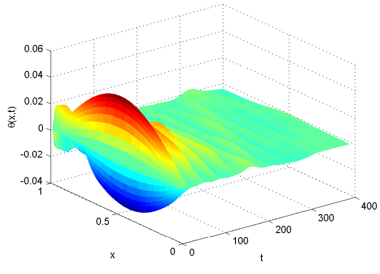

Figure 4.9:
Evolution in time of the function ω (Exemple1)

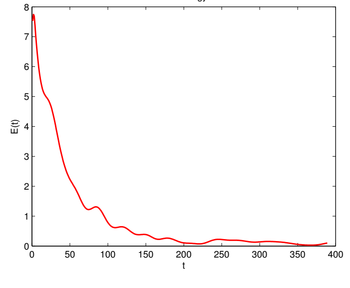

Figure 4.10:
Evolution in time of the function θ (Exemple1)

In each above numerical example, the graphics presented in the Figures 1–4, 6–9 show the evolution in time of the approximations solutions u, φ, ω and θ on the interval $[0, T]$, for different choices of the system parameters and of the initial data. Furthermore, the Figure 5 and Figure 10 show that the approximate energy (4.5) decays in an exponential manner in the case where $\chi_1 = 0$ and decays in polynomial manner where $\chi_1 \neq 0$, which confirms the main theoretical result obtained.

Conclusion

In this manuscript, we established a decay rate of the solution of porous thermoelastic system with microtemperature effects under some assumptions on system parameters. We showed that the system is well-posed using semigroup theory. Also, we proved that the energy of the considered system decreased in exponential and polynomial manner and this depends on the obtained rate. Finally, we validate the theoritical results by carrying out some numerical experiments.

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