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Analytical and Numerical Study of the Non-linear Burgers' Equation

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"Success is the sum of small efforts, repeated day in and day out."
Robert Collier

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Imene Segmane and Reffas Aya.✍

Dedication

To the girl I used to be. Quiet, curious, dreaming behind wide eyes. You made it. You are no longer surviving, you are becoming, I have become the voice, the strength and the peace you longed for.

To my mother, the soul of our home, your tenderness shaped the gentlest parts of me, your prayers held me together when I didn't even know I was falling apart.

To my father, my first protector, in your calm, I found safety, when the world grew too loud, your silence made sense of it all.

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This page carries every piece of me and of everyone who ever loved me enough to believe I'd get here.This is for us.

By Imene Segmane. ✍

To little Aya, the dreamy child who used to hide the world in her notebooks and smile despite everything. You once believed the road was longer than your strength but you walked it with quiet courage. Today, I write to you, not because you were weak, but because you were patient, you grew, and you bloomed.

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By Aya Reffas. ✍

Étude analytique et numérique de l'équation non-linéaire de Burgers

Résumé

Dans ce travail, nous étudions l'équation non linéaire de Burgers à l'aide de la transformation de Cole-Hopf, qui permet de les convertir en équations paraboliques linéaires. Cette linéarisation facilite considérablement l'analyse et la résolution du problème. Pour résoudre l'équation linéaire obtenue, nous utilisons la transformation de Fourier. La solution des équations de Burgers est ensuite déduite par la combinaison de la solution de l'équation linéaire et de la transformation inverse de Cole-Hopf.
Une analyse discrète est ensuite effectuée, complétée par des simulations numériques visant à évaluer l'efficacité de la méthode proposée.

Mots Clés: Équation de Burgers, transformation de Cole–Hopf, transformée de Fourier, schéma explicite, schéma implicite, méthodes numériques.

ANALYTICAL AND NUMERICAL STUDY OF THE NON-LINEAR BURGERS' EQUATION

Abstract

In this work, we study the nonlinear Burgers' equation using the Cole-Hopf transformation, which allows converting it into linear parabolic equations. This linearization significantly facilitates the analysis and resolution of the problem. To solve the resulting linear equation, we use the Fourier transform. The solution to the Burgers' equation is then obtained by combining the solution of the linear equation with the inverse Cole-Hopf transformation. A discrete analysis is subsequently performed, complemented by numerical simulations aimed at evaluating the effectiveness of the proposed method.

Keywords: Burgers' equation, Cole–Hopf transformation, Fourier transform, explicit scheme, implicit scheme, numerical methods.

دراسة تحليلية و عددية لمعادلة برقر اللاخطية

ملخص

في هذا العمل، تم تناول دراسة معادلة برغر غير الخطية من خلال تطبيق تحويلة كول هوبف، التي تُتيح تحويل المعادلة إلى معادلة خطية من النوع القطعي المكافئ. وقد أدى هذا التحويل إلى تبسيط كبير في تحليل المسألة وحلّها. لحل المعادلة الخطية الناتجة، تم استخدام تحويل فورييه، بعد ذلك، يتم الحصول على حل معادلة برغر من خلال الجمع بين حل المعادلة الخطية وتطبيق التحويل العكسي لكولهوبف. تم دعم هذا العمل بدراسة عددية، ترافقها محاكاة رقمية تهدف إلى تقييم فعالية المنهجية المقترحة.

الكلمات المفتاحية: معادلة برغر، تحويل كول هوبف، تحويل فورييه، الأسلوب الصريح، الأسلوب الضمني، الطرق العددية.

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NOTATIONS

$\mathbb{R}^n$: The real vector space of dimension n with $n \geq 2$.

Ω : Open subset of $\mathbb{R}^n$.

$\partial\Omega$: The boundary of Ω.

$M_n(\mathbb{R})$: The set of square matrices.

$\phi(t)$: denotes $\phi(0,t)$, i.e., the function ϕ evaluated at $x = 0$.

For $V = (v_1, v_2, ..., v_n)^t \in \mathbb{R}^n$, we define the norms $||.||_1, ||.||_2, ||.||_\infty$:

$$||V||_1 = \sum_{i=1}^{n} |v_i| \, .$$

$$||V||_2 = \left(\sum_{i=1}^{n} v_i^2 \right)^{\frac{1}{2}} .$$

$$||V||_\infty = \max_{1 \leq i \leq n} |v_i|,$$

and for a matrix $\mathcal{A} \in M_n(\mathbb{R})$ with coefficients a_{ij}, we define:

$$||\mathcal{A}||_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |a_{ij}| \right) .$$

$$||\mathcal{A}||_2 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} .$$

$$||\mathcal{A}||_\infty = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|a_{ij}|\right).$$

Eq : Equation.

PDE: Partial differential equation.

ODE: Ordinary differential equation.

FDM: Finite difference method.

F.T: Fourier transform.

F.T^{-1}: The inverse Fourier transform.

General Introduction

General Introduction

PARTIAL differential equations play a fundamental role in modeling physical, biological, and technological phenomena. They describe the evolution of variables that depend on both time and space, such as temperature, velocity, concentration, or pressure. Among these equations, some stand out due to their apparent simplicity and the richness of their dynamic behavior. This is the case for the Burgers' equation, which serves as a basic model in fluid mechanics and in other applied science fields.

Burgers' equation, first introduced by Harry Bateman in 1918 and later thoroughly analyzed by the Dutch physicist Johannes Martinus Burgers in 1948, is a PDE that combines two fundamental effects nonlinear convection and viscous diffusion. It is often regarded as a simplified form of the Navier–Stokes equations, and is used to illustrate phenomena such as shock wave formation and energy dissipation. Due to its particular structure, it appears in a wide range of contexts, including gas dynamics, acoustics, turbulence, traffic flow theory, and even cosmology.

From a mathematical perspective, Burgers' equation provides an excellent testing ground, as it admits both exact solutions and efficient numerical approximations. However, its nonlinear nature makes direct analytical solutions challenging. In this context, a powerful analytical technique comes into play: the Cole–Hopf transformation, which linearizes the equation by reducing it to a simple heat equation. This transformation enables the construction of an explicit solution using classical tools such as the Fourier transform.

This thesis addresses the solution of the one-dimensional viscous Burgers' equation using both analytical and numerical methods, with particular focus on the Cole–Hopf transformation as a key analytical tool. it is organized into three main chapters, in addition to the introduction and conclusion.

➤ **Chapter One**, presents the mathematical foundations essential for understanding the subject of this study, along with a review of the main tools that will be employed throughout this thesis.

➤ **Chapter two**, is devoted to deriving the exact solution of the one dimensional viscous Burger's equation. We apply the Cole–Hopf transformation, which linearizes the nonlinear equation by converting it into the classical heat equation. We then use the Fourier transform to solve the heat equation and obtain the analytical solution of the original Burger's equation.

➤ **Chapter Three**, explores numerical methods for approximating the solution of the viscous Burgers' equation, specifically the explicit and implicit finite difference schemes. The study is then complemented by numerical experiments aimed at justifying the effectiveness of this transformation.

Chapter 1

Preliminaries

Chapter 1

Preliminaries

In this chapter, we introduce the fundamental concepts necessary for understanding and developing the numerical methods that will be employed throughout this thesis. It begins by covering key definitions and concepts related to partial differential equations (PDEs). Next, we discuss their classification based on structural properties, this classification helps in understanding the different types of PDEs. Then we present essential mathematical tools and transformations, including the Finite Difference Method, the Cole–Hopf Transformation and the Fourier Transform.

1.1 Definitions

1.1.1 Partial differential equation (PDE)

Definition 1.1. A partial differential equation (or briefly a PDE) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables.
A partial differential equation can be written as:

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots\right) = 0. \tag{1.1}$$

1.1.2 Linear and nonlinear partial differential equations

Definition 1.2. A partial differential equation (PDE) is said to be linear if it can be written in the general form:

$$Lu = f. \tag{1.2}$$

Where, L is a linear differential operator applied to an unknown function, u and f is a given function (often called the source term).
The operator L is linear if it satisfies the following properties:

Additivity: $L(u+v) = Lu + Lv$.

Homogeneity: $L(cu) = cL(u)$, for any scalar c.

If these properties are not satisfied, the PDE is said to be nonlinear.

Example 1.1. **Linear PDE:** The one-dimensional **heat equation** on the interval $0 < x < M$:

$$\frac{\partial u}{\partial t} = r\frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

Where, $u(x,t)$ represents the temperature at position x and time t, $r > 0$ is the thermal diffusivity constant.

Example 1.2. **Nonlinear PDE:** The inviscid Burgers' equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1.4}$$

This equation introduces nonlinearity through the term $u\frac{\partial u}{\partial x}$. It appears in fluid mechanics and models wave propagation, shock formation, and simplified turbulence.

Definition 1.3. **Order of a PDE**
The order of a partial differential equation is the highest order of partial derivative of the unknown function that appears in the equation.

1.1.3 The viscous Burgers' equation

Definition 1.4. The Burgers' equation is a nonlinear PDE given by:

$$u_t + uu_x = ru_{xx}. \tag{1.5}$$

Where, $u(x,t)$ is the function to be determined, interpreted as velocity in fluid dynamics and $r > 0$ is the viscosity coefficient, which determines the strength of diffusion.

- $u_t = \frac{\partial u}{\partial t}$ denotes the time derivative.

- $u_x = \frac{\partial u}{\partial x}$ represents the first spatial derivative.
- $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ is the second spatial derivative.

This equation consists of two main terms:

- The **convective term** uu_x, which models nonlinear wave propagation and can lead to shock formation.
- The **diffusive term** ru_{xx}, which acts to smooth out steep gradients by introducing viscosity.

1.1.4 The initial and boundary conditions

To uniquely determine the solution of a partial differential equation, appropriate initial and boundary conditions must be specified.

Initial Conditions:
These specify the value of the unknown function $u(x,t)$ at the initial time $t = 0$. Typically, the initial condition is expressed as:

$$u(x,0) = u_0(x),\ x \in \Omega,$$

where, Ω denotes the spatial domain and $u_0(x)$ is a known function defining the initial state of the system.

Boundary Conditions:
These are imposed on the boundary $\partial\Omega$ of the spatial domain to govern the behavior of the solution at the edges. Boundary conditions vary according to the problem's nature and commonly include:

- *Dirichlet conditions:* specify the value of the function on the boundary, i.e.,
$$u(x,t) = g(x,t),\ x \in \partial\Omega.$$
- *Neumann conditions:* specify the value of the derivative (typically the normal derivative) of the function on the boundary, i.e.,
$$\frac{\partial u}{\partial n}(x,t) = f(x,t),\ x \in \partial\Omega.$$
- *Robin (mixed) conditions:* represent a linear combination of the function and its derivative on the boundary.

1.2 Classification of PDE

Definition 1.5. In the study of partial differential equations (PDE), it is essential to understand that there are various types of equations depending on how the derivatives are combined. These types reflect the nature of the physical phenomena they model, such as diffusion, wave propagation, or steady-state behavior. Mathematically, PDEs are classified based on the structure of their second-order terms.

Elliptic, Parabolic and Hyperbolic PDE

Let us consider the general form of a second-order linear partial differential equation with two independent variables x and y,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \tag{1.6}$$

where,

- $u = u(x, y)$ is the unknown function.
- A, B, C, D, E, F and G are coefficients, which are often assumed to be constants for simplification, especially A, B, C as they determine the type of the equation.

The classification of this PDE depends on the discriminant $B^2 - 4AC$. This discriminant is similar to the one used in quadratic equations, as it determines the mathematical nature and behavior of the equation's solutions.

- If $B^2 - 4AC < 0$, the PDE is called: **Elliptic**.
  Elliptic equations usually describe steady-state phenomena, such as the distribution of temperature in a stationary solid.
- If $B^2 - 4AC = 0$, the PDE is called: **Parabolic**.
  Parabolic equations typically describe diffusion processes or gradual changes over time.
- If $B^2 - 4AC > 0$, the PDE is called: **Hyperbolic**.
  Hyperbolic equations often model wave propagation and dynamic systems.

Examples:

- Let consider the Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.7}$$

  We have, $B^2 - 4AC = -4 < 0$, so this equation is **elliptic**.

- The Heat equation of the form:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \text{ with } \nu > 0, \tag{1.8}$$

  is **parabolic**, because $B^2 - 4AC = 0$.

- The wave equation of the form:

$$\frac{\partial^2 u}{\partial y^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \text{ with } c > 0, \tag{1.9}$$

  is **hyperbolic**, because $B^2 - 4AC = 4c^2 > 0$.

1.3 Cole-Hopf Transformation

1.3.1 Definition

The Cole–Hopf transformation is a mathematical technique used to convert a special class of parabolic partial differential equations, particularly those with nonlinear terms, into a linear heat equation. This transformation simplifies the analysis and solution of nonlinear PDEs by reducing them to a linear form.

1.3.2 Principle of the technique

The method uses a change of variables designed to eliminate the nonlinear terms from the original equation. Specifically, the Cole–Hopf transformation introduces a new dependent variable, typically by expressing the original unknown function as the logarithmic derivative of another function. This substitution converts the nonlinear PDE into a linear heat equation, which is well understood and significantly easier to solve.
Once the equation is linearized, the general solution can be obtained using

the fundamental solution of the heat equation, followed by inverting the transformation to recover the original variable.

In the following chapter, we will examine this technique in detail, explore its applications, and demonstrate how it can be used.

Remark 1.1. In infinite space, where the spatial domain extends without any boundary constraints from $-\infty$ to ∞, the Hopf-Cole transformation can be applied, and therefore the Burger's equation can be solved.
The Hopf-Cole transformation is not restricted to infinite domains only. The transformation remains valid and the equation is solvable in case of finite spatial domains such as $x = 0$ to $x = M$. As long as the Dirichlet type are the boundary conditions.

However, the Hopf-Cole transformation is no longer applicable when:

- $\frac{\partial u}{\partial x}$ is fixed: Instead of a fixed value u, one boundary enforces a constant total flux condition.
- $u = 0$: The other boundary imposes a homogeneous Dirichlet condition.

In this case, the Hopf-Cole transformation cannot solve the Burger's equation because it relies on turning both the equation and its boundary conditions into a form that matches the linear heat equation. When a flux boundary condition is specified in which it doesn't align with the heat equation's natural structure. Therefore, the transformation no longer works.

1.4 Finite Difference Method

Definition 1.6. The finite difference method (FDM) is a numerical approach for solving PDEs, including Burger's equation. This method replaces derivatives with discrete approximations based on grid points in space and time.

1.4.1 Fundamentals of Finite Difference Method

Finite difference method has their origins in the work of Leonhard Euler during the 18th century. The idea is straightforward: approximate derivatives by differences over small intervals, replacing continuous calculus with algebraic approximations on discrete grids.

For a function $u(x,t)$, the derivative at a point (x,t) and is defined as:

$$\frac{\partial u}{\partial x}(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}. \tag{1.10}$$

$$\frac{\partial u}{\partial t}(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}. \tag{1.11}$$

In numerical simulations, Δx and Δt cannot be infinitesimal, so this expression is approximated by selecting a finite Δx and a finite Δt, introducing an associated *truncation error*.

Using Taylor series, $u(x+\Delta x,t)$ and $u(x,t+\Delta t)$ can be expanded as:

$$u(x+\Delta x,t) = u(x,t) + \Delta x \frac{\partial u}{\partial x}(x,t) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x,t) + \ldots. \tag{1.12}$$

$$u(x,t+\Delta t) = u(x,t) + \Delta t \frac{\partial u}{\partial t}(x,t) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}(x,t) + \ldots. \tag{1.13}$$

Rearranging gives a finite difference form for the first derivative:

$$\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} = \frac{\partial u}{\partial x}(x,t) + O(\Delta x). \tag{1.14}$$

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{\partial u}{\partial t}(x,t) + O(\Delta t). \tag{1.15}$$

Remark 1.2. The approximation of the derivatives $\frac{\partial u}{\partial x}(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$ is of order 1, which indicates that the truncation error $O(\Delta x)$ and $O(\Delta t)$ tends toward zero as it is considered to be of the first power of Δx and Δt. The power of Δx and Δt with which the truncation error tends to zero is called the order of the method.

We denote u_i as the discrete value of $u(x)$ at the point x_i, that is $u_i = u(x_i)$. Similarly, for the derivative of $u(x)$ at the node x_i, we denote:

$$\left(\frac{\partial u}{\partial x}\right)_{x=x_i} = \left(\frac{\partial u}{\partial x}\right)_i = u_i'. \tag{1.16}$$

This notation is used equivalently for all higher-order derivatives of the quantity u.

1.4.2 First Derivative Approximations

Several finite difference schemes exist to approximate the first derivative, each balancing accuracy and computational cost.

Forward Difference

The forward difference scheme uses function values at points x_i and x_{i+1}. The first derivative at x_i is approximated by:

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1} - u_i}{\Delta x}. \tag{1.17}$$

The second derivative can be approximated using:

$$\left(\frac{d^2 u}{dx^2}\right)_i \approx \frac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta x)^2}. \tag{1.18}$$

Backward Difference

The backward difference scheme uses values at x_i and x_{i-1}.
The first derivative at x_i is given by:

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_i - u_{i-1}}{\Delta x}. \tag{1.19}$$

And the second derivative is approximated as:

$$\left(\frac{\partial^2 u}{dx^2}\right)_i \approx \frac{u_i - 2u_{i-1} + u_{i-2}}{(\Delta x)^2}. \tag{1.20}$$

Central Difference

The central difference scheme, using the points x_{i-1} and x_{i+1}.
The first derivative at x_i is approximated by:

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}. \tag{1.21}$$

And the second derivative is approximated as:

$$\left(\frac{d^2 u}{dx^2}\right)_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}. \tag{1.22}$$

Definition. The space $L^1(\Omega)$ consists of all integrable functions f defined on a domain Ω, such that

$$\int_\Omega |f(x)|\,dx < \infty.$$

It is equipped with the norm

$$\|f\|_{L^1} = \int_\Omega |f(x)|\,dx,$$

and forms a Banach space.

1.5 Fourier transformation

Definition 1.7. The Fourier transformation is a fundamental mathematical technique that translates a function from its original spatial or temporal domain into the frequency domain. This transformation is especially useful in the analysis of differential equations, such as heat equation, as it simplifies many mathematical operations.

1.5.1 Fourier transformation

Let $\phi(x,t)$ be a function defined on $\mathbb{R}$. The Fourier transform is given by:

$$\hat{\phi}(k_x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x,t) e^{-ik_x x}\,dx. \tag{1.23}$$

Here, k_x represents the wave number, which relates to the frequency components of $\phi(x,t)$. The function $\hat{\phi}(k_x,t)$ represents the signal in frequency space.

1.5.2 Inverse Fourier transformation

To reconstruct the original function $\phi(x,t)$ from its Fourier transform, the inverse formula is used:

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\phi}(k_x,t) e^{ik_x x}\,dk_x. \tag{1.24}$$

This relationship ensures that the original function can be fully recovered from its transformed version.

1.5.3 Properties of the Fourier transformation

The Fourier transformation has several essential properties:

- **Linearity:**

$$\mathcal{F}\{a\phi_1(x,t) + b\phi_2(x,t)\} = a\hat{\phi}_1(k_x,t) + b\hat{\phi}_2(k_x,t). \tag{1.25}$$

- **Shift (translation):**

$$\mathcal{F}\{\phi(x - x_0, t)\} = e^{-ik_x x_0}\hat{\phi}(k_x,t). \tag{1.26}$$

- **Scaling:**

$$\mathcal{F}\{\phi(ax,t)\} = \frac{1}{|a|}\hat{\phi}\left(\frac{k_x}{a}, t\right). \tag{1.27}$$

- **Differentiation:**

$$\mathcal{F}\left\{\frac{\partial^n \phi}{\partial x^n}\right\} = (ik_x)^n\hat{\phi}(k_x,t). \tag{1.28}$$

Fundamental Theorems

Theorem 1.1. ***Parseval's Theorem:***
If $\phi_1(x,t)$ and $\phi_2(x,t)$ are square-integrable, their energy is preserved under the Fourier transform:

$$\int_{-\infty}^{+\infty} \phi_1(x,t)\overline{\phi_2(x,t)}\,dx = \int_{-\infty}^{+\infty} \hat{\phi}_1(k_x,t)\overline{\hat{\phi}_2(k_x,t)}\,dk_x. \tag{1.29}$$

Theorem 1.2. ***Convolution Theorem:***
For two integrable functions ϕ_1 and ϕ_2:

$$\mathcal{F}\{\phi_1 * \phi_2\} = \hat{\phi}_1(k_x,t)\cdot\hat{\phi}_2(k_x,t), \tag{1.30}$$

where $*$ denotes the convolution operation.

Theorem 1.3. ***Riemann-Lebesgue Lemma Theorem:***
If $\phi(x,t)$ is an integrable function, i.e., $\phi \in L^1(\mathbb{R})$, then its Fourier transform tends to zero at infinity,

$$\lim_{|k_x|\to\infty} \hat{\phi}(k_x,t) = 0. \tag{1.31}$$

This important theorem ensures that the energy content of an integrable signal fades out at extreme frequencies, which is critical when studying the physical properties and stability of solutions to PDEs.

Chapter 2

Analytical study of the one dimentional Burger's Equation

CHAPTER 2

ANALYTICAL STUDY OF THE ONE DIMENTIONAL BURGERS' EQUATION

In this chapter, we aim to establish the analytical solution for the one dimensional Burgers' equation. To achieve this, we focus on transforming the nonlinear Burgers' equation into a linear heat equation using the Cole-Hopf transformation. This powerful technique simplifies the problem, making it more tractable. Once the heat equation is solved, we apply the inverse Cole-Hopf transformation to reconstruct the solution of the original Burgers' equation. This approach is illustrated in the following Figure. 2.1.

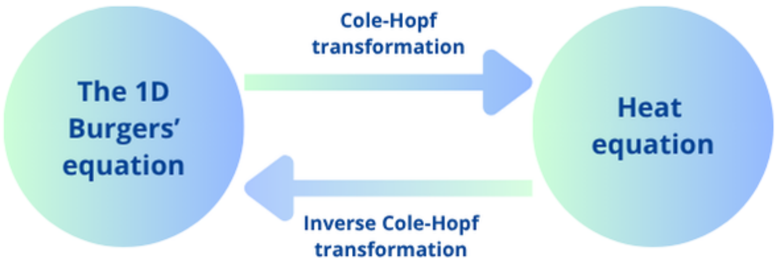

Figure 2.1: Visual Summary of the Cole-Hopf Transformation for the Burgers' Equation.

2.1 Position of problem

We consider the Burger's equation as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = r\frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

where $u = u(x,t)$, $r > 0$ is the viscosity coefficient, $x \in [0,b], t > 0$.

Subject to the initial and the boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ x \in [0,b], \\ \\ u(0,t) = 0 = u(b,t),\ t > 0. \end{cases} \tag{2.2}$$

2.2 A linearized Cole-Hopf transformation

In this section, we introduce the Cole-Hopf transformation in order to linearize the Burgers' equation (2.1).

2.2.1 Linearisation of a Burgers' equation

The Cole-Hopf transformation is performed in two steps:

First step

Suppose that $u = \psi_x$ thus Eq.(2.1) becomes:

$$\psi_{xt} + (\psi_x)\psi_{xx} = r\left(\psi_{xxx}\right), \tag{2.3}$$

which can be rewritten as:

$$\psi_{xt} + \frac{\partial}{\partial x}\left(\frac{1}{2}\psi_x^2\right) = r\left(\psi_{xxx}\right), \tag{2.4}$$

we integrate Eq.(2.4) with respect to x, we obtain:

$$\psi_t + \left(\frac{1}{2}\psi_x^2\right) = r\left(\psi_{xx}\right) + \eta(t), \tag{2.5}$$

where $\eta(t)$ is arbitrary function depending of t.

Second step

Introducing now, the transformation $\psi = -2r \ln \phi$, we obtain:

$$u = -2r\frac{\phi_x}{\phi}. \tag{2.6}$$

The derivatives of the function ψ are:

$$\psi_t = -2r\frac{\phi_t}{\phi}, \quad \psi_x = -2r\frac{\phi_x}{\phi}, \quad \psi_{xx} = -2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}. \tag{2.7}$$

Substituting the derivatives ψ_t, ψ_x and ψ_{xx} in Eq.(2.5), we obtain:

$$\left(-2r\frac{\phi_t}{\phi}\right) + \left(\frac{1}{2}\left(-2r\frac{\phi_x}{\phi}\right)^2\right) = r\left(-2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}\right) + \eta(t). \tag{2.8}$$

Eq.(2.8) can be reduced to:

$$\frac{\partial \phi}{\partial t} = r\phi_{xx} + \zeta(t)\phi, \tag{2.9}$$

where $\zeta(t) = \dfrac{-\eta(t)}{2r}$.

Let's give the following theorem which shows that the cancel of function $\zeta(t)$ in Eq.(2.9) has no effect on the solution of Eq. (2.6).

Theorem 2.1. *Let $\phi(x,t)$ be the solution of Eq.(2.9), $u(x,t)$ is defined in (2.6), then the solutions u is independent of $\zeta(t)$.*

Proof. Let

$$\beta(t) = \int \zeta(t)dt,$$

then,

$$\beta'(t) = \zeta(t).$$

Multiply by $e^{-\beta(t)}$ to both sides of Eq.(2.9), yields:

$$\frac{\partial \phi}{\partial t}e^{-\beta(t)} = r\phi_{xx}e^{-\beta(t)} + \zeta(t)\phi e^{-\beta(t)}. \tag{2.10}$$

Eq.(2.10) becomes:

$$\frac{\partial \phi}{\partial t}e^{-\beta(t)} - \zeta(t)\phi e^{-\beta(t)} = r\phi_{xx}e^{-\beta(t)}. \tag{2.11}$$

Then,

$$\frac{\partial}{\partial t}\left(e^{-\beta(t)}\phi\right) = r\phi_{xx}e^{-\beta(t)}. \tag{2.12}$$

Let $\psi(x,t) = e^{-\beta(t)}\phi(x,t)$, then $\psi(x,t)$ satisfies the following linear equation:

$$\frac{\partial \psi}{\partial t} = r\psi. \tag{2.13}$$

We can see that the difference between the solution of Eq.(2.9) and Eq.(2.13) is the factor $e^{-\beta(t)}$. Therefore, we have:

$$u(x,t) = \frac{\phi_x}{\phi} = \frac{e^{-\beta(t)}\phi_x}{e^{-\beta(t)}\phi} = \frac{\psi_x}{\psi}. \tag{2.14}$$

It is clear that the solutions $u(x,t)$ and is independent of the function $\zeta(t)$.

□

In order to simplify the study, we can take $\zeta(t) = 0$ in Eq.(2.9). Then it is written as:

$$\frac{\partial \phi}{\partial t} = r\phi_{xx}. \tag{2.15}$$

2.2.2 Determination of initial and boundary conditions

In the order to determinate the initial condition (IC) and boundary condition (BC), of the Eq.(2.6), we use,

$$\frac{\phi_x}{\phi} = \frac{u(x,t)}{-2r}. \tag{2.16}$$

Integrating both sides of Eq.(2.16) with respect to x, we obtain:

$$\phi(x,t) = \phi(t)\exp\left(\frac{-1}{2r}\int_0^x u(s,t)ds\right), \tag{2.17}$$

where $\phi(t)$ is constant of integration, and at $t = 0$ in Eq.(2.17), we obtain then the initial condition:

$$\phi(x,0) = \phi(0)\exp\left(\frac{-1}{2r}\int_0^x u(s,0)ds\right). \tag{2.18}$$

In order to show that $\phi(0)$ has no effect on the solution of the Burger's equation, we set $\tilde{\phi} = c\phi$, with c a constant, and state the following proposition.

Proposition 2.1. [33] *Let $\tilde{\phi}$ be the solution of the heat equation* (2.15), *let $\tilde{u}$ be the solution defined in the relation* (2.6), *therefore $\tilde{u}(x,t)$ are independants of the constant c.*

Proof. From the Eq.(2.6), we have:

$$\tilde{u}(x,t) = -2r\frac{(\tilde{\phi})_x}{\tilde{\phi}} = -2r\frac{c(\phi)_x}{c\phi} = -2r\frac{(\phi)_x}{\phi} = u(x,t).$$

Therefore, the proof is complete. To simplify the studying, we can consider, $\phi(0) = 1$, which gives us:

$$\phi_0(x) = \exp\left(\frac{-1}{2r}\int_0^x u_0(s)ds\right). \tag{2.19}$$

Now, the transformed boundary condition (BC), can reduced to:

$$\begin{aligned}\phi_x &= -\frac{1}{2r}u(x,t)\phi(x,t),\ (x,t) \in (\partial\Omega \times (0,T)).\\ &= 0,\ (x,t) \in (\partial\Omega \times (0,T)).\end{aligned} \tag{2.20}$$

So finally we have obtained the following linear problem concerning heat equation as an Initial-Boundary value problem with Neumann boundary conditions.

$$\begin{cases} Eq.: & \dfrac{\partial\phi}{\partial t} = r\phi_{xx}. \\ IC: & \phi_0(x) = \exp\left(\dfrac{-1}{2r}\displaystyle\int_0^x u_0(s)ds\right). \\ BC: & \phi_x = 0,\ (x,t) \in (\partial\Omega \times (0,T)). \end{cases} \tag{2.21}$$

2.3 Analytical solution

We introduce the Fourier transform (F.T):

$$\widehat{\phi}(k_x,t) \overset{F.T}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(x,t)e^{-ik_x x}dx, \tag{2.22}$$

and the inverse Fourier transformation (F.T^{-1}):

$$\phi(x,t) \overset{F.T^{-1}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \widehat{\phi}(k_x,t)e^{ik_x.x}\,dk_x. \tag{2.23}$$

First, we apply the F.T to the term ϕ_t:

$$\begin{aligned} \text{F.T}\left(\frac{\partial \phi}{\partial t}\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial t}\, e^{-ik_x t}\, dt \\ &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi\, e^{-ik_x t}\, dt\right) \\ &= \frac{\partial \hat{\phi}(k_x, t)}{\partial t}. \end{aligned} \tag{2.24}$$

Next, we examine the F.T to the term ϕ_x:

$$F.T(\phi_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial x} e^{-ik_x x} dx. \tag{2.25}$$

Integrating by part with respect to x, then we obtain:

$$F.T(\phi_x) = \frac{1}{\sqrt{2\pi}} \phi(x,t) e^{-ik_x x}\Big|_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}}(ik_x) \int_{-\infty}^{+\infty} \phi(x,t) e^{-ik_x x} dx.$$

According to [22], the boundary conditions for the heat equation on the infinite interval: $\phi = 0$ as $|x| = \infty$, so we get:

$$F.T(\phi_x) = (ik_x)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x,t) e^{-ik_x x} dx = (ik_x)\widehat{\phi}(k_x, t). \tag{2.26}$$

We can use this result to obtain the F.T of the term ϕ_{xx}:

$$\begin{aligned} F.T(\phi_{xx}) =& F.T\left((\phi_x)_x\right), \\ =& ik_x \times F.T\left(\phi_x\right), \\ =& -\left(k_x\right)^2 \widehat{\phi}(k_x, t). \end{aligned} \tag{2.27}$$

Substituting the above results into Eq.(2.15), we obtain:

$$\frac{\partial \widehat{\phi}}{\partial t} = -r k_x^2 \widehat{\phi}. \tag{2.28}$$

Thus, the solution of Eq.(2.28) is given by:

$$\widehat{\phi} = A(k_x)\, e^{(-r k_x^2) t}, \tag{2.29}$$

where, $A(k_x) = \widehat{\phi}_0(k_x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \phi_0(y) e^{-ik_x x} dy$ is the integration constant.

Applying $\mathrm{F.T}^{-1}$ to Eq.(2.29), then we obtain:

$$
\begin{aligned}
\phi(x,t) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{ik_x.x}\, \widehat{\phi}_0(k_x) e^{rk_x^2)t} dk_x, \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \phi_0(y) \left[\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ik_x.(y-x)\ -(rk_x^2)t} dk_x \right] dy.
\end{aligned} \tag{2.30}
$$

By using the program of Maple, the solution of Eq.(2.21) is given by:

$$
\phi(x,t) = \frac{1}{2\sqrt{\pi r t}} \int\limits_{-\infty}^{+\infty} \exp\left[\frac{-(x-y)^2}{4rt} \right] \phi_0(y) dy. \tag{2.31}
$$

To calculate the analytical solution of Eq.(2.1), we calculate first:

$$
\phi_x(x,t) = \frac{1}{\sqrt{2\pi r t}} \int\limits_{-\infty}^{+\infty} c\ \exp\left[\frac{-(x-y)^2}{4rt} \right] \phi_0(y) dy, \tag{2.32}
$$

where $c = \frac{-2(x-y)}{4rt}$.

Once the functions $\phi(x,t)$ and $\phi_x(x,t)$ are known and by using (2.6), therefore the solutions is:

$$
u(x,t) = \frac{\int\limits_{-\infty}^{+\infty} c'\ \exp\left[\frac{-(x-y)^2}{4rt} - \frac{1}{2r} \int_0^y u_0(s) ds \right] dy}{\int\limits_{-\infty}^{+\infty} \exp\left[\frac{-(x-y)^2}{4rt} - \frac{1}{2r} \int_0^y u_0(s) ds \right] dy}, \tag{2.33}
$$

where, $c' = \left[(x-y) t^{-1} \right]$.

2.4 Numerical evaluation of Analytical solution

We consider the initial condition function $u_0(x) = sinx$ and find the solution over the bounded spatial domain $[0, 2\pi]$ at different time steps. For the above initial condition we get the following analytical solution of Burger's equation:

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} (x-y) \;\; \exp\left[\frac{-(x-y)^2}{4rt} - \frac{1}{2r}\cos y\right] dy}{t \int_{-\infty}^{+\infty} \exp\left[\frac{-(x-y)^2}{4rt} - \frac{1}{2r}\cos y\right] dy}. \tag{2.34}$$

For very small r, both numerator and denominator of (2.34) get more closed to zero or infinity which becomes very difficult to handle. So considering the value of r arbitrarily very small, we cannot perform our numerical experiment. We consider the value of r as 0.1.

Again, for very small t, both numerator and denominator get much closed to zero and thus difficult to handle numerically. We have found that for minimum value 0.1 of t the calculation is possible.

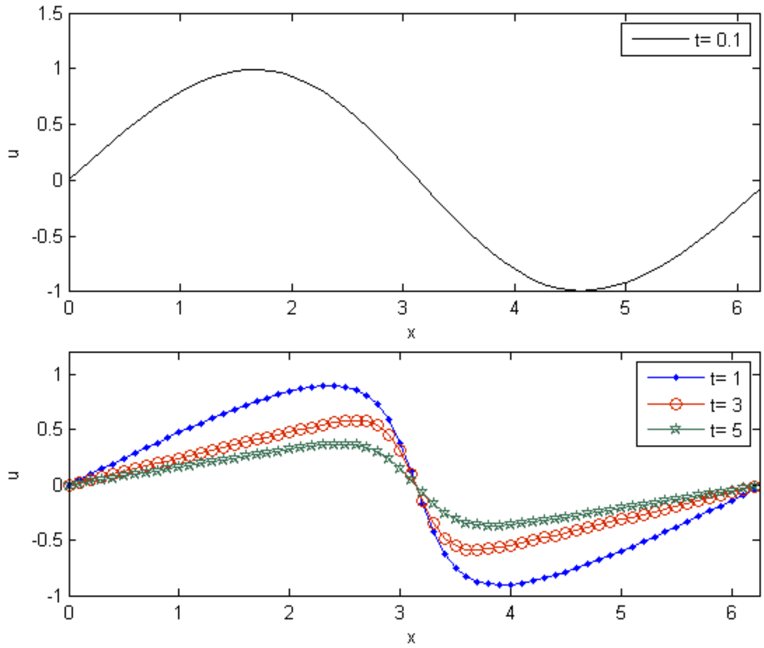

Figure 2.2: Analytical solution of Burgers' equation at different time steps.

Chapitre 3

Numerical Study Of the one dimentional Burger's Equation

CHAPTER 3

NUMERICAL STUDY OF THE ONE DIMENTIONAL BURGERS' EQUATION

In this chapter, we address the numerical solution of the Burgers' equation using finite difference methods. Both explicit and implicit schemes are implemented within the framework of the Cole–Hopf transformation. A comparative analysis is carried out to evaluate their performance in terms of accuracy and computational efficiency. Numerical simulations are presented to illustrate the effectiveness of the Cole–Hopf approach in solving the problem.

3.1 Discrete Analogues

We discretize the domain Ω by the finite difference method (FDM) into nx, each of length $\Delta x = (b-a)/nx$ along the x-axis, and define the discrete mesh points (x_i, t_n) by $(a + i\Delta x, n\Delta t)$, where $i = 0, ..., nx$ and $n = 0, ..., nt$. $\Delta t = T/nt$.

3.1.1 An explicit scheme

In order to find an explicit scheme, we use a simple forward in time and centered in space discretization at point (x_i, t_n).

We discretize $\dfrac{\partial \phi}{\partial t}$ and $\dfrac{\partial^2 \phi}{\partial x^2}$ at any discrete point (x_i, t_n) as follows:

$$\frac{\partial \phi}{\partial t} \approx \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}. \tag{3.1}$$

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}. \tag{3.2}$$

We insert (3.1) and (3.2) in Eq.(2.21), we obtain the following explicit scheme:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = r\left(\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right).$$

So that, for every interior point (x_i, t_n), with $i = 1, ..., nx - 1$, we obtain:

$$\phi_i^{n+1} = \alpha\phi_{i-1}^n + (1 - 2\alpha)\phi_i^n + \alpha\phi_{i+1}^n, \tag{3.3}$$

where,

$$\alpha = \frac{r\Delta t}{\Delta x^2}.$$

To find boundary values of ϕ, let us consider the BC at two points (one is boundary and the other is the nearest point of boundary according to our discretisation) described as:

$$\phi_x(x_i, t_n) \simeq \frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} = 0 = \frac{\phi_i^n - \phi_{i-1}^n}{\Delta x}, \tag{3.4}$$

which can be rewritten as:

$$\phi_{i+1}^n = \phi_i^n \quad \text{and} \quad \phi_{i-1}^n = \phi_i^n. \tag{3.5}$$

Substituting this constraint into Eq.(3.3) at the boundary points, we obtain respectively:

$$\begin{aligned} \phi_0^{n+1} &= \alpha\phi_1^n + (1 - \alpha)\phi_0^n, \\ \phi_{nx}^{n+1} &= \alpha\phi_{nx-1}^n + (1 - \alpha)\,\phi_{nx}^n. \end{aligned} \tag{3.6}$$

3.1.2 An implicit scheme

By using a simple forward in time and centered in space discretization at point (x_i, t_n), we discretize $\frac{\partial \phi}{\partial t}$ and $\frac{\partial^2 \phi}{\partial x^2}$ as follows:

$$\frac{\partial \phi}{\partial t} \approx \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}. \tag{3.7}$$

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}. \tag{3.8}$$

We insert (3.7) and (3.8) in Eq.(2.21), we obtain the following implicit scheme:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = r\left(\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right).$$

Which can rewrite as:

$$-\alpha\phi_{i-1}^{n+1} + \beta\phi_i^{n+1} - \alpha\phi_{i+1}^{n+1} = \phi_i^n, \tag{3.9}$$

where,

$$\alpha = \frac{r\Delta t}{\Delta x^2}, \quad \beta = (1+2\alpha)\,.$$

Since we have given Neumann boundary conditions, so we have not been able to find boundary values directly as discussed in the previous section. From Eq.(3.4), we have the following relations:

$$\phi_1^{n+1} = \phi_0^{n+1} \quad \text{and} \quad \phi_{nx}^{n+1} = \phi_{nx-1}^{n+1}. \tag{3.10}$$

In order to calculate the values of $\phi_1^{n+1}, \phi_2^{n+1}, \phi_3^{n+1}, ..., \phi_{nx-1}^{n+1}$, we substitute $i = 1, 2, 3, ..., nx-1$ in the finite difference equation and obtain the following linear system:

$$\begin{cases} (1+\gamma)\phi_1^{n+1} - \gamma\phi_2^{n+1} = \phi_1^n, \\ -\gamma\phi_1^{n+1} + (1+2\gamma)\phi_2^{n+1} - \gamma\phi_3^{n+1} = \phi_2^n, \\ -\gamma\phi_2^{n+1} + (1+2\gamma)\phi_3^{n+1} - \gamma\phi_4^{n+1} = \phi_3^n, \\ \quad\vdots \qquad\qquad\qquad \vdots \\ -\gamma\phi_{nx-3}^{n+1} + (1+2\gamma)\phi_{nx-2}^{n+1} - \gamma\phi_{nx-1}^{n+1} = \phi_{nx-2}^n, \\ -\gamma\phi_{nx-2}^{n+1} + (1+\gamma)\phi_{nx-1}^{n+1} = \phi_{nx-1}^n. \end{cases} \tag{3.11}$$

This system can be rewritten in matrix form as:

$$A\mathbf{X} = \mathbf{B}, \tag{3.12}$$

where, the coefficient matrix A is given by:

$$A = \begin{pmatrix} 1+\gamma & -\gamma & 0 & 0 & \dots & 0 \\ -\gamma & 1+2\gamma & -\gamma & 0 & \dots & 0 \\ 0 & -\gamma & 1+2\gamma & -\gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\gamma & 1+2\gamma & -\gamma \\ 0 & 0 & 0 & 0 & -\gamma & 1+\gamma \end{pmatrix} \tag{3.13}$$

The unknown vector **X** is:

$$\mathbf{X} = \begin{pmatrix} \phi_1^{n+1} \\ \phi_2^{n+1} \\ \vdots \\ \phi_{nx-1}^{n+1} \end{pmatrix}, \tag{3.14}$$

and the right-hand side vector **B** is:

$$\mathbf{B} = \begin{pmatrix} \phi_1^{n} \\ \phi_2^{n} \\ \vdots \\ \phi_{nx-1}^{n} \end{pmatrix}. \tag{3.15}$$

3.1.3 Calculating the required solution

The solution of Burgers' equation can be obtained by the inverse Cole-Hopf transformation.
Let $D_x\phi_i^n$ denote the derivative of ϕ, at point (x_i, t_n) with respect to x. Then, $D_x\phi_i^n$ can be calculated from the first order centered difference formula, for $i = 1, .., nx-1$.

$$D_x\phi_i^n = \frac{\partial \phi}{\partial x} \simeq \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x}. \tag{3.16}$$

Note that the derivatives: $D_x\phi_0^n$ and $D_x\phi_{nx}^n$ at the end points are known.

Once the approximated values of ϕ and ϕ_x are known at any discrete point (x_i, t_n), then the approximated values of u at discrete points can be calculated from the following discrete version of Eq.(2.6), for $i = 1, .., nx$,

$$u_i^n = -2r\frac{D_x\phi_i^n}{\phi_i^n}. \tag{3.17}$$

3.2 Numerical experiment using MATLAB and discussion

In this section, we discuss an example to test the performance and accuracy of the method. The numerical results arrived by this method are compared with analytic solution for various values of T. To show the accuracy of the method, both the relative error L_1-norm is given by:

$$||Erreuru||_1 = \frac{||u_a - u_n||_1}{||u_a||_1}, \tag{3.18}$$

where, u_a represents the analytical solution (2.33) and u_n represents the numerical solution (3.17). We use the Matleb program to calculate the u_n.

Considering the initial conditions:

$$u(x,0) = \sin(x),\ x \in [0, 2\pi], \tag{3.19}$$

and boundary conditions:

$$u(0,t) = u(2\pi,t) = 0,\ t > 0. \tag{3.20}$$

Let's give in Figure 3.2 and Figure 3.3 respectively the graphs of the numerical solution using explicit and implicit schemes for solving the diffusion equation and Burgers equation at different time steps. For simulation, we take the following data, $r = 0.1$.

Discussion:

As time progresses, the peak values of the solution decrease, which is characteristic of diffusion processes where the solution tends to flatten over time.

Figure 3.4 illustrates the numerical solutions of Burgers' equation at different time steps using both explicit and implicit schemes. Both methods show similar overall behavior.

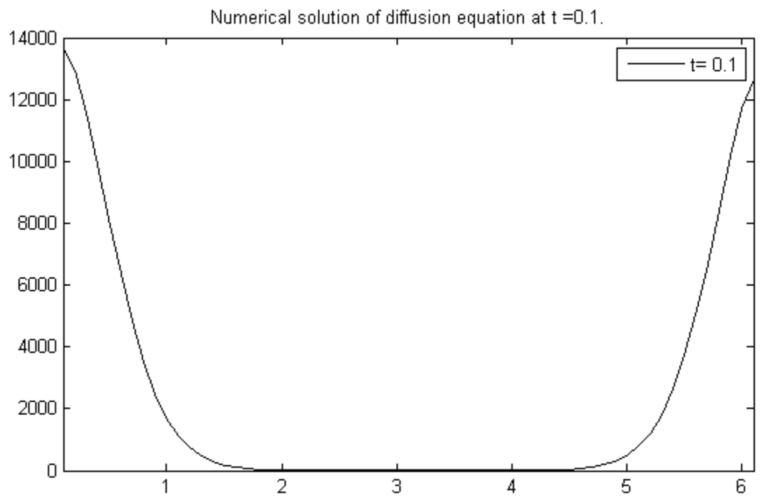

Figure 3.1: Numerical solution of diffusion equation at t=0.1.

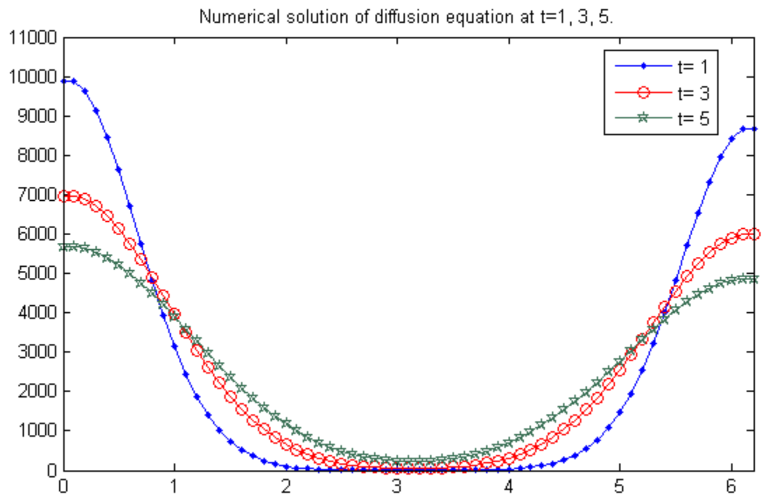

Figure 3.2: Numerical solution of diffusion equation at different time.

Now, we show in the Figure 3.5 a comparison between analytical and numerical solutions of the 1D Burgers' Equation.

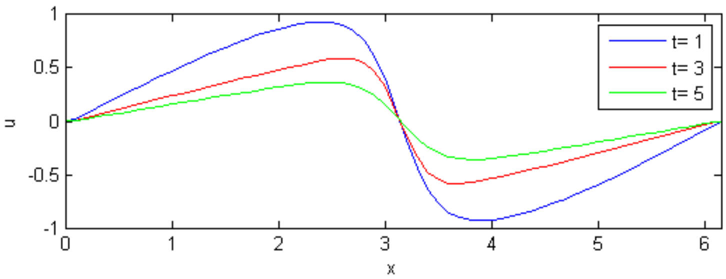

Figure 3.3: Numerical solution of Burgers' equation at different time.

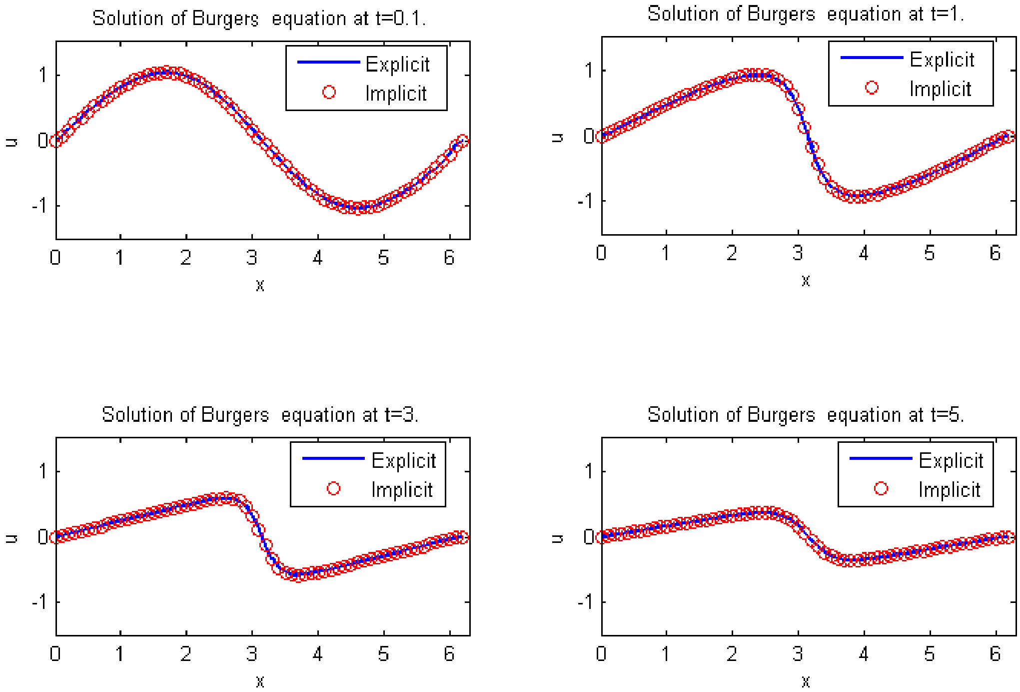

Figure 3.4: Numerical solution of Burgers' equation using explicit and implicit schemes at different time.

Discussion:

There is a good agreement between the analytical and numerical solutions at all time steps, demonstrating that the numerical scheme is capable of accurately capturing the dynamics of Burgers' equation. The minor differences observed are expected and fall within acceptable numerical error margins.

After computation of relative errors, we show the convergence of each scheme by plotting relative errors for different pairs of $(\Delta x, \Delta t)$.

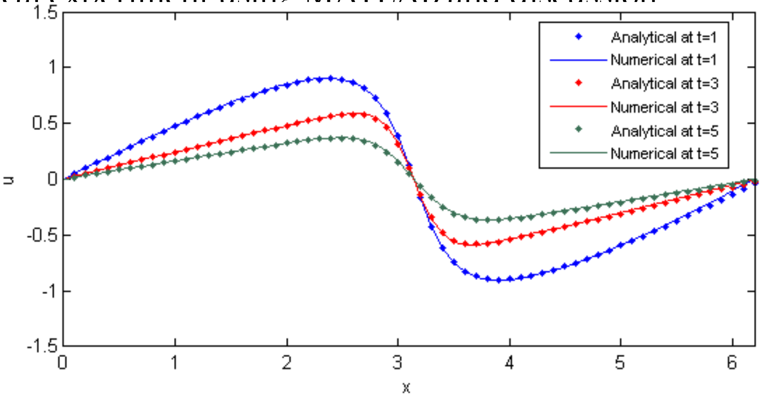

Figure 3.5: Comparison between analytical and numerical solutions of Burgers' equation at different time.

Discussion:

Both schemes show a similar trend, the relative error decreases initially, reaches a minimum, and then slightly increases again. Also, as expected, smaller values of h and k lead to lower relative errors, showing improved accuracy in both methods. However, the implicit scheme consistently produces slightly lower errors compared to the explicit scheme, especially for larger step sizes. This confirms the common advantage of implicit methods, they are generally more stable and accurate, particularly for stiff problems or larger time steps.

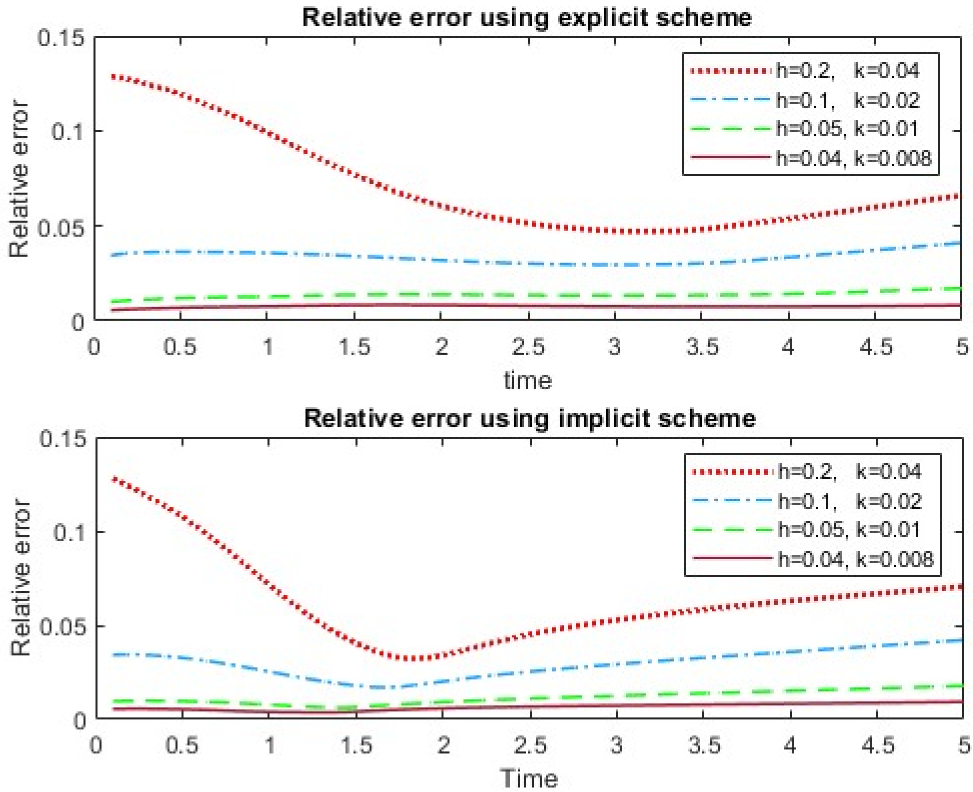

Figure 3.6: Comparison of relative error in explicit and implicit schemes for Ssolving Burgers' equation

CONCLUSION

CONCLUSION

In conclusion, after studying the nonlinear Burgers' equation using the Cole-Hopf transformation, we were able to convert it into a linear parabolic equation, which greatly facilitated its analysis and solution. We used the Fourier transform to solve the resulting linear equation, and then applied the inverse transformation to retrieve the original solution of the Burgers equation.

We also performed a numerical analysis of the discretized model, followed by numerical simulations to evaluate the effectiveness of the method. The results showed a very close agreement between the analytical and numerical solutions, with an error that is very small and almost negligible, which confirms the accuracy and efficiency of the adopted approach.

Therefore, we conclude that the Cole-Hopf transformation is an effective tool for studying this type of nonlinear equation, especially when combined with appropriate numerical techniques, making the adopted methodology a successful and reliable choice for solving such problems.

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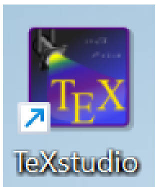
Appendix

APPENDIX I

☞ LaTeX program



LaTeX is a free tool for writing and formatting academic theses, widely used in scientific and engineering fields for its strong support of equations and scientific symbols. Developed by Leslie Lamport in 1985 as an extension of TeX by Donald Knuth, it allows users to insert tables, images, and generate contents automatically. While powerful, it can be challenging for beginners due to its markup-based syntax.

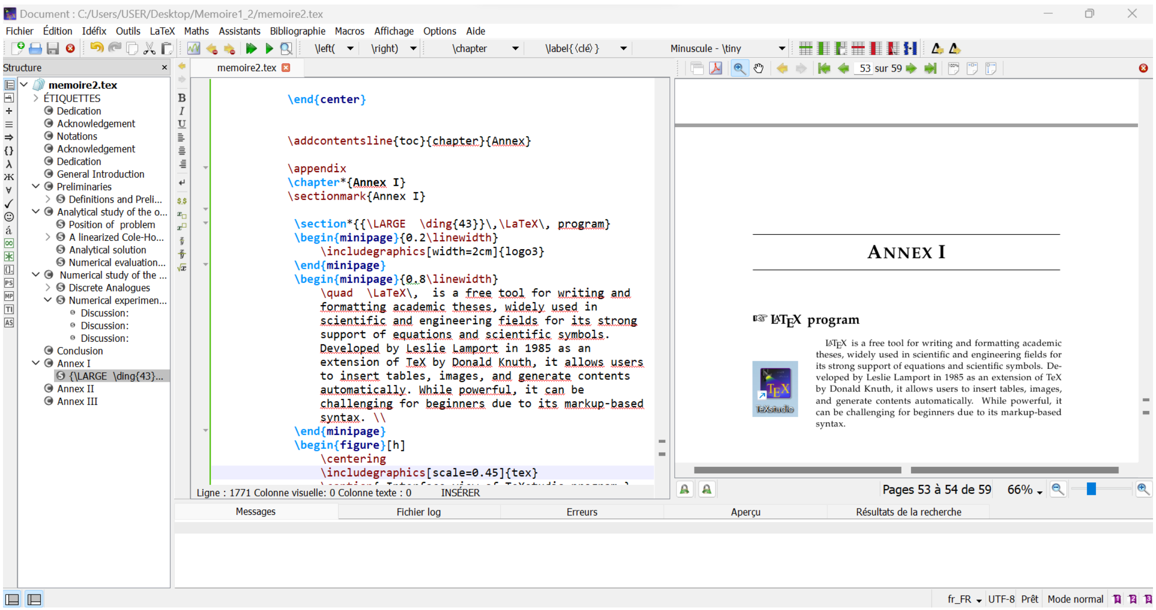


Figure 7: Interface view of TeXstudio program.

☞ Maple program

Maple is a symbolic and numeric computing environment developed by Maplesoft, designed to perform complex mathematical computations, including algebraic manipulation, calculus, differential equations, and matrix operations. Unlike languages focused primarily on numerical methods, Maple excels in symbolic computation, meaning it can solve equations, simplify expressions, and perform exact arithmetic with variables and functions. It includes a vast library of mathematical functions and tools for algebra, geometry, and applied mathematics, and it can also generate high-quality plots and interactive visualizations. With its easy-to-use interface and support for mathematical notation, Maple is widely used in education, research, and engineering to explore, analyze, and visualize mathematical problems with precision and flexibility.

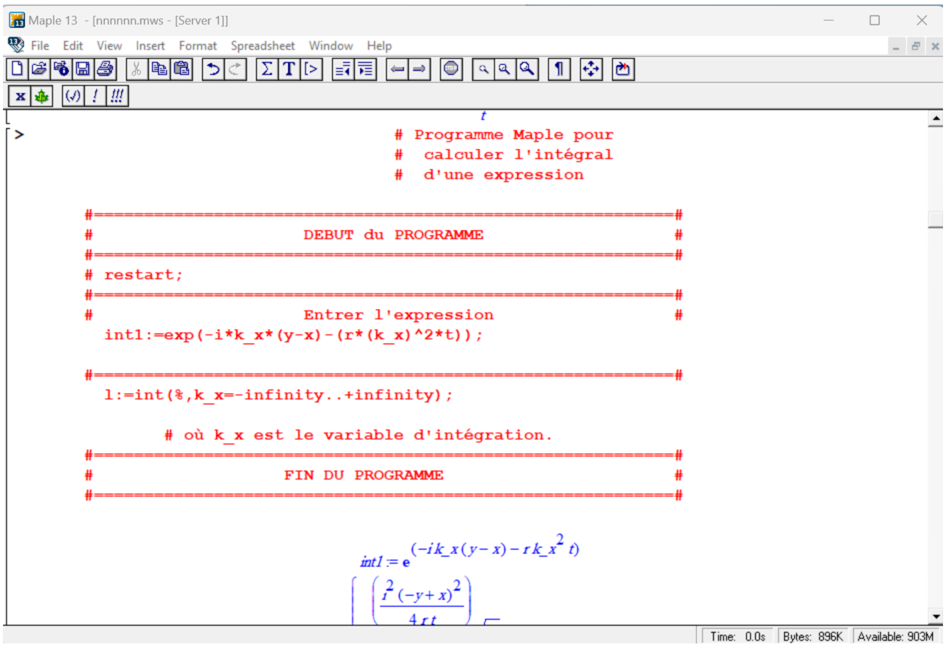


Figure 8: Interface view of Maple program.

☞ Matlab program

MATLAB is a high-level programming language and computing environment developed by MathWorks, primarily used for numerical and scientific computations. It is built around matrix and array operations, which makes it especially useful for tasks involving linear algebra, data analysis, and algorithm development. MATLAB offers a wide range of built-in functions and specialized toolboxes that support fields like engineering, physics, and applied mathematics. In addition to its strong computational capabilities, MATLAB provides powerful tools for visualizing data through 2D and 3D plots. Its user-friendly interface, combined with its versatility, has made it a popular choice in both academic research and industry for solving complex mathematical problems and simulating real-world systems.

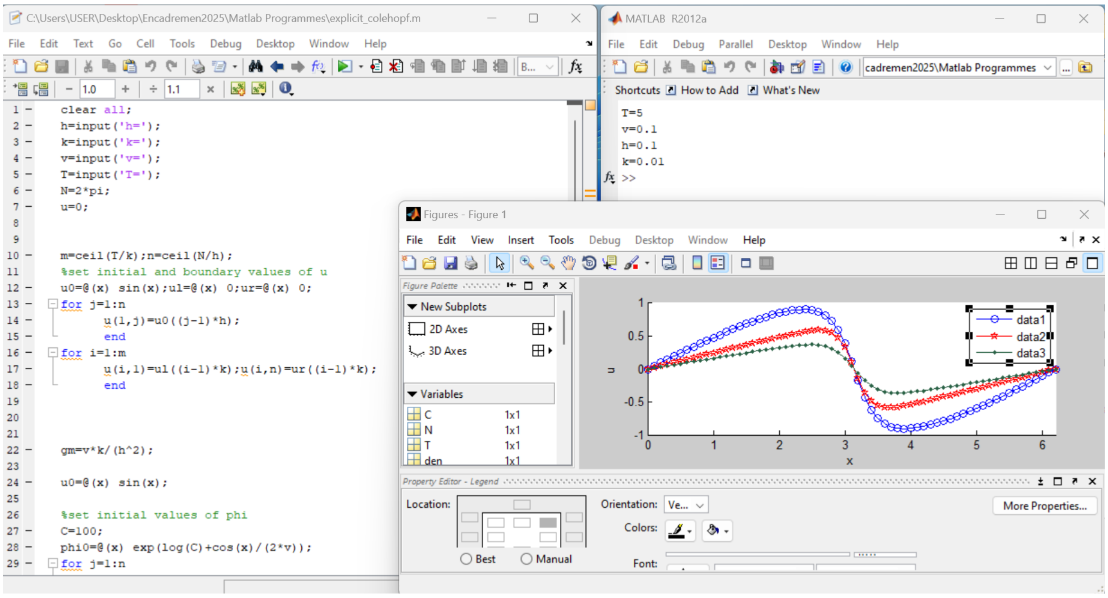


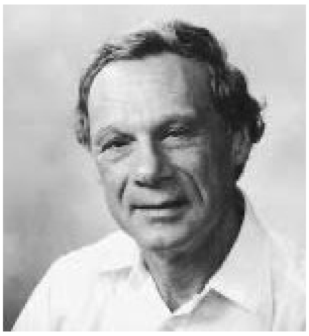
Figure 9: Interface view of Matlab program.

APPENDIX II

HISTORICAL BACKGROUND OF KEY CONTRIBUTORS

Johannes Martinus Burgers

JOHANNES MARTINUS BURGERS (1895 – 1981) A Dutch physicist and mathematician, Burgers introduced a nonlinear model that combines diffusion and compression, known as the Burgers' equation. This equation became a cornerstone in understanding the nonlinear behavior of fluids and shock waves.



Julian D. Cole

JULIAN D. COLE (1925 – 1994) An American applied mathematician, Cole co-developed the Hopf–Cole transformation, which made it easier to handle the nonlinearity in Burgers' equation. His contributions helped bridge mathematical theory and practical fluid dynamics.

Eberhard Hopf

EBERHARD HOPF (1902 – 1983) A brilliant German mathematician skilled in transforming equations. Along with his colleague, he introduced a method to convert the nonlinear Burgers' equation into a linear heat equation. His insight was that nonlinearity is not a dead end—it can be tamed through a clever transformation into a simpler analytical form.

Joseph Fourier

JOSEPH FOURIER (1768 – 1830) A French scientist who realized that complex spatial and temporal phenomena could be decomposed into simple frequencies. Through Fourier analysis, he paved the way for solving differential equations by transforming functions from the spatial domain to the frequency domain—making the equations far more tractable.

Harry Bateman

HARRY BATEMAN (1882 – 1946) One of the earliest to study differential equations in a purely physical context. Bateman provided general solutions and early analytical frameworks that were later built upon, leaving a lasting impact on transformation methods and the analysis of diffusion and wave equations.