



People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific
Research
Higher Normal School of Technological Education-
Skikda



Department of Mathematics and computer science

Dissertation

Presented to obtain a degree in Mathematics as a teacher of middle school

Entitled

**Existence results and Ulam stability for some problem of
fractional differential equations**

Presented by:

Louzat Safa

Board of Jury:

Chair man: Ayach Moussab	MAA	ENSET SKIKDA
Supervisor: Bentimama Wiam	MAA	ENSET SKIKDA
Examiner: Boulares Salah	MCB	ENSET SKIKDA
Examiner: Benhiouna Salah	MCB	ENSET SKIKDA

2024-2025

Acknowledgment

First and foremost, I thank and praise Allah for his guidance and support, and for granting me the strength and ability to complete this humble work.

I would like to express my deepest gratitude and appreciation to my supervisor Dr. Bentimama Wiam, and to my professors Dr. Kenef Asma, Dr. Khochmane Housseem Eddine, Dr. Boucenna Djalal, and Dr. Meziani Mohamed Sief Eddine, for their valuable guidance, continuous support, and constant encouragement throughout the course of this research. Their extensive knowledge, patience, expertise, and sincerity played a significant role in shaping this thesis. I have learned a great deal from them, not only academically, but also through their character, principles, and values. I sincerely ask Allah to reward them abundantly and bless them.

I also extend my heartfelt thanks and appreciation to the members of the examiners, Dr. Boularas Salah, Dr. Benhiouna Salah, and Dr. Ayach Mossab, for their evaluation, constructive feedback, and valuable suggestions regarding this modest work.

Finally, I offer my sincere thanks to everyone who contributed to the success of this research.

Dedicate

To my virtuous mother, who sacrificed her life for me.

To my noble father, who has done the impossible for me since the day I was born.

To my siblings Abir, Ghada, and Badr Eddine, who have always been my support in every little and big thing.

To the closest people to my heart, my dearest friends Bouaziz Assala, Lechheb Asma, and Slimani Nourhene, who taught me the true meaning of friendship.

To my classmates Wadia Soumia and Mansar Asmaa for their help and support.

To everyone who ever taught me a word, and to all those who accompanied me on my academic journey especially my teacher Dr. Kenef Asma.

To everyone who has ever done me a favor or helped me in any way may Allah reward them all with goodness.

To myself, for all the patience, effort, and perseverance all of which were only possible by Allah's guidance and grace. Praise be to him, first and last.

Résumé

L'objectif de ce travail est d'étudier l'existence et l'unicité de la solution d'un problème impliquant la dérivée fractionnaire généralisée de Mittag-Leffler, ainsi que d'un problème hybride mixte intégrant des dérivées fractionnaires du premier type au sens de Riemann et de Caputo, dans le cadre de l'opérateur p -Laplacien. La stabilité de la solution a été analysée selon les concepts de stabilité d'Ulam-Hyers, de stabilité généralisée d'Ulam-Hyers, de stabilité d'Ulam-Hyers-Rassias et de stabilité généralisée d'Ulam-Hyers-Rassias. Les résultats concernant l'existence et l'unicité ont été établis en appliquant des théorèmes du point fixe, tels que ceux de Banach, Krasnoselskii et Schaefer.

Mots-clés: Intégrale fractionnaire généralisée de Mittag-Leffler, dérivée fractionnaire généralisée de Mittag-Leffler, intégrale fractionnaire, dérivée fractionnaire de Riemann-Liouville, dérivée fractionnaire de Caputo, équations différentielles fractionnaires hybrides mixtes de premier type, opérateur p -Laplacien, existence et unicité de la solution, théorème du point fixe, stabilité de la solution, espaces de Banach.

Abstract

The objective of this work is to study the existence and uniqueness of the solution to a problem involving the generalized Mittag-Leffler fractional derivative, and to a mixed hybrid problem that includes first kind fractional derivatives in the senses of Riemann and Caputo, within the framework of the p -Laplace operator. The stability of the solution has been examined according to the concept of Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability. The existence and uniqueness results are established by applying fixed point theorems such as those of Banach, Krasnoselskii, and Schaefer.

Keywords: Generalized Mittag-Leffler fractional integral, generalized Mittag-Leffler fractional derivative, fractional integral, The Riemann-Liouville fractional derivative, the Caputo fractional derivative, mixed hybrid fractional differential equations of type one, p -Laplacian operator, existence and uniqueness of solution, fixed point theorem, stability of solution, Banach spaces.

Contents

Introduction	vi
1 preliminaries	6
1.1 Generalities about Functional Spaces	6
1.2 Generalities about Functional Analysis	7
1.3 Fractional calculus	9
1.3.1 Fractional Integral and derivatives of Riemann-Liouville	9
1.3.2 Fractional derivatives of Caputo	10
1.3.3 Fractional Integral and derivatives of Mittag-Leffler	11
1.4 Some fixed point theorems	12
1.5 Type of stability	13
2 Existence, uniqueness and stability of some Mittag-Leffler generalized problems	16
2.1 Introduction	16
2.2 Existence, uniqueness and stability	18
2.2.1 Existence and uniqueness results	19
2.2.2 Stability results	28
2.3 Example	31
3 Existence, uniqueness and stability of first type hybrid fractional differential equations involving the p-Laplace operator	34
3.1 Introduction	34
3.2 Existence, uniqueness and stability results	35
3.2.1 Existence and uniqueness results	38
3.2.2 Stability results	46

3.3 Example	51
Conclusion	53
Bibliography	57

Introduction

Fractional calculus is a specialized branch of mathematics that extends the classical notions of differentiation and integration to non-integer (fractional) orders within the fields of real and complex numbers. This area of study provides theoretical foundations and analytical tools for examining the existence and behavior of solutions to fractional differential equations that incorporate fractional derivatives and are commonly referred to as fractional equations.

These equations are widely applied across numerous scientific and engineering disciplines, including physics, chemistry, mechanics, electronics, biology, control theory, signal and image processing, traffic modeling, seismic oscillations, blood flow dynamics, and more (see sources [19], [20], [22], [23], [29], [40].)

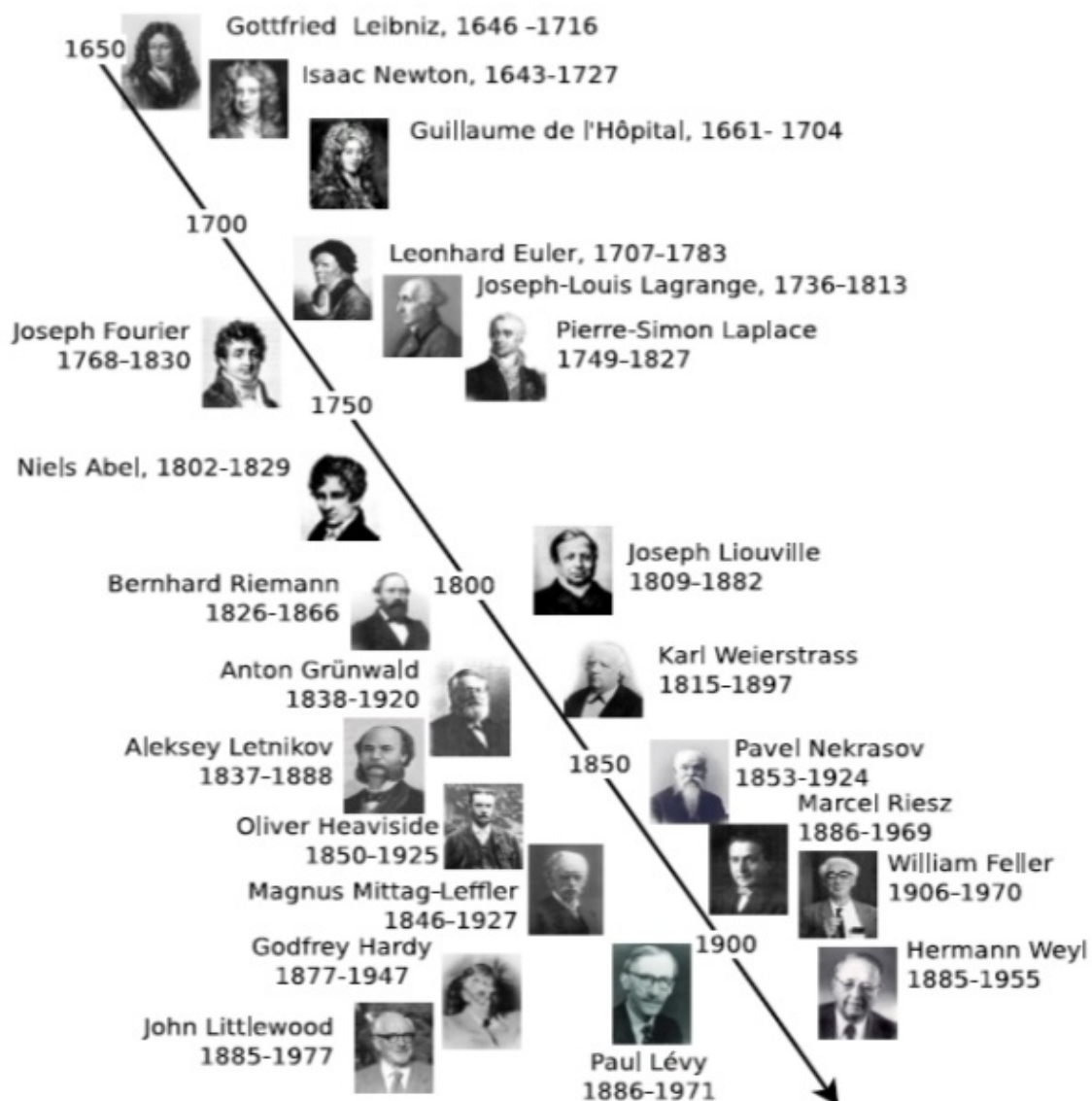
Although fractional calculus has gained popularity as a modern field of application, its conceptual origins are quite old. Historians trace its beginnings to a letter dated September 30, 1695, in which Marquis de l'Hôpital asked Gottfried Wilhelm Leibniz about the meaning of a derivative of non-integer order. The question: What would be the result if the order is $\frac{1}{2}$?, was intellectually provocative for its time.

In 1819, Silvestre François Lacroix discussed the idea of fractional derivatives in his mathematical works. Later, in 1823, Niels Henrik Abel applied fractional calculus to solve the tautochrone problem, marking the first known physical application. Throughout the 19th century, mathematicians such as Joseph Liouville, Laurent, and others expanded the theory through generalized operators.

In the early 20th century, Oliver Heaviside applied fractional derivatives to transmission line theory, while Dumont, in 1963, explored applications in elasticity. In 1974, Brian Ross organized the first international conference on fractional calculus and its applications. This momentum culminated in 1993, when Kenneth Arrau and Miller Ross published the influential book *An Introduction to the Fractional Calculus and*

Fractional Differential Equations.

A historical timeline maps the development of fractional calculus from the 17th century to the 1970s. Since the early 20th century, this field has grown significantly, thanks to contributions from researchers such as Bernhard Riemann, Michele Caputo, Edwin Spanier, Mohamed Benchohra, and Anatoly Kilbas. Today, fractional derivatives have evolved from theoretical curiosities to powerful modeling tools for systems with memory and hereditary properties, offering capabilities beyond those of classical integer-order models [8], [21], [28], [30]. The figure is represent timeline of main scientists in the area fractional calculus.



As fractional calculus becomes increasingly relevant, it has attracted wide academic

interest in recent decades, with numerous books and scholarly articles dedicated to its development. Once regarded as a purely theoretical branch, it is now recognized for its ability to model non-local and complex natural behaviors across a wide range of scientific and engineering domains. [10], [16], [31].

In particular, the existence and uniqueness of solutions to fractional differential equations remain an active area of research. Techniques used to investigate these properties include the method of upper and lower solutions, Mawhin's continuation theory, and fixed point theorems. Solving such equations whether analytically or numerically requires careful handling, as computational approximations can introduce errors. Even small perturbations in the differential equation may lead to significant deviations in the solution, potentially compromising its accuracy or relevance [13], [14], [15].

There are several types of fractional derivatives, each with distinct formulations suited for specific applications. Among the most widely used are the: Riemann–Liouville derivative (1847), Caputo derivative (1967), Grünwald–Letnikov derivative (1867), Atangana–Baleanu–Caputo (Mittag-Leffler) derivative (2016).

The Riemann–Liouville derivative extends classical differentiation using fractional integrals. The Caputo derivative is particularly useful in solving initial value problems, as it uses integer-order derivatives of the function. The Grünwald–Letnikov derivative, based on difference quotients, is especially suitable for numerical methods and time series modeling.

Other notable types include: Hadamard derivative (1892), Hilfer derivative (2000), Caputo–Fabrizio derivative (2015).

The ABC derivative uses the Mittag-Leffler function as its kernel rather than a singular power-law kernel. This gives it a non-local, non-singular structure, allowing for higher accuracy in modeling real world systems with memory. It has been applied to model the spread of COVID-19, considering past infection history, and to analyze heat transfer in materials that retain thermal memory, employs a non-singular, non-local kernel. Unlike the power-law kernels used in classical formulations, the Mittag-Leffler kernel offers a more realistic representation of memory effects. This makes it especially valuable for modeling viscoelastic materials, anomalous diffusion, and biological systems [6], [7], [9], [16], [24], [33], [34], [38].

Another important class involves mixed Riemann-Liouville and Caputo derivatives, useful in hybrid models that require both local and non-local behaviors. Recently, mixed type hybrid fractional differential equations of the first kind have emerged, incorporating p-Laplace operators that blend the features of both Riemann and Caputo derivatives for more flexible and accurate modeling [2], [3], [4], [18], [25], [37].

Due to the difficulty of obtaining exact solutions particularly in nonlinear problems approximation methods are commonly used. However, only stable approximations are acceptable, making stability analysis essential. The idea of stability in functional equations was introduced by Ulam in a 1940 lecture, with the first solution proposed by Hyers in 1941, leading to the concept of Ulam–Hyers stability. This was later generalized by Rassias in 1978 to handle mappings with variable dependence. Many studies since then have explored both Ulam–Hyers and Ulam–Hyers–Rassias stability, especially in operator theory (see [5], [10], [31], [32]).

This work has been divided into three chapters as following

In the first Chapter (1) Introduces foundational concepts related to spaces, definitions, theorems, and properties of fractional calculus. In addition to some fixed point theorems that I relied on in the following chapters, and some stability concepts according to the Ulam-Hyers definition.

In the second Chapter (2) We studied the existence, uniqueness, and stability of solution for the evolution problem under the generalized fractional derivative of the Mittag-Leffler type .

$$\begin{cases} {}^{ML}D_{0+}^{\varepsilon,h} x(\theta) = \Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)), & \theta \in \Delta \\ x(0) = \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta. \end{cases}$$

Where

- $\Delta = [0, T]$.
- Zx is the controls function.

$$Zx(\theta) = \int_0^\theta \frac{(\theta-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} x(\eta) d\eta.$$

- ${}^{ML}D_{0+}^{\varepsilon,h}$ the generalized Mittag-Leffler fractional derivative of order $0 < \varepsilon \leq 1$.

- $h \in C^1(\Delta, \mathbb{R}_+)$ is increasing function, such that $h'(t) \neq 0$;
 $\forall t \in [0, T]$ and h' positive defined on Δ .
- $\Lambda, \Phi : \Delta \mapsto \mathbb{R}$ are bounded linear operator and $\Omega : \Delta \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is nonlinear continuous function.
- $x' \in L(\Delta, \mathbb{R})$.

The third Chapter (3) concerns the existence, uniqueness, and stability of solution for the problem related to first-type hybrid fractional differential equations

$$\left\{ \begin{array}{l} {}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) = g \left(t, x(t), {}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) \right), \quad t \in J, \\ \phi_P \left({}^R D_{0+}^{\beta} (x(0)) \right) = \varphi f(0, x(0)), \\ \lim_{t \rightarrow 0} \left(t^{1-\beta} x(t) \right) = 0. \end{array} \right.$$

Where

- $J = [0, 1]$ and $\varphi \in \mathbb{R}$.
- ${}^C D_{0+}^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$.
- ${}^R D_{0+}^{\beta}$ is the Riemann-Liouville fractional derivative of order $0 < \beta < 1$.
- $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}^2, \mathbb{R})$ are two nonlinear and continuous functions.
- ϕ_P is the p-Laplacian operator.

preliminaries

This chapter aims to establish the essential theoretical foundations required for a clear and thorough understanding of the subject matter of this work. It is structured into four main sections. The first section offers a concise overview of function spaces. The second section is dedicated to the definitions of fractional derivatives of the Riemann-Liouville, Caputo and Mittag-Leffler types highlighting their fundamental properties and key theorems. The third section outlines the principal fixed point theorems employed in this research as a central mathematical tool to prove the existence and uniqueness of solutions. Finally, the chapter concludes with a discussion of the Ulam-Hyers and Ulam-Hyers-Rassias stability theorems, which play a vital role in understanding the stability behavior of solutions to fractional differential equations.

1.1 Generalities about Functional Spaces

This section, we present some tools and preliminaries notions of analysis used throughout this memoir.

- Let $\Delta = [a, b]$, $(-\infty < a < b < \infty)$ be a finite interval of the real axis $\mathbb{R} = (-\infty, +\infty)$.
- Let $C(\Delta, \mathbb{R}) = C(\Delta)$ be the Banach space of all continuous functions from Δ to \mathbb{R} with the norm

$$\|u\|_{C(\Delta)} = \sup_{t \in \Delta} |u(t)|.$$

- We denote by $C^m(\Delta, \mathbb{R})$ the space of functions f that are m times continuously differentiable on Δ , with the norm

$$\|u\|_{C^m(\Delta)} = \sum_{k=0}^m \|u^{(k)}\|_{C(\Delta)}, \quad m \in \mathbb{N}.$$

- Let $L_p(\Delta, \mathbb{R}) = L_p(\Delta)$ for $1 \leq p < \infty$ denote the set of real-valued Lebesgue measurable functions f on Δ for which

$$\|f\|_{L_p(\Delta)} < \infty, \text{ where}$$

$$\|f\|_{L_p(\Delta)} = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}.$$

And

$$\|f\|_{L_\infty([a, b])} = \text{ess sup}_{t \in \Delta} |f(t)| = \inf \{M > 0, |f(t)| \leq M, \text{ a.e. on } \Delta\}.$$

In particular, for $p = 1$, $L_1(\Delta, \mathbb{R}) = L_1(\Delta)$.

- Denote by $AC(\Delta, \mathbb{R}) = AC(\Delta)$, The space of all functions f which are absolutely continuous on Δ . It is Known that $AC(\Delta)$ coincides with the space of primitives of Lebesgue summable functions, i.e.

$$f \in AC(\Delta) \iff f(x) = c + \int_a^x \varphi(t) dt, \quad \varphi(t) \in L_1(\Delta).$$

For $n \in \mathbb{N}$, we define $AC^n(\Delta)$ by

$$AC^n(\Delta) = \left\{ f : \Delta \longrightarrow \mathbb{R}, \text{ and } f^{(n-1)} \in AC(\Delta) \right\}.$$

For more details, see [21], [11].

1.2 Generalities about Functional Analysis

Let $\Delta = [a, b]$ be a finit interval of the real axis \mathbb{R} . Let $(E, \|\cdot\|)$ be a Banach space .

Definition 1.1. We say that an operator $A : E \longrightarrow E$ is continuous on E if and only if

$$\forall (u_n)_n \subset E, \left(\lim_{n \rightarrow +\infty} u_n = u \right) \implies \left(\lim_{n \rightarrow +\infty} A(u_n) = A(u) \right).$$

Definition 1.2. [39]. We say an operator $A : E \longrightarrow E$ is a contraction on E if there exists a real number k such that $0 < k < 1$ satisfies

$$\forall x, y \in E, \|Ax - Ay\| \leq k \|x - y\|.$$

Definition 1.3. [19]. An operator $A : E \longrightarrow E$ is called compact if the image of each bounded set $B \subset E$ is relatively compact i.e $\left(\overline{A(B)}\right)$ is compact).

Theorem 1.1. (Arzela-Ascoli theorem) [21]. Let H be a subset of $C(\Delta)$, equipped with the Chebyshev norm. Then H is relatively compact in $C(\Delta)$ if and only if, H is equicontinuous and uniformly bounded.

Definition 1.4. [19]. An operator $A : E \longrightarrow E$ is called completely continuous operator if it is continuous and compact.

We recall that a family H of continuous functions on Δ into \mathbb{R} is called

- Uniformly bounded if there exists a constant $M > 0$ such that

$$\|f\|_{C(\Delta)} = \sup_{t \in \Delta} |f(t)| \leq M, \forall f \in H.$$

- Equicontinuous on Δ , if, for every $\varepsilon > 0$, there exists some $\eta > 0$ such that for all $f \in H$ and all $t_1, t_2 \in \Delta$, with $|t_1 - t_2| < \eta$, we have

$$|f(t_1) - f(t_2)| < \varepsilon.$$

The criteria for compactness for sets in the space of continuous functions $C(\Delta)$ is the following.

Definition 1.5. (p-Laplacian operator)[26]. The p-Laplacian operator ϕ_p is defined on \mathbb{R} as

$$\phi_p(u) = \begin{cases} |u|^{p-2} u, & u \neq 0, \\ 0, & u = 0, \end{cases}$$

where $p > 1$.

Some basics properties of the p-Laplacian operator are given in the following lemmas.

Lemma 1.1. [27]. The p-Laplacien operator ϕ_p is a homeomorphism from \mathbb{R} to \mathbb{R} . Moreover, $(\phi_p)^{-1}$ is continuous, sends bounded sets to bounded sets, and is defined as follow

$$(\phi_p)^{-1}(u) = \begin{cases} \phi_q(u), & u \neq 0, \\ 0, & u = 0, \end{cases}$$

such that $\frac{1}{p} + \frac{1}{q} = 1$, and $q > 0$.

Lemma 1.2. [26]. Let $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$ be a p -Laplacian operator.

(i) If $1 < p \leq 2$, $xy > 0$, $|x|, |y| \geq m > 0$, then

$$|\phi_p(x) - \phi_p(y)| \leq (p-1) m^{p-2} |x - y|.$$

(ii) If $p > 2$, $|x|, |y| \leq M$, then

$$|\phi_p(x) - \phi_p(y)| \leq (p-1) M^{p-2} |x - y|.$$

Where m and M are two strictly positive constants.

1.3 Fractional calculus

1.3.1 Fractional Integral and derivatives of Riemann-Liouville

Definition 1.6. We define the Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f \in L^1(\Delta)$ defined by

$$(I_{a^+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a.$$

Such that $\Gamma(\cdot)$ is defined as follows

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \exp(-t) dt, \quad \alpha > 0.$$

Which achieves some properties, including

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \quad \alpha > 0.$$

Definition 1.7. The Riemann – Liouville fractional derivative of order $\alpha > 0$ of the function $f \in L^1(\Delta)$, defined by

$$\left({}^R D_{a^+}^\alpha f \right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad t > a.$$

With $n = [\alpha] + 1$, where $[\alpha]$ is an integer part of α .

In the case where $\alpha = n$ for $\alpha \in \mathbb{N}$, we have

$$\left({}^R D_{a^+}^n f \right)(t) = f^{(n)}(t),$$

where $f^{(n)}$ is the ordinary derivative of f of order n .

Lemma 1.3. [21]. Let $\alpha \in \mathbb{R}_+$. If $f(t) \in L^1(\Delta)$ and $f_{n-\alpha}(t) \in AC^n(\Delta)$, then

$$\begin{aligned} \left(I_{a^+}^\alpha {}^R D_{a^+}^\alpha f \right) (t) &= f(t) - \sum_{k=1}^n \frac{f_{n-\alpha}^{(n-k)}(a)}{\Gamma(\alpha - k + 1)} (t - a)^{\alpha - k}. \\ \left({}^R D_{a^+}^\alpha I_{a^+}^\alpha f \right) (t) &= f(t). \end{aligned}$$

- In particular, if $0 < \alpha < 1$, then

$$\left(I_{a^+}^\alpha {}^R D_{a^+}^\alpha f \right) (t) = f(t) - k(t - a)^{\alpha - 1}.$$

Where $k = \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}$ with $f_{1-\alpha}(x) = \left(I_{a^+}^{1-\alpha} f \right) (x)$.

1.3.2 Fractional derivatives of Caputo

Definition 1.8. The Caputo fractional derivative of order $\alpha > 0$ of the function $f \in L^1(\Delta)$ defined by

$$\left({}^C D_{a^+}^\alpha f \right) (t) = \left({}^R D_{a^+}^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t - a)^k \right] \right) (t), \quad t \in \Delta.$$

Where $n = [\alpha] + 1$, for $\alpha \notin \mathbb{N}$, where $[\alpha]$ is an integer part of order α .

In the case where $\alpha = n$ for $\alpha \in \mathbb{N}$, we have

$$\left({}^C D_{a^+}^n f \right) (t) = f^{(n)}(t).$$

Lemma 1.4. [21]. Let $\alpha \in \mathbb{R}_+$. If $f \in AC^n(\Delta)$ or $f \in C^n(\Delta)$, then

$$\left(I_{a^+}^\alpha {}^C D_{a^+}^\alpha f \right) (t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t - a)^k.$$

Where $n = [\alpha] + 1$, for $\alpha \notin \mathbb{N}$, and $\alpha = n$ for $\alpha \in \mathbb{N}$.

- In case $0 < \alpha \leq 1$ and $f \in AC(\Delta)$ or $f \in C(\Delta)$ then

$$\left(I_{a^+}^\alpha {}^C D_{a^+}^\alpha f \right) (t) = f(t) - f(a).$$

- Soit $\alpha > 0$, et $f \in L^\infty(\Delta)$ then

$$\left({}^C D_{a^+}^\alpha I_{a^+}^\alpha f \right) (t) = f(t),$$

Lemma 1.5. [21]. For $\alpha > 0$, if $f \in AC^n(\Delta)$ or $f \in C^n(\Delta)$, then

- The solution of the fractional differential equation ${}^R D_{a+}^\alpha f(t) = 0$ is given by

$$u(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \dots + k_n t^{\alpha-n}.$$

- The solution of the fractional differential equation ${}^C D_{a+}^\alpha f(t) = 0$ is given by

$$u(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}.$$

Where $k_i, c_j \in \mathbb{R}$, for $i = 1, \dots, n$ and $j = 0, 1, \dots, n-1$. $t \in \Delta$. $n = [\alpha] + 1$.

1.3.3 Fractional Integral and derivatives of Mittag-Leffler

Definition 1.9. [35], [36]. Let $\varepsilon \in (0, 1)$, $h \in T_k$, $x \in L^1(\Delta, \mathbb{R})$. The generalized Mittag-Leffler fractional integral of the real-valued defined by

$${}^{ML} I_{a+}^{\varepsilon, h} x(\theta) = \frac{1-\varepsilon}{ML(\varepsilon)} x(\theta) + \frac{\varepsilon}{ML(\varepsilon) \Gamma(\varepsilon)} \int_{a+}^{\theta} h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} x(\eta) d\eta.$$

Where

- $ML(\varepsilon)$ is a normalization constant satisfying $ML(0) = ML(1) = 1$.
- T_k the set of real-valued function and $h \in C(\Delta, \mathbb{R}_+)$ satisfies h is increasing, $h'(\theta) \neq 0$, $\forall \theta \in \Delta$ and $h'(\theta)$ is positive defined on (Δ) .

In particular when we choose $h(x) = x$, for all $x \in L^1(\Delta, \mathbb{R})$, we obtain a well know fractional derivative called Mittag-Leffler fractional integral defined by

$${}^{ML} I_{a+}^{\varepsilon} x(\theta) = \frac{(1-\varepsilon)}{ML(\varepsilon)} x(\theta) + \frac{\varepsilon}{ML(\varepsilon) \Gamma(\varepsilon)} \int_a^{\theta} (\theta - \eta)^{\varepsilon-1} x(\eta) d\eta.$$

Definition 1.10. [35], [36]. Let $\varepsilon \in (0, 1)$, $h \in T_k$. The generalized Mittag-Leffler fractional derivative in the caputo sense of the real-valued defined by

$${}^{ML} D_{a+}^{\varepsilon, h} (x(\theta)) = \frac{ML(\varepsilon)}{(1-\varepsilon)} \int_a^{\theta} x'(\eta) E_{\varepsilon, 1} \left(-\frac{\varepsilon}{(1-\varepsilon)} (h(\theta) - h(\eta))^{\varepsilon} \right) d\eta.$$

Where

- $ML(\varepsilon)$ is a normalization constant satisfying $ML(0) = ML(1) = 1$.
- T_k the set of real-valued function and $h \in C(\Delta, \mathbb{R}_+)$ satisfies h is increasing, $h'(\theta) \neq 0, \forall \theta \in \Delta$ and $h'(\theta)$ is positive defined on Δ .
- $x, x' \in L^1([a, b], \mathbb{R})$.

Such that $E_{\varepsilon, \beta}(\cdot)$ is the two parameter Mittag-Leffler function defined by

$$E_{\varepsilon, \beta}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(\varepsilon k + \beta)}.$$

- If $\beta = 1$ this function coincides with the classical Mittag-Leffler function

$$E_{\varepsilon, 1}(\theta) = E_{\varepsilon}(\theta).$$

$$E_{\varepsilon, 1}\left(-\frac{\varepsilon}{(1-\varepsilon)}(h(\theta) - h(\eta))^\varepsilon\right) = \sum_{k=0}^{\infty} \left(-\frac{\varepsilon}{1-\varepsilon}\right)^k \frac{(h(\theta) - h(\eta))^{k\varepsilon}}{\Gamma(\varepsilon k + 1)},$$

where $E_{\varepsilon, 1}(\theta)$ is positive defined with $\varepsilon \in [0, 1]$ and $\theta \in \mathbb{R}$.

In particular when we choose $h(x) = x$, for all $x \in \Delta$, we obtain a well know fractional derivative called Mittag-Leffler fractional derivative in the caputo sense defined by

$${}^{ML}D_{a^+}^\varepsilon x(\theta) = \frac{ML(\varepsilon)}{(1-\varepsilon)} \int_a^\theta x'(\eta) E_{\varepsilon}\left(-\frac{\varepsilon}{(1-\varepsilon)}(\theta - \eta)^\varepsilon\right) d\eta.$$

Lemma 1.6. [35]. *Let $x' \in L^1(\Delta, \mathbb{R})$, $\varepsilon \in [0, 1]$, $h \in T_k$. Then the following assertion is satisfied*

$${}^{ML}I_{a^+}^{\varepsilon, h}\left({}^{ML}D_{a^+}^{\varepsilon, h}x(\theta)\right) = x(\theta) - x(a).$$

1.4 Some fixed point theorems

Fixed point theorems serves as fundamental mathematical tools for establishing the existence of solutions to differential equations. In this work, the main results are established through the application of the fixed point method, which involves reformulating the original problem into a fixed point problem. The fixed points obtained from the reformulated problem represent the solution to the original problem.

We start with the Krasnoselskii fixed point theorem which only gives the existence of a fixed point without its uniqueness .

Theorem 1.2. (*Krasnoselskii fixed point theorem*) [12], [1]. Let H be a closed bounded and convex nonempty subset of a Banach space E . Suppose that A and B map H into E such that

- (i) A is continuous and compact;
- (ii) B is a contraction mapping;
- (iii) $x, y \in H$, implies $Ax + By \in H$;

Then there exists $x \in H$ with $x = Ax + Bx$.

Banach's fixed point theorem guarantees the existence and uniqueness of a fixed point for a contraction mapping in a Banach space. Proven by Stefan Banach in 1922, the theorem fundamentally relies on the concept of contraction mappings.

Theorem 1.3. (*Banach's fixed point theorem*) [17]. Let Δ be a non-empty closed subset of a Banach space E , then any contraction mapping A of Δ into itself has a unique fixed point. i.e.,

$$\exists! x \in E, Ax = x.$$

The third fixed point theorem that we will state is that Schaefer's fixed point theorem that gives the existence of at least one fixed point.

Theorem 1.4. (*Schaefer's fixed point theorem*) [1]. Let E be a Banach space, and $A : E \rightarrow E$ a continuous and completely continuous operator. If the set

$$\Omega = \{u \in E : u = \lambda A(u), \text{ for some } \lambda \in (0, 1)\}$$

is bounded, then A has at least one fixed point in E .

1.5 Type of stability

We adopt the definitions [10], [5], [31]. Ulam-Hyers stability, generalised Ulam-Hyers stability, Ulam-Hyers -Rassias stability and generalised Ulam-Hyers -Rassias stability for the following fractional differential equation be

Let $H : C(\Delta) \rightarrow C(\Delta)$ be an operator satisfying

$$Hx = x, \text{ for } x \in C(\Delta) . \tag{1.1}$$

Definition 1.11. We say that equation (1.1) is Ulam-Hyers stable if there exists a real number $C_q > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C(\Delta, \mathbb{R})$ of the inequality

$$|x(\theta) - Hx(\theta)| \leq \epsilon, \text{ for } \theta \in \Delta. \quad (1.2)$$

There exists a solution $\bar{x} \in C(\Delta, \mathbb{R})$ of equation (1.1) with

$$|x(\theta) - \bar{x}(\theta)| \leq C_q \epsilon, \quad \theta \in \Delta. \quad (1.3)$$

Definition 1.12. The equation (1.1) is generalised Ulam-Hyers stable if there exists $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$, such that for each solution $x \in C(\Delta, \mathbb{R})$ of the inequality (1.2) there exists a solution $\bar{x} \in C(\Delta, \mathbb{R})$ of the equation (1.1)

$$|x(\theta) - \bar{x}(\theta)| \leq \varphi(\epsilon), \quad \theta \in \Delta. \quad (1.4)$$

Remark 1.1. The function $x \in C(\Delta, \mathbb{R})$ is a solution of the inequality (1.2) if and only if there exists a function $\alpha \in C(\Delta, \mathbb{R})$ (which depend on \bar{x}) such that

- (i) $|\alpha(\theta)| \leq \epsilon, \quad \forall \theta \in \Delta.$
- (ii) $Hx(\theta) = x(\theta) + \alpha(\theta), \quad \forall \theta \in \Delta.$

Definition 1.13. The equation (1.1) is Ulam-Hyers -Rassias stable with respect to $\phi \in C(\Delta, \mathbb{R}_+)$ if there exists a real number $C_q > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C(\Delta, \mathbb{R})$ of inequality

$$|x(\theta) - Hx(\theta)| \leq \phi(\theta) \epsilon, \text{ for } \theta \in \Delta. \quad (1.5)$$

There exists a solution $\bar{x} \in C(\Delta, \mathbb{R})$ of equation (1.1) with

$$|x(\theta) - \bar{x}(\theta)| \leq C_q \phi(\theta) \epsilon, \quad \forall \theta \in \Delta. \quad (1.6)$$

Definition 1.14. The equation (1.1) is generalised Ulam-Hyers -Rassias stable with respect to $\phi \in C(\Delta, \mathbb{R}_+)$ if there exists a real number $C_{q,\phi} > 0$ such that for each solution $x \in C(\Delta, \mathbb{R})$ of inequality

$$|x(\theta) - Hx(\theta)| \leq \phi(\theta), \text{ for } \theta \in \Delta. \quad (1.7)$$

There exists a solution $\bar{x} \in C(\Delta, \mathbb{R})$ of equation (1.1) with

$$|x(\theta) - \bar{x}(\theta)| \leq C_{q,\phi} \phi(\theta), \quad \forall \theta \in \Delta. \quad (1.8)$$

Remark 1.2. The function $x \in C(\Delta, \mathbb{R})$ is a solution of the inequality (1.2) if and only if there exists a function $\alpha \in C(\Delta, \mathbb{R})$ (which depend on \bar{x}) such that

- (i) $|\alpha(\theta)| \leq \phi(\theta)\epsilon, \forall \theta \in \Delta.$
- (ii) $Hx(\theta) = x(\theta) + \alpha(\theta), \forall \theta \in \Delta.$

Existence, uniqueness and stability of some Mittag-Leffler generalized problems

2.1 Introduction

In recent years, fractional calculus and integration have garnered considerable attention due to their growing importance and wide range of applications across various scientific and engineering disciplines. As a result, numerous researchers have focused on exploring their potential, particularly in areas such as physics, mechanics, and optics. For further details, readers may refer to [30], [40].

Recently, the ABC (Mittag-Leffler) derivative has been used in modeling various real-world phenomena, including several infectious diseases such as the Ebola virus, smoking dynamics, and Leptospirosis; see references [6],[24].

In the following, we consider a general nonlocal Cauchy problem for an evolution equation under the influence of a nonsingular and nonlocal fractional derivative of the Mittag-Leffler type.

Another noteworthy area of research is the study of fractional derivatives of the Mittag-Leffler type, which were investigated in a specific class of problems in the reference [34].

$$\left\{ \begin{array}{l} {}^{ML}D_{0+}^{\varepsilon} x(\theta) = \Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)), \quad \theta \in \Delta \\ x(0) = x_0 + \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta. \end{array} \right.$$

Where

- $\Delta = [0, T]$.

- Zx is the controls function.

$$Zx(\theta) = \int_0^\theta \frac{(\theta - \eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} x(\eta) d\eta.$$

- $\Lambda, \Phi : \Delta \mapsto \mathbb{R}$ are bounded linear operator and $\Omega : \Delta \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is nonlinear continuous function.

By using the the Krasnoselskii's and Banach fixed point theorem, the researchers proved the existence and uniqueness of the solution to the preceding problem.

In this chapter, we will present a study on the existence, uniqueness, and stability of the solution for the evolution problem under the generalized fractional derivative of the Mittag-Leffler type as follows

$$\begin{cases} {}^{ML}D_{0+}^{\varepsilon, h} x(\theta) = \Phi(\theta) x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)), & \theta \in \Delta \\ x(0) = \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta. \end{cases} \quad (S)$$

Where

- $\Delta = [0, T]$.

- Zx is the controls function.

$$Zx(\theta) = \int_0^\theta \frac{(\theta - \eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} x(\eta) d\eta.$$

- ${}^{ML}D_{0+}^{\varepsilon, h} x$ the generalized Mittag-Leffler fractional derivative of order $0 < \varepsilon \leq 1$.

- $h \in C^1(\Delta, \mathbb{R}_+)$ is increasing function, such that $h'(t) \neq 0$, $\forall t \in [0, T]$ and h' positive defined on Δ .

- $\Lambda, \Phi : \Delta \mapsto \mathbb{R}$ are bounded linear operator and $\Omega : \Delta \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is nonlinear continuous function.

- $x' \in L(\Delta, \mathbb{R})$.

For that, we transform the problem (S) to equivalent integral equation, then we prove the existence of a unique solution by the help of Banach and krasnoselskii's fixed point theorems, in the end we prove the solution is Ulam-Hyres stability, generalized Ulam-Hyres stability, Ulam-Hyres-Rassias stability and generalized Ulam-Hyres-Rassias stability.

2.2 Existence, uniqueness and stability

First of all, we define the solution of the problem (S).

Definition 2.1. We say that the function x from $C(\Delta, \mathbb{R})$ is a solution of the problem (S) if it satisfies the equation

$${}^{ML}D_{0+}^{\varepsilon, h} x(\theta) = \Phi(\theta) x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)), \quad \theta \in \Delta, \quad (2.1)$$

and the condition

$$x(0) = \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta. \quad (2.2)$$

Lemma 2.1. Let $0 < \varepsilon \leq 1$ and $Y : [0, T] \rightarrow \mathbb{R}$ is a continuous function, then the function x is a solution of the fractional order differential equation

$${}^{ML}D_{0+}^{\varepsilon, h} x(\theta) = Y(\theta), \quad \theta \in \Delta, \quad (2.3)$$

and satisfies the initial condition (2.2). If and only if x is a solution of the fractional order integral equation

$$x(\theta) = x_0 + \frac{(1-\varepsilon)}{ML(\varepsilon)} Y(\theta) + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} Y(\eta) d\eta, \quad (2.4)$$

where

$$Y(\theta) = \Phi(\theta) x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)), \quad \theta \in \Delta. \quad (2.5)$$

Proof. Let x be a solution of equation (2.1), it satisfies the initial condition (2.2), by applying the fractional integral ${}^{ML}I_{0+}^{\varepsilon, h}$ to the equation (2.3) and according to Lemma (1.6), we get

$$x(\theta) = x_0 + \frac{(1-\varepsilon)}{ML(\varepsilon)} Y(\theta) + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} Y(\eta) d\eta,$$

and since

$$x_0 = x(0) = \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta.$$

Then

$$\begin{aligned} x(\theta) &= \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta + \frac{(1-\varepsilon)}{ML(\varepsilon)} [\Phi(\theta) x(\theta) \\ &\quad + \Omega(\theta, x(\theta), Zx(\theta))] + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\ &\quad \times \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} [\Phi(\eta) x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta. \end{aligned}$$

□

Let $X = C(\Delta, \mathbb{R})$ be a Banach space with norm define as

$$\|x\|_{\infty} = \sup_{\theta \in \Delta} |x(\theta)|, \quad x \in X.$$

We define operator $T : X \rightarrow X$ as

$$Tx(\theta) = Fx(\theta) + Gx(\theta), \quad \theta \in \Delta,$$

such that

$$Fx(\theta) = \frac{(1-\varepsilon)}{ML(\varepsilon)} [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta))], \quad \theta \in \Delta,$$

and

$$\begin{aligned} Gx(\theta) &= \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\ &\quad \times \int_0^{\theta} h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta. \end{aligned}$$

2.2.1 Existence and uniqueness results

In this subsection we will study the existence and the uniqueness of the solution of the problem (S) using the Banach's and Krasnoselskii's fixed point theorems.

Therefore, we impose the following conditions

(A₁) There exists constant $K_{\Lambda} > 0$ such that for any $x, \bar{x} \in X$ one has

$$|\Lambda(x) - \Lambda(\bar{x})| \leq K_{\Lambda} |x - \bar{x}|.$$

(A₂) There exists constants $C_{\Lambda} > 0$, $M_{\Lambda} \geq 0$ such that

$$|\Lambda(x)| \leq C_{\Lambda} |x|^{q_1} + M_{\Lambda}, \quad \forall x \in \mathbb{R},$$

where $q_1 \in (0, 1)$.

(A₃) There exists constants $L_{\Omega}^1, L_{\Omega}^2 > 0$ such that

$$|\Omega(\theta, x, v) - \Omega(\theta, \bar{x}, \bar{v})| \leq L_{\Omega}^1 |x - \bar{x}| + L_{\Omega}^2 |v - \bar{v}|,$$

for all $(\theta, x, \bar{x}, v, \bar{v}) \in \Delta \times \mathbb{R}^4$.

(A₄) There exists constants $C_{\Omega}^1, C_{\Omega}^2 > 0, M_{\Omega} \geq 0$ such that

$$|\Omega(\theta, x, v)| \leq C_{\Omega}^1 |x|^{q_2} + C_{\Omega}^2 |v|^{q_2} + M_{\Omega},$$

for all $(\theta, x, v) \in \Delta \times \mathbb{R}^2$ and $q_2 \in (0, 1)$.

Since Φ is bounded linear functional, so there exists constant $K_{\Phi} > 0$, such that

$$|\Phi(\theta)| \leq K_{\Phi}, \theta \in \Delta.$$

Let us define a set

$$E = \{x \in X : \|x\|_{\infty} \leq R\}.$$

Where $R > 0$ is chosen such that

$$R > \max \left\{ 4B_1; (4B_3)^{\frac{1}{1-q_1}}; (4B_4)^{\frac{1}{1-q_2}} \right\}.$$

Where

$$\begin{aligned} B_1 &= \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^{\varepsilon}}{ML(\varepsilon)\Gamma(\varepsilon)} \right) M_{\Omega} + \frac{T^{\varepsilon}}{\Gamma(\varepsilon+1)} M_{\Lambda} \right], \\ B_3 &= \left[\frac{T^{\varepsilon}}{\Gamma(\varepsilon+1)} C_{\Lambda} \right], \\ B_4 &= \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^{\varepsilon}}{ML(\varepsilon)\Gamma(\varepsilon)} \right) \left(C_{\Omega}^1 + C_{\Omega}^2 \left(\frac{T^{\varepsilon}}{\Gamma(\varepsilon+1)} \right)^{q_2} \right) \right]. \end{aligned}$$

It is clear that E is a bounded, closed and convex subset of X .

- We will now demonstrate the existence of the solution of the problem (S) using the Krasnoselskii's fixed point theorem.

Theorem 2.1. *Assuming that condition (A₁), (A₂), (A₃) and (A₄) are hold and if $\Pi \in (0, 1)$ and $B_2 < \frac{1}{4}$ such that*

$$\Pi = \frac{(1-\varepsilon)}{ML(\varepsilon)} \left[K_{\Phi} + L_{\Omega}^1 + L_{\Omega}^2 \frac{T^{\varepsilon}}{\Gamma(\varepsilon+1)} \right]. \quad (2.7)$$

And

$$B_2 = \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^{\varepsilon}}{ML(\varepsilon)\Gamma(\varepsilon)} \right) K_{\Phi} \right]. \quad (2.8)$$

Then the problem (S) has at least one solution in E .

Proof. For this the proof will be done in some steps.

Step 1 We prove that the operator G is continuous.

We consider the sequence $(u_n)_n$ converging to u in $C(\Delta, \mathbb{R})$.

Then, for every $\theta \in \Delta$, and according to the conditions (A_1) and (A_3) , we have

$$\begin{aligned}
|Gu_n(\theta) - Gu(\theta)| &= \left| \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(u_n(\eta)) d\eta + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right. \\
&\quad \times \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)u_n(\eta) + \Omega(\eta, u_n(\eta), Zu_n(\eta))] d\eta \\
&\quad \left. - \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(u(\eta)) d\eta - \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right. \\
&\quad \times \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)u(\eta) + \Omega(\eta, u(\eta), Zu(\eta))] d\eta \Big| \\
&\leq \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} |\Lambda(u_n(\eta)) - \Lambda(u(\eta))| d\eta \\
&\quad + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \times [|\Phi(\eta)||u_n(\eta) - u(\eta)| \\
&\quad + |\Omega(\eta, u_n(\eta), Zu_n(\eta)) - \Omega(\eta, u(\eta), Zu(\eta))|] d\eta.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\Omega(\eta, u_n(\eta), Zu_n(\eta)) - \Omega(\eta, u(\eta), Zu(\eta))| &\leq L_\Omega^1 |u_n(\eta) - u(\eta)| \\
&\quad + L_\Omega^2 |Zu_n(\eta) - Zu(\eta)| \\
&\leq L_\Omega^1 |u_n(\eta) - u(\eta)| \\
&\quad + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} |u_n(\eta) - u(\eta)| \\
&\leq \left(L_\Omega^1 + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right) \\
&\quad \times |u_n(\eta) - u(\eta)|.
\end{aligned}$$

Through substitution we obtain

$$\begin{aligned}
|Gu_n(\theta) - Gu(\theta)| &\leq \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} K_\Lambda |u_n(\eta) - u(\eta)| + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\
&\quad \times \left[K_\Phi + L_\Omega^1 + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right] |u_n(\eta) - u(\eta)| \\
&\leq \left[\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} K_\Lambda + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \left(K_\Phi + L_\Omega^1 + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right) \right] \\
&\quad \times \|u_n - u\|_\infty.
\end{aligned}$$

If, $\|u_n - u\|_\infty \rightarrow 0$ as $n \rightarrow +\infty$ then, $\|Gu_n(\theta) - Gu(\theta)\|_\infty \rightarrow 0$.

Hence, G is continuous.

Step 2 We prove that the operator G is uniformly bounded in $C(\Delta, \mathbb{R})$.

Let $x \in C(\Delta, \mathbb{R})$ then, for all $\theta \in \Delta$, and according to the conditions

$(A_2) - (A_4)$, and, we have

$$|Gx(\theta)| \leq \left| \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta \right| + \left| \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right. \\ \left. \times \int_0^{\theta_1} h'(\eta) (h(\theta_1) - h(\eta))^{\varepsilon-1} [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta \right|.$$

On the other hand, we have

$$|\Omega(\theta, x(\theta), Zx(\theta))| \leq C_\Omega^1 |x(\theta)|^{q_2} + C_\Omega^2 |Zx(\theta)|^{q_2} + M_\Omega \\ \leq C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega.$$

Through substitution we obtain

$$|Gx(\theta)| \leq \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} [C_\Lambda \|x\|^{q_1} + M_\Lambda] + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\ \times \left[\int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} d\eta \right] \\ \times \left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right] \\ \leq \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} [C_\Lambda \|x\|^{q_1} + M_\Lambda] + \frac{1}{ML(\varepsilon)\Gamma(\varepsilon)} [(h(T) - h(0))^\varepsilon] \\ \times \left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right] \\ \leq C_{\Omega,\Lambda} \|x\|^{q_3} + ML_{\Omega,\Lambda} \\ \leq l.$$

Therefore

$$\|Gx\|_\infty \leq l.$$

Where

$$C_{\Omega,\Lambda} = \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} C_\Lambda + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \left[C_\Omega^1 + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \right]. \\ ML_{\Omega,\Lambda} = \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} M_\Lambda + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} [K_\Phi R + M_\Omega]. \\ q_3 = \min\{q_1, q_2\}. \\ l = C_{\Omega,\Lambda} \|x\|^{q_3} + ML_{\Omega,\Lambda}.$$

The operator G is uniformly bounded set in $C(\Delta, \mathbb{R})$.

Step 3 We prove that The operator G is equicontinuous set in $C(\Delta, \mathbb{R})$.

Further let $\theta_1, \theta_2 \in \Delta$ such that $\theta_1 > \theta_2$, and according to the condition (A_4) .

Then we have

$$\begin{aligned}
|Gx(\theta_1) - Gx(\theta_2)| &= \left| \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right. \\
&\quad \times \int_0^{\theta_1} h'(\eta) (h(\theta_1) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta \\
&\quad - \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta - \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\
&\quad \times \int_0^{\theta_2} h'(\eta) (h(\theta_2) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta \Big| \\
&\leq \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \left(\left| \int_0^{\theta_1} h'(\eta) (h(\theta_1) - h(\eta))^{\varepsilon-1} d\eta \right. \right. \\
&\quad \left. \left. - \int_0^{\theta_2} h'(\eta) (h(\theta_2) - h(\eta))^{\varepsilon-1} d\eta \right| \right) \\
&\quad \times |\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))| \\
&\leq \frac{[(h(\theta_1) - h(0))^\varepsilon - (h(\theta_2) - h(0))^\varepsilon]}{ML(\varepsilon)\Gamma(\varepsilon)} \\
&\quad \times \left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right] \\
&\leq \frac{\left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right]}{ML(\varepsilon)\Gamma(\varepsilon)} \\
&\quad \times [(h(\theta_1) - h(0))^\varepsilon - (h(\theta_2) - h(0))^\varepsilon].
\end{aligned}$$

On the other hand, we have

$$g(\theta) = (h(\theta) - h(0))^\varepsilon,$$

Through substitution we obtain

$$\begin{aligned}
|Gx(\theta_1) - Gx(\theta_2)| &\leq \frac{\left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right]}{ML(\varepsilon)\Gamma(\varepsilon)} \\
&\quad \times g'(\theta) (\theta_1 - \theta_2),
\end{aligned}$$

when $\theta_1 \rightarrow \theta_2$ the right-hand side of the last inequality tends to zero.

$$\|Gx(\theta_1) - Gx(\theta_2)\|_\infty \leq \varepsilon.$$

Hence G is equicontinuous set in $C(\Delta, \mathbb{R})$.

Consequently, according to three steps and to the Ascoli-Arzelà theorem, G is completely continuous.

Step 4 We prove that $Fx(\theta) + Gx(\theta) \in E$ for any $x \in C(\Delta, \mathbb{R})$.

Then, for all $\theta \in \Delta$, and according to the conditions $(A_2) - (A_4)$ we have

$$\begin{aligned}
 |Fx(\theta) + Gx(\theta)| &= \left| \frac{(1-\varepsilon)}{ML(\varepsilon)} [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta))] \right. \\
 &\quad + \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\
 &\quad \times \int_0^\theta h'(\eta)(h(\theta) - h(\eta))^{\varepsilon-1} [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta \left. \right| \\
 &\leq \frac{(1-\varepsilon)}{ML(\varepsilon)} [|\Phi(\theta)||x(\theta)| + |\Omega(\theta, x(\theta), Zx(\theta))|] \\
 &\quad + \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} |\Lambda(x(\eta))| d\eta + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\
 &\quad \times \int_0^\theta h'(\eta)(h(\theta) - h(\eta))^{\varepsilon-1} [|\Phi(\eta)||x(\eta)| + |\Omega(\eta, x(\eta), Zx(\eta))|] d\eta \\
 &\leq \frac{(1-\varepsilon)}{ML(\varepsilon)} \left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right] \\
 &\quad + \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} [C_\Lambda \|x\|^{q_1} + M_\Lambda] + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \\
 &\quad \times \left[K_\Phi R + C_\Omega^1 \|x\|^{q_2} + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \|x\|^{q_2} + M_\Omega \right] \\
 &\leq \left[\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right] M_\Omega + \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} M_\Lambda \\
 &\quad + \left[\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right] K_\Phi R + \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} C_\Lambda \|x\|^{q_1} \\
 &\quad + \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right) \left(C_\Omega^1 + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \right) \right] \|x\|^{q_2} \\
 &\leq \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right) M_\Omega + \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} M_\Lambda \right] \\
 &\quad + \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right) K_\Phi \right] R + \left[\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} C_\Lambda \right] R^{q_1} \\
 &\quad + \left[\left(\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \right) \left(C_\Omega^1 + C_\Omega^2 \left(\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right)^{q_2} \right) \right] R^{q_2} \\
 &\leq R.
 \end{aligned}$$

Thus, $Fx(\theta) + Gx(\theta) \in E$.

Step 5 We prove that the operator F is contraction in $C(\Delta, \mathbb{R})$.

Let $x, \bar{x} \in C(\Delta, \mathbb{R})$, and for all $\theta \in \Delta$, and according to the condition (A_3) we have

$$\begin{aligned} |Fx(\theta) - F\bar{x}(\theta)| &= \left| \frac{(1-\varepsilon)}{ML(\varepsilon)} [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta))] \right. \\ &\quad \left. - \frac{(1-\varepsilon)}{ML(\varepsilon)} [\Phi(\theta)\bar{x}(\theta) + \Omega(\theta, \bar{x}(\theta), Z\bar{x}(\theta))] \right| \\ &\leq \frac{(1-\varepsilon)}{ML(\varepsilon)} (|\Phi(\theta)||x(\theta) - \bar{x}(\theta)| \\ &\quad + |\Omega(\theta, x(\theta), Zx(\theta)) - \Omega(\theta, \bar{x}(\theta), Z\bar{x}(\theta))|). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |Zx(\theta) - Z\bar{x}(\theta)| &= |I_0^\varepsilon x(\theta) - I_0^\varepsilon \bar{x}(\theta)| \\ &= I_0^\varepsilon |x(\theta) - \bar{x}(\theta)| \\ &= \frac{\theta^\varepsilon}{\Gamma(\varepsilon+1)} |x(\theta) - \bar{x}(\theta)|, \quad \theta \in \Delta \\ &\leq \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} |x(\theta) - \bar{x}(\theta)|. \end{aligned}$$

$$\begin{aligned} |\Omega(\theta, x(\theta), Zx(\theta)) - \Omega(\theta, \bar{x}(\theta), Z\bar{x}(\theta))| &\leq L_\Omega^1 |x(\theta) - \bar{x}(\theta)| \\ &\quad + L_\Omega^2 |Zx(\theta) - Z\bar{x}(\theta)| \\ &\leq L_\Omega^1 |x(\theta) - \bar{x}(\theta)| \\ &\quad + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} |x(\theta) - \bar{x}(\theta)|. \end{aligned}$$

Through substitution we obtain

$$\begin{aligned} |Fx(\theta) - F\bar{x}(\theta)| &\leq \frac{(1-\varepsilon)}{ML(\varepsilon)} \left[K_\Phi |x(\theta) - \bar{x}(\theta)| + L_\Omega^1 |x(\theta) - \bar{x}(\theta)| \right. \\ &\quad \left. + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} |x(\theta) - \bar{x}(\theta)| \right] \\ &\leq \frac{(1-\varepsilon)}{ML(\varepsilon)} \left[K_\Phi \|x - \bar{x}\|_\infty + L_\Omega^1 \|x - \bar{x}\|_\infty \right. \\ &\quad \left. + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \|x - \bar{x}\|_\infty \right] \\ &\leq \frac{(1-\varepsilon)}{ML(\varepsilon)} \left[K_\Phi + L_\Omega^1 + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right] \|x - \bar{x}\|_\infty \\ &\leq \Pi \|x - \bar{x}\|_\infty, \end{aligned}$$

thus

$$\|Fx - F\bar{x}\|_\infty \leq \Pi \|x - \bar{x}\|_\infty.$$

Then the operator F is a contraction.

- According to the krasnoselskii's fixed point theorems the problem (S) admits at least one solution $x \in E$.

□

- We will now demonstrate the uniqueness of the solution of the problem (S) using the Banah fixed point theorem.

Theorem 2.2. *Assume that the assumptions (A_1) through (A_3) are satisfied, and that there exists a constant Σ such that $0 < \Sigma < 1$, and a constant satisfying*

$$0 < \Sigma + \Pi < 1. \quad (2.9)$$

Where

$$\Sigma = \left[\frac{T^\varepsilon}{\Gamma(\varepsilon + 1)} K_\Lambda + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \left(K_\Phi + L_\Omega^1 + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon + 1)} \right) \right]. \quad (2.10)$$

Then T has unique solution in $C(\Delta, \mathbb{R})$.

Proof. It is clear that the fixed point of T is a solution of the problem (S). We will now demonstrate the uniqueness of a fixed point for T through the proof that T is a contraction.

Let $x, \bar{x} \in C(\Delta, \mathbb{R})$, then for all $\theta \in \Delta$, we have

$$|Tx(\theta) - T\bar{x}(\theta)| \leq |Fx(\theta) - F\bar{x}(\theta)| + |Gx(\theta) - G\bar{x}(\theta)|.$$

On the other hand, we have

$$\begin{aligned}
|Gx(\theta) - G\bar{x}(\theta)| &= \left| \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta \right. \\
&\quad + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta \\
&\quad - \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(\bar{x}(\eta)) d\eta \\
&\quad - \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \left. [\Phi(\eta)\bar{x}(\eta) + \Omega(\eta, \bar{x}(\eta), Z\bar{x}(\eta))] d\eta \right| \\
&\leq \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} |\Lambda(x(\eta)) - \Lambda(\bar{x}(\eta))| d\eta \\
&\quad + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\varepsilon-1} d\eta \\
&\quad \times [|\Phi(\eta)| |x(\eta) - \bar{x}(\eta)| \\
&\quad + |\Omega(\eta, x(\eta), Zx(\eta)) - \Omega(\eta, \bar{x}(\eta), Z\bar{x}(\eta))|] \\
&\leq \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} K_\Lambda \|x - \bar{x}\|_\infty + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} [K_\Phi \|x - \bar{x}\|_\infty \\
&\quad + L_\Omega^1 \|x - \bar{x}\|_\infty + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \|x - \bar{x}\|_\infty] \\
&\leq \left[\frac{T^\varepsilon}{\Gamma(\varepsilon+1)} K_\Lambda + \frac{(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \left(K_\Phi + L_\Omega^1 + L_\Omega^2 \frac{T^\varepsilon}{\Gamma(\varepsilon+1)} \right) \right] \\
&\quad \times \|x - \bar{x}\|_\infty \\
&\leq \Sigma \|x - \bar{x}\|_\infty.
\end{aligned}$$

Therefore

$$\|G(x) - G(\bar{x})\|_\infty \leq \Sigma \|x - \bar{x}\|_\infty.$$

Through substitution we obtain

$$\begin{aligned}
|Tx(\theta) - T\bar{x}(\theta)| &\leq \leq \Pi \|x - \bar{x}\|_\infty + \Sigma \|x - \bar{x}\|_\infty \\
&\leq (\Pi + \Sigma) \|x - \bar{x}\|_\infty.
\end{aligned}$$

Therefore

$$\|Tx - T\bar{x}\|_\infty \leq (\Pi + \Sigma) \|x - \bar{x}\|_\infty.$$

The operator T is a contraction, and according to Banach fixed point theorem, the problem (S) admits a unique solution in $C(\Delta, \mathbb{R})$. \square

2.2.2 Stability results

In this section, we study a type of stability for problem (S), known as Ulam-Hyres stability, generalized Ulam-Hyres stability, Ulam-Hyres-Rassias stability and generalized Ulam-Hyres-Rassias stability.

Lemma 2.2. *If x is a solution of the following fractional differential inequality, for $\epsilon > 0$*

$$\left| {}^{ML}D_{0+}^{\epsilon, h}(x(\theta)) - \Phi(\theta)x(\theta) - \Omega(\theta, x(\theta), Zx(\theta)) \right| \leq \epsilon, \theta \in \Delta. \quad (2.10)$$

Then x is also a solution satisfies the following inequality

$$|x(\theta) - Tx(\theta)| \leq \nabla \epsilon, \theta \in \Delta. \quad (2.11)$$

Where

$$\nabla = \left[\frac{(1 - (1 - (h(T) - h(0))^\epsilon) \epsilon)}{ML(\epsilon)} \right].$$

Proof. Let $x \in C(\Delta, \mathbb{R})$ be a solution of the inequality (2.10), for $\epsilon > 0$, using the lemma (2.1) and Remark (1.1) related to the continuous function α where $|\alpha(\theta)| < \epsilon$, for all $\theta \in \Delta$. Then we have

$$\begin{aligned} x(\theta) &= \int_0^T \frac{(T-\eta)^{\epsilon-1}}{\Gamma(\epsilon)} \Lambda(x(\eta)) d\eta + \frac{(1-\epsilon)}{ML(\epsilon)} [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)) + \alpha(\theta)] \\ &\quad + \frac{\epsilon}{ML(\epsilon)\Gamma(\epsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\epsilon-1} [\Phi(\eta)x(\eta) \\ &\quad + \Omega(\eta, x(\eta), Zx(\eta)) + \alpha(\eta)] d\eta, \end{aligned}$$

$$\begin{aligned} Tx(\theta) &= \int_0^T \frac{(T-\eta)^{\epsilon-1}}{\Gamma(\epsilon)} \Lambda(x(\eta)) d\eta + \frac{(1-\epsilon)}{ML(\epsilon)} [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta))] \\ &\quad + \frac{\epsilon}{ML(\epsilon)\Gamma(\epsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\epsilon-1} [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta. \end{aligned}$$

According to Remark (1.1) we obtain

$$\begin{aligned}
|x(\theta) - Tx(\theta)| &= \left| \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta + \frac{(1-\varepsilon)}{ML(\varepsilon)} \right. \\
&\quad \times [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta)) + \alpha(\theta)] \\
&\quad + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta)(h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta)) + \alpha(\eta)] d\eta \\
&\quad - \int_0^T \frac{(T-\eta)^{\varepsilon-1}}{\Gamma(\varepsilon)} \Lambda(x(\eta)) d\eta + \frac{(1-\varepsilon)}{ML(\varepsilon)} \\
&\quad \times [\Phi(\theta)x(\theta) + \Omega(\theta, x(\theta), Zx(\theta))] \\
&\quad - \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta)(h(\theta) - h(\eta))^{\varepsilon-1} \\
&\quad \times [\Phi(\eta)x(\eta) + \Omega(\eta, x(\eta), Zx(\eta))] d\eta \left. \right| \\
&= \left| \frac{(1-\varepsilon)}{ML(\varepsilon)} \alpha(\theta) + \frac{\varepsilon}{ML(\varepsilon)\Gamma(\varepsilon)} \int_0^\theta h'(\eta)(h(\theta) - h(\eta))^{\varepsilon-1} \alpha(\eta) d\eta \right| \\
&\leq \left[\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{\varepsilon(h(T) - h(0))^\varepsilon}{ML(\varepsilon)\varepsilon\Gamma(\varepsilon)} \right] \alpha(\theta) \\
&\leq \left[\frac{(1-\varepsilon)}{ML(\varepsilon)} + \frac{\varepsilon(h(T) - h(0))^\varepsilon}{ML(\varepsilon)} \right] \alpha(\theta) \\
&\leq \left[\frac{(1 - (1 - [h(T) - h(0)]^\varepsilon) \varepsilon)}{ML(\varepsilon)} \right] \epsilon \\
&\leq \nabla \epsilon.
\end{aligned}$$

And this satisfies the inequality (2.11) □

Theorem 2.3. *Suppose that conditions $(A_1) - (A_3)$ and (2.9) satisfied, then the problem (S) is Ulam-Hyers stable and also generalized Ulam-Hyers stable.*

Proof. Under conditions $(A_1) - (A_3)$ and (2.9), the problem (S) has a unique solution in $C(\Delta, \mathbb{R})$, and Let $x \in C(\Delta, \mathbb{R})$ be a solution of the inequality (2.10). Let $\bar{x} \in C(\Delta, \mathbb{R})$ the unique solution of the inequality (2.3).

Then for all $\theta \in \Delta$, we have

$$\begin{aligned}
|x(\theta) - \bar{x}(\theta)| &\leq \sup_{\theta \in \Delta} |x(\theta) - Tx(\theta)| + \sup_{\theta \in \Delta} |Tx(\theta) - \bar{x}(\theta)| \\
&\leq \nabla \epsilon + (\Sigma + \Pi) \|x - \bar{x}\|_\infty \\
&\leq \frac{\nabla}{(1 - [\Sigma + \Pi])} \epsilon.
\end{aligned}$$

Then

$$\|x - \bar{x}\|_{\infty} \leq \frac{\nabla}{(1 - [\Sigma + \Pi])} \epsilon.$$

Setting

$$C_q = \frac{\nabla}{(1 - [\Sigma + \Pi])},$$

we obtain

$$\|x - \bar{x}\|_{\infty} \leq C_q \epsilon.$$

Consequently, the problem (S) is Ulam-Hyers stable. If we take $\varphi(\epsilon) = C_q \epsilon$, $\varphi(0) = 0$, then we get the generalized Ulam-Hyers stability of the problem (S). \square

Lemma 2.3. *If x is a solution of the following fractional differential inequality, for $\epsilon > 0$*

$$\left| {}^{ML}D_{0^+}^{\epsilon, h}(x(\theta)) - \Phi(\theta)x(\theta) - \Omega(\theta, x(\theta), Zx(\theta)) \right| \leq \phi(\theta)\epsilon, \theta \in \Delta. \quad (2.12)$$

Then x is also a solution to the following inequality

$$|x(\theta) - Tx(\theta)| \leq \left[\frac{(1 - (1 - (h(T) - h(0))^\epsilon)\epsilon)}{ML(\epsilon)} \right] \phi(\theta)\epsilon, \theta \in \Delta. \quad (2.13)$$

Where

$$\nabla = \left[\frac{(1 - (1 - (h(T) - h(0))^\epsilon)\epsilon)}{ML(\epsilon)} \right].$$

Proof. Let $x \in C(\Delta, \mathbb{R})$ be a solution of the inequality (2.12), for $\epsilon > 0$, using the lemma (2.1) and Remark (1.2) related to the continuous function α where $|\alpha(\theta)| < \phi(\theta)\epsilon$, for all $\theta \in \Delta$.

According to Remark (1.2) and conditions (A_1) and (A_3) , we obtain

$$\begin{aligned} |x(\theta) - Tx(\theta)| &= \left| \frac{(1 - \epsilon)}{ML(\epsilon)} \alpha(\theta) + \frac{\epsilon}{ML(\epsilon)\Gamma(\epsilon)} \int_0^\theta h'(\eta) (h(\theta) - h(\eta))^{\epsilon-1} \alpha(\eta) d\eta \right| \\ &\leq \left[\frac{(1 - \epsilon)}{ML(\epsilon)} + \frac{\epsilon (h(T) - h(0))^\epsilon}{ML(\epsilon)\epsilon\Gamma(\epsilon)} \right] \alpha(\theta) \\ &\leq \left[\frac{(1 - \epsilon)}{ML(\epsilon)} + \frac{\epsilon (h(T) - h(0))^\epsilon}{ML(\epsilon)} \right] \alpha(\theta) \\ &\leq \left[\frac{(1 - (1 - [h(T) - h(0)]^\epsilon)\epsilon)}{ML(\epsilon)} \right] \phi(\theta)\epsilon \\ &\leq \nabla \phi(\theta)\epsilon. \end{aligned}$$

And this satisfies the inequality (2.13). \square

Theorem 2.4. *Suppose that conditions $(A_1) - (A_3)$ and (2.9) are satisfied, then the problem (S) is Ulam-Hyers-Rassias stable and also generalized Ulam-Hyers-Rassias stable.*

Proof. Let $x \in C(\Delta, \mathbb{R})$ be a solution of the inequality (2.12).

Let $\bar{x} \in C(\Delta, \mathbb{R})$ the unique solution of the inequality (2.3).

Then for all $\theta \in \Delta$, we have

$$\begin{aligned} |x(\theta) - \bar{x}(\theta)| &\leq |x(\theta) - Tx(\theta)| + |Tx(\theta) - \bar{x}(\theta)| \\ &\leq \nabla\phi(\theta)\epsilon + (\Sigma + \Pi)\|x - \bar{x}\|_\infty \\ &\leq \frac{\nabla}{(1 - [\Sigma + \Pi])}\phi(\theta)\epsilon. \end{aligned}$$

Setting

$$C_q = \frac{\nabla}{(1 - [\Sigma + \Pi])},$$

we obtain

$$\|x - \bar{x}\|_\infty \leq C_q \phi(\theta)\epsilon.$$

Hence problem (S) is Ulam-Hyers-Rassias stable. If we take $C_{q, \phi} = C_q \epsilon$, then we get the generalised Ulam-Hyers-Rassias stable of the problem (S). \square

2.3 Example

corresponding to the proposed problem, we provide the following example that demonstrate consistence to the main theorems.

$$\begin{cases} {}^{ML}D^{\frac{1}{2}, t^3} x(\theta) = \frac{\exp(-\theta)x(\theta)}{10\pi} + \frac{1}{500\pi} \left[\sin\left(|x|^{\frac{2}{3}}\right) + \cos\left(|Zx|^{\frac{2}{3}}\right) \right] + 4\theta^2 + 9, \\ x(0) = \int_0^1 \frac{(1-\eta)^{-\frac{1}{2}} |x|^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{100} d\eta. \end{cases} \quad (K)$$

Where

$$Zx(\theta) = \int_0^\theta \frac{(\theta - \eta)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} x(\eta) d\eta.$$

Φ, Λ, Ω are continuous

$$\begin{aligned} \Phi(\theta) &= \frac{\exp(-\theta)}{10\pi}, \\ \Lambda(x) &= \frac{|x|^{\frac{1}{2}}}{100}, \\ \Omega(\theta, x(\theta), Zx(\theta)) &= \frac{1}{500\pi} \left[\sin\left(|x|^{\frac{2}{3}}\right) + \cos\left(|Zx|^{\frac{2}{3}}\right) \right] + 4\theta^2 + 9. \end{aligned}$$

h continuous and increasing in $[0, 1]$

$$h(t) = t^3.$$

Let $x, \bar{x} \in X$ and $\theta \in \Delta$ we have

$$|\Phi(\theta)| \leq \frac{1}{10\pi}.$$

$$|\Omega(\theta, x(\theta), Zx(\theta))| \leq \frac{1}{500\pi} \left[|x|^{\frac{2}{3}} + |Zx|^{\frac{2}{3}} \right] + 13.$$

$$|\Lambda(x)| \leq \frac{1}{100} |x|^{\frac{1}{2}}.$$

$$|\Lambda(x) - \Lambda(\bar{x})| \leq \frac{1}{100} |x - \bar{x}|.$$

$$|\Omega(\theta, x(\theta), Zx(\theta)) - \Omega(\theta, \bar{x}(\theta), Z\bar{x}(\theta))| \leq \frac{1}{500\pi} [|x - \bar{x}| + |Zx - Z\bar{x}|].$$

Thus Ω and Λ satisfy all conditions (A_1) , (A_2) , (A_3) , and (A_4) for

$$\begin{aligned} K_\Lambda &= C_\Lambda = \frac{1}{100}, M_\Lambda = 0, M_\Omega = 13, q_1 = \frac{1}{2}, q_2 = \frac{2}{3}, \\ C_\Omega^1 &= C_\Omega^2 = L_\Omega^1 = L_\Omega^2 = \frac{1}{500\pi}, K_\Phi = \frac{1}{10\pi}, \varepsilon = \frac{1}{2}. \end{aligned}$$

$$ML(\varepsilon) = 1 - \varepsilon + \frac{\varepsilon}{\Gamma(\varepsilon)} = \frac{0.5(\sqrt{\pi} + 1)}{\sqrt{\pi}}.$$

$$B_2 \simeq 4.3312 \times 10^{-2} < \frac{1}{4}.$$

$$\Pi \simeq 0.0647971124 < 1.$$

By theoreme (2.1) the probleme (K) has one solution at least further.

$$\Sigma \simeq 0.0843995034 < 1.$$

As

$$\Pi + \Sigma \simeq 0.1491906274 < 1.$$

Hence by theoreme (2.2) the considered problem possess unique solution.

$$\nabla \simeq 1.2786173882 > 0.$$

$$C_q \simeq 1.5028247565 > 0.$$

Therefore, according to the theorem (2.3), and theorem (2.4) the problem (K) is stable in the sense of Ulam-Hyres stability, generalized Ulam-Hyres stability, Ulam-Hyres-Rassias stability and generalized Ulam-Hyres-Rassias stability.

$$B_1 = 17.689.$$

$$B_3 \simeq 1.1284 \times 10^{-2}.$$

$$B_4 \simeq 1.8051 \times 10^{-3}.$$

$$R > \max \left\{ 4B_1; (4B_3)^{\frac{1}{1-q_1}}; (4B_4)^{\frac{1}{1-q_2}} \right\}.$$

$$R > \max \left\{ 70.756; 2.0373 \times 10^{-3}; 3.7643 \times 10^{-7} \right\}.$$

$$R > 70.756.$$

The problem (K) has at least one solution x satisfying $\|x\|_\infty \leq R$

where $R > 70.756$.

Existence, uniqueness and stability of first type hybrid fractional differential equations involving the p-Laplace operator

3.1 Introduction

In this chapter, we will present the study of the existence, uniqueness, and stability of the solution for the problem related to first-type hybrid fractional differential equations involving the p-Laplace operator as follows

$$\begin{cases} {}^C D_{0+}^{\alpha} \left(\frac{\phi_P({}^R D_{0+}^{\beta}(x(t)))}{f(t, x(t))} \right) = g \left(t, x(t), {}^C D_{0+}^{\alpha} \left(\frac{\phi_P({}^R D_{0+}^{\beta}(x(t)))}{f(t, x(t))} \right) \right), & t \in J, \\ \phi_P({}^R D_{0+}^{\beta}(x(0))) = \varphi f(0, x(0)), \\ \lim_{t \rightarrow 0} (t^{1-\beta} x(t)) = 0. \end{cases} \quad (P)$$

Where

- $J = [0, 1]$ and $\varphi \in \mathbb{R}$.
- ${}^C D_{0+}^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$.
- ${}^R D_{0+}^{\beta}$ is the Riemann-Liouville fractional derivative of order $0 < \beta < 1$.
- $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}^2, \mathbb{R})$ are two nonlinear and continuous functions.
- ϕ_P is the p – Laplacian operator.

For that, we transform the problem (P) to equivalent integral equation, then we prove the existence of a unique solution by the help of Banach and Schaefer's fixed point theorems, in the end we prove the solution is Ulam-Hyres and Ulam-Hyres-Rassias stable.

3.2 Existence, uniqueness and stability results

First of all, we define the solution of the problem (P).

Definition 3.1. We say that the function $x(t)$ from $C(J, \mathbb{R})$ is a solution of the problem (P) if it satisfies the equation

$${}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) = g \left(t, x(t), {}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) \right), \quad t \in J, \quad (3.1)$$

and the conditions

$$\lim_{t \rightarrow 0} (t^{1-\beta} x(t)) = 0, \quad (3.2)$$

and

$$\phi_P \left({}^R D_{0+}^{\beta} (x(0)) \right) = \varphi f(0, x(0)). \quad (3.3)$$

Lemma 3.1. *If $0 < \alpha, \beta < 1$ and $h : J \rightarrow \mathbb{R}$ is a continuous function, then the function x is a solution to the first-order hybrid fractional differential equation in the presence of the p -Laplace operator.*

$${}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) = h(t), \quad t \in J, \quad (3.4)$$

and satisfies the two initial conditions (3.2), (3.3).

If and only if x is a solution of the fractional order integral equation

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\phi_q \left(\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} h(r) dr \right) \right] ds, \quad t \in J, \quad (3.5)$$

Proof. For x to be a solution to equation (3.1), by applying the fractional integral of order $\alpha : I_{0+}^\alpha$ to the equation (3.4) and according to Lemma (1.4) and Lemma (1.5), we get

$$\begin{aligned} I_{0+}^\alpha \left[{}^C D_{0+}^\alpha \left(\frac{\phi_P \left({}^R D_{0+}^\beta (x(t)) \right)}{f(t, x(t))} \right) \right] &= I_{0+}^\alpha h(t), \\ \frac{\phi_P \left({}^R D_{0+}^\beta (x(t)) \right)}{f(t, x(t))} - c_0 &= I_{0+}^\alpha h(t) \\ \frac{\phi_P \left({}^R D_{0+}^\beta (x(t)) \right)}{f(t, x(t))} &= c_0 + I_{0+}^\alpha h(t) \end{aligned}$$

and hence

$$\frac{\phi_P \left({}^R D_{0+}^\beta (x(t)) \right)}{f(t, x(t))} = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

from condition (3.3)

$$\begin{aligned} c_0 &= \frac{\phi_P \left({}^R D_{0+}^\beta (x(0)) \right)}{f(0, x(0))} = \varphi \\ c_0 &= \varphi, \end{aligned}$$

substituting c_0 gives

$$\frac{\phi_P \left({}^R D_{0+}^\beta (x(t)) \right)}{f(t, x(t))} = \varphi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

hence

$$\phi_P \left({}^R D_{0+}^\beta (x(t)) \right) = \varphi f(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

by applying ϕ_p^{-1} to both sides of the previous equation and according to Lemma (1.1) we obtain

$$\begin{aligned} \phi_p^{-1} \phi_P \left({}^R D_{0+}^\beta (x(t)) \right) &= \phi_p^{-1} \left[\varphi f(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right], \\ {}^R D_{0+}^\beta (x(t)) &= \phi_q \left[\varphi f(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right], \end{aligned}$$

by applying the fractional integral of order $\beta (I_{0+}^\beta)$ and according to Lemma (1.3) and

Lemma (1.5), we get

$$\begin{aligned}
I_{0+}^{\beta} \left[{}^R D_{0+}^{\beta} (x(t)) \right] &= I_{0+}^{\beta} \left[\phi_q \left(\varphi f(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right) \right], \\
x(t) - k t^{\beta-1} &= \phi_q I_{0+}^{\beta} \left[\varphi f(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right], \\
x(t) &= k t^{\beta-1} + \phi_q \left[I_{0+}^{\beta} \varphi f(t, x(t)) + I_{0+}^{\beta} \left(\frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right) \right], \\
x(t) &= k t^{\beta-1} + \phi_q \left[\frac{\varphi}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \right. \\
&\quad \left. \left(f(s, x(s)) \int_0^s (s-r)^{\alpha-1} h(r) dr \right) ds \right], \\
x(t) &= k t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \right. \\
&\quad \left. h(r) dr \right] ds
\end{aligned}$$

from condition (3.2) we get

$$k = \lim_{t \rightarrow 0} t^{1-\beta} x(t) = 0,$$

thus

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} h(r) dr \right] ds,$$

such that

$$h(t) = g \left(t, x(t), {}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) \right), \quad t \in J.$$

It is clear that the problem with the first-order hybrid mixed fractional differential equations in the presence of the p-Laplace operator (3.4), (3.3), (3.2) satisfies the integral equation (3.5). \square

Where $X = C(J, \mathbb{R})$ be a Banach space with norm define as

$$\|x\|_{\infty} = \sup_{t \in J} |x(t)|, \quad \text{for } x(t) \in X.$$

We define operator $G : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ by

$$Gx(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right] ds, \quad t \in J.$$

Such that

$$\sigma_x(t) = g \left(t, x(t), {}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) \right), \quad t \in J.$$

3.2.1 Existence and uniqueness results

In this subsection, we will use the Schaefer's fixed point theorem and Banach fixed point theorem to study the existence and the uniqueness of the solution for problem (P) under certain conditions imposed on the function g and f .

Therefore, we impose the following conditions

(H₁) There exists a constant $M_g \in \mathbb{R}_+^*$ such that

$$|g(t, x, y)| \leq M_g, \quad \forall (t, x, y) \in J \times \mathbb{R}^2,$$

(H₂) There exists a constant $M_f \in \mathbb{R}_+^*$ such that

$$|f(t, x)| \leq M_f, \quad \forall (t, x) \in J \times \mathbb{R},$$

(H₃) There exists a constant $k_3 \in (0, 1)$ and $k_1, k_2 \in \mathbb{R}_+^*$ such that

$$|f(t, x)| - |f(t, \bar{x})| \leq k_1 |x - \bar{x}|, \quad \forall (t, x, \bar{x}) \in J \times \mathbb{R}^2,$$

$$|g(t, x, y) - g(t, \bar{x}, \bar{y})| \leq k_2 |x - \bar{x}| + k_3 |y - \bar{y}|, \quad \forall (t, x, \bar{x}, y, \bar{y}) \in J \times \mathbb{R}^4,$$

Let $r > 0$ and define the ball

$$B_r = \{x \in C(J, \mathbb{R}) : \|x\|_\infty \leq r\}.$$

It is clear that B_r is a bounded, closed and convex subset of $C(J, \mathbb{R})$.

- We will now demonstrate the existence of the solution of the problem (P) using the Schaefer's fixed point theorem.

Theorem 3.1. *Let $1 < p \leq 2$ then $q > 2$. Assuming that condition (H₁), (H₂) and (H₃) are satisfied. Then the problem (P) has at least one solution in B_r .*

Proof. For this proof will be done in some steps.

Step 1 We prove that the operator G is continuous. We consider the sequence $(u_n)_n$ converging to u in $C(J, \mathbb{R})$.

Then every $t \in J$, and according to the conditions (H_1) , (H_2) and (H_3) we have

$$\begin{aligned}
|Gu_n(t) - Gu(t)| &= \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi \left[{}_q\varphi f(s, u_n(s)) + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \right. \right. \\
&\quad \left. \left. \times \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr \right] \times ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, u(s)) + \frac{f(s, u(s))}{\Gamma(\alpha)} \right. \right. \\
&\quad \left. \left. \times \int_0^s (s-r)^{\alpha-1} \sigma_u(r) dr \right] ds \right| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left| \phi_q \left[\varphi f(s, u_n(s)) + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr \right] \right. \\
&\quad \left. - \phi_q \left[\varphi f(s, u(s)) + \frac{f(s, u(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_u(r) dr \right] \right| ds,
\end{aligned}$$

from conditions (H_1) and (H_2) , we find that

$$\begin{aligned}
\left| \varphi f(s, u_n(s)) + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr \right| &\leq |\varphi| |f(s, u_n(s))| + \frac{|f(s, u_n(s))|}{\Gamma(\alpha)} \\
&\quad \times \int_0^s (s-r)^{\alpha-1} |\sigma_{u_n}(r)| dr \\
&\leq |\varphi| M_f + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} M_g dr \\
&\leq |\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha+1)} = M.
\end{aligned}$$

According to the lemma (1.2), we have

$$\begin{aligned}
&\left| \phi_q \left[\varphi f(s, u_n(s)) + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr \right] - \phi_q \left[\varphi f(s, u(s)) + \frac{f(s, u(s))}{\Gamma(\alpha)} \right. \right. \\
&\quad \left. \left. \times \int_0^s (s-r)^{\alpha-1} \sigma_u(r) dr \right] \right| \\
&\leq (q-1) M^{q-2} \left| \varphi f(s, u_n(s)) + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr \right. \\
&\quad \left. - \varphi f(s, u(s)) - \frac{f(s, u(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \int_0^s (s-r)^{\alpha-1} \sigma_u(r) dr \right|.
\end{aligned}$$

Through substitution we obtain

$$\begin{aligned}
|Gu_n(t) - Gu(t)| &\leq \frac{(q-1) M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left| \varphi f(s, u_n(s)) + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) \right. \\
&\quad \left. dr - \varphi f(s, u(s)) - \frac{f(s, u(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_u(r) dr \right| ds \\
&\leq \frac{(q-1) M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\varphi (f(s, u_n(s)) - f(s, u(s)))|
\end{aligned}$$

$$\begin{aligned}
 & + \frac{f(s, u_n(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr - \frac{f(s, u(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr \\
 & + \left. \frac{f(s, u(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_{u_n}(r) dr - \frac{f(s, u(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_u(r) dr \right| ds \\
 \leq & \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [|\varphi| |f(s, u_n(s)) - f(s, u(s))| \\
 & + \frac{|f(s, u_n(s)) - f(s, u(s))|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_{u_n}(r)| dr \\
 & + \frac{|f(s, u(s))|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_{u_n}(r) - \sigma_u(r)| dr] ds \\
 \leq & \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [k_1 |\varphi| |u_n(s) - u(s)| \\
 & + \frac{k_1 |u_n(s) - u(s)|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_{u_n}(r)| dr \\
 & + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_{u_n}(r) - \sigma_u(r)| dr] ds.
 \end{aligned}$$

From condition (H_3) , we find that

$$\begin{aligned}
 |\sigma_{u_n}(r) - \sigma_u(r)| & \leq |g(r, u_n(r), \sigma_{u_n}(r)) - g(r, u(r), \sigma_u(r))| \\
 & \leq k_2 |u_n(r) - u(r)| + k_3 |\sigma_{u_n}(r) - \sigma_u(r)| \\
 & \leq \frac{k_2}{(1-k_3)} |u_n(r) - u(r)|.
 \end{aligned}$$

By substitution, we find

$$\begin{aligned}
 |Gu_n(t) - Gu(t)| & \leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [k_1 |\varphi| |u_n(s) - u(s)| \\
 & + \frac{k_1 M_g}{\Gamma(\alpha+1)} |u_n(s) - u(s)| + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \frac{k_2}{(1-k_3)} |u_n(r) - u(r)| dr] \\
 & \leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [k_1 |\varphi| \|u_n - u\|_\infty \\
 & + \frac{k_1 M_g}{\Gamma(\alpha+1)} \|u_n - u\|_\infty + \frac{M_f k_2}{\Gamma(\alpha+1)(1-k_3)} \|u_n - u\|_\infty] ds \\
 & \leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \left[k_1 |\varphi| + \frac{k_1 M_g}{\Gamma(\alpha+1)} + \frac{M_f k_2}{\Gamma(\alpha+1)(1-k_3)} \right] \|u_n - u\|_\infty \\
 & \quad \times \int_0^t (t-s)^{\beta-1} ds, \\
 & \leq \frac{(q-1)M^{q-2}}{\Gamma(\beta+1)} \left[k_1 |\varphi| + \frac{k_1 M_g}{\Gamma(\alpha+1)} + \frac{k_2 M_f}{(1-k_3)\Gamma(\alpha+1)} \right] \|u_n - u\|_\infty.
 \end{aligned}$$

If $\|u_n - u\|_\infty \rightarrow 0$ as $n \rightarrow +\infty$ then $\|Gu_n(t) - Gu(t)\|_\infty \rightarrow 0$. Hence, G is continuous.

Step 2 We prove that the operator G uniformly bounded in $C(J, \mathbb{R})$. Let $x \in C(J, \mathbb{R})$ then, for all $t \in J$, and according to conditions (H_1) - (H_2) , we have

$$\begin{aligned}
|Gx(t)| &= \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right] ds \right| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[|\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \right] ds \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[|\varphi| M_f + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} M_g dr \right] ds \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha+1)} \right] ds \\
&\leq \frac{1}{\Gamma(\beta+1)} \phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha+1)} \right] \\
&\leq \frac{1}{\Gamma(\beta+1)} \phi_q \left[\left(|\varphi| + \frac{M_g}{\Gamma(\alpha+1)} \right) M_f \right].
\end{aligned}$$

According to definition (1.5), we have

$$\begin{aligned}
\phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha+1)} \right] &= \phi_q(M) \\
&= |M|^{q-2} M \\
&\leq M^{q-1}.
\end{aligned}$$

Through substitution we obtain

$$\begin{aligned}
|Gx(t)| &\leq \frac{M^{q-1}}{\Gamma(\beta+1)} \\
&\leq l.
\end{aligned}$$

Therefore

$$\|G(x(t))\|_\infty \leq l.$$

Where

$$l = \frac{M^{q-1}}{\Gamma(\beta+1)}.$$

The operator G is uniformly bounded set in $C(J, \mathbb{R})$.

Step 3 We prove that the operator G is equicontinuous set in $C(J, \mathbb{R})$.

Further let $t_1, t_2 \in J$ such that $t_1 > t_2$, and according to conditions (H_1) - (H_2) .

Then we have

$$\begin{aligned}
 |Gx(t_1) - Gx(t_2)| &= \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} \phi_q \left[\begin{array}{l} \varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \end{array} \right] ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} \varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \end{array} \right] ds \right| \\
 &\leq \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right. \\
 &\quad \left. + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right| \\
 &\leq \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right| \\
 &\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right| \\
 &\leq \left| \frac{1}{\Gamma(\beta)} \int_{t_2}^{t_1} (t_2 - s)^{\beta-1} \phi_q \left[\begin{array}{l} |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \times \\ \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \end{array} \right] ds \right| \\
 &\quad \left| + \frac{1}{\Gamma(\beta)} \int_0^{t_1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] \right. \\
 &\quad \left. \times \phi_q \left[|\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \right] ds \right| \\
 &\leq \left| \frac{-(t_2 - t_1)^\beta}{\Gamma(\beta + 1)} \phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha + 1)} \right] \right. \\
 &\quad \left. + \frac{t_1^\beta - t_2^\beta + (t_2 - t_1)^\beta}{\Gamma(\beta + 1)} \phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha + 1)} \right] \right| \\
 &\leq \frac{t_1^\beta - t_2^\beta}{\Gamma(\beta + 1)} \phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha + 1)} \right]
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{t_1^\beta - t_2^\beta}{\Gamma(\beta + 1)} \phi_q \left[\left(|\varphi| + \frac{M_g}{\Gamma(\alpha + 1)} \right) M_f \right]. \\ &\leq \frac{M^{q-1}}{\Gamma(\beta + 1)} (t_1^\beta - t_2^\beta). \end{aligned}$$

Whene $t_1 \rightarrow t_2$ the right-hand side of the last inequality tends to zero

$$\|Gx(t_1) - Gx(t_2)\|_\infty \leq \epsilon.$$

Hence G is equicontinuous set in $C(J, \mathbb{R})$.

Consequently, according to three steps and according to the Ascoli-Arzelà theorem, G is completely continuous.

Step 4 We prove that the set Ω is bounded, where $\Omega = \{x \in C(j, \mathbb{R}), x = \lambda G(x), \forall \lambda \in]0, 1[\}$.

Let $x \in \Omega$. Then, for all $t \in J$, and according to conditions (H_1) - (H_2) we have

$$\begin{aligned} x(t) &= \lambda G(x(t)) \\ &= \lambda \left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right] ds \right) \\ &\leq \frac{1}{\Gamma(\beta + 1)} \phi_q \left[|\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha + 1)} \right] \\ &\leq \frac{1}{\Gamma(\beta + 1)} \phi_q \left[\left(|\varphi| + \frac{M_g}{\Gamma(\alpha + 1)} \right) M_f \right] \\ &\leq \frac{M^{q-1}}{\Gamma(\beta + 1)} \\ &\leq R. \end{aligned}$$

Where

$$R = \frac{M^{q-1}}{\Gamma(\beta + 1)} \leq \infty$$

So Ω is a bounded set.

- Thus, according to Schaefer's fixed point theorem, we conclude that the problem (P) has at least one solution. \square

- We will now demonstrate the uniqueness of the solution of the problem (P) using the Banach fixed point theorem.

Theorem 3.2. *Let $1 < p \leq 2$ then $q > 2$. We assume that the conditions (H_1) and (H_2) and (H_3) are hold and if $0 < \eta < 1$, such that*

$$\eta = \frac{(q-1)}{\Gamma(\beta + 1)} M^{q-2} \left[k_1 |\varphi| + \frac{k_1 M_g}{\Gamma(\alpha + 1)} + \frac{M_f k_2}{\Gamma(\alpha + 1)(1 - k_3)} \right] < 1, \quad (3.6)$$

where

$$M = |\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha + 1)},$$

and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then the problem (P) admits a unique solution in $C(J, \mathbb{R})$.

Proof. It is clear that the fixed point of G is a solution of problem (P). We will now demonstrate the uniqueness of a fixed point for G through the proof that G is a contraction.

Let $x, y \in C(J, \mathbb{R})$, then for all $t \in J$, we have

$$\begin{aligned} |Gx(t) - Gy(t)| &= \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right] ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left[\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_y(r) dr \right] ds \right| \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left| \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right] \right. \\ &\quad \left. - \phi_q \left[\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_y(r) dr \right] \right| ds. \end{aligned}$$

From condition (H_1) and (H_2) , we find that

$$\begin{aligned} \left| \varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right| &\leq |\varphi| |f(s, x(s))| + \frac{|f(s, x(s))|}{\Gamma(\alpha)} \\ &\quad \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \\ &\leq |\varphi| M_f + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} M_g dr \\ &\leq |\varphi| M_f + \frac{M_f M_g}{\Gamma(\alpha + 1)} = M. \end{aligned}$$

According to the lemma (1.2), we have

$$\begin{aligned} &\left| \phi_q \left[\varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right] - \phi_q \left[\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \right. \right. \\ &\quad \left. \left. \int_0^s (s-r)^{\alpha-1} \sigma_y(r) dr \right] \right| \\ &\leq (q-1) M^{q-2} \left| \varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr - \right. \end{aligned}$$

$$\left. \varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \int_0^s (s-r)^{\alpha-1} \sigma_y(r) dr \right|.$$

Through substitution we obtain

$$\begin{aligned}
|Gx(t) - Gy(t)| &\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left| \varphi f(s, x(s)) + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \right. \\
&\quad \left. - \varphi f(s, y(s)) - \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_y(r) dr \right| ds \\
&\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\varphi (f(s, x(s)) - f(s, y(s)))| \\
&\quad + \frac{f(s, x(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr - \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr \\
&\quad + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_x(r) dr - \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \sigma_y(r) dr \Big| ds \\
&\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [|\varphi| |f(s, x(s)) - f(s, y(s))| \\
&\quad + \frac{|f(s, x(s)) - f(s, y(s))|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \\
&\quad + \frac{|f(s, y(s))|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_x(r) - \sigma_y(r)| dr] ds \\
&\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [k_1 |\varphi| |x(s) - y(s)| \\
&\quad + \frac{k_1 |x(s) - y(s)|}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_x(r)| dr \\
&\quad + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} |\sigma_x(r) - \sigma_y(r)| dr] ds.
\end{aligned}$$

From condition (H_3) , we find that

$$\begin{aligned}
|\sigma_x(r) - \sigma_y(r)| &\leq |g(r, x(r), \sigma_x(r)) - g(r, y(r), \sigma_y(r))| \\
&\leq k_2 |x(r) - y(r)| + k_3 |\sigma_x(r) - \sigma_y(r)| \\
&\leq \frac{k_2}{(1-k_3)} |x(r) - y(r)|.
\end{aligned}$$

By substitution, we find

$$\begin{aligned}
 |Gx(t) - Gy(t)| &\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [k_1 |\varphi| |x(s) - y(s)| \\
 &\quad + \frac{k_1 M_g}{\Gamma(\alpha+1)} |x(s) - y(s)| + \frac{M_f}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} \frac{k_2}{(1-k_3)} |x(r) - y(r)| dr] ds \\
 &\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [k_1 |\varphi| \|x - y\|_\infty \\
 &\quad + \frac{k_1 M_g}{\Gamma(\alpha+1)} \|x - y\|_\infty + \frac{M_f k_2}{\Gamma(\alpha+1)(1-k_3)} \|x - y\|_\infty] ds \\
 &\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta)} \left[k_1 |\varphi| + \frac{k_1 M_g}{\Gamma(\alpha+1)} + \frac{M_f k_2}{\Gamma(\alpha+1)(1-k_3)} \right] \|x - y\|_\infty \\
 &\quad \times \int_0^t (t-s)^{\beta-1} ds, \\
 &\leq \frac{(q-1)M^{q-2}}{\Gamma(\beta+1)} \left[k_1 |\varphi| + \frac{k_1 M_g}{\Gamma(\alpha+1)} + \frac{k_2 M_f}{(1-k_3)\Gamma(\alpha+1)} \right] \|x - y\|_\infty, \\
 &\leq \eta \|x - y\|_\infty.
 \end{aligned}$$

Therefore

$$|G(x) - G(y)| \leq \eta \|x - y\|_\infty.$$

The operator G is a contraction, and according to Banach fixed point theorem, the problem (P) admits a unique solution in $C(J, \mathbb{R})$. \square

3.2.2 Stability results

In this section, we study a type of stability for problem (P) , known as Ulam-Hyres stability, generalized Ulam-Hyres stability, Ulam-Hyres-Rassias stability and generalized Ulam-Hyres-Rassias stability.

Lemma 3.2. *Let $1 < p \leq 2$ then $q > 2$. Suppose that condition (H_1) is satisfied. If y is a solution of the fractional differential inequality given, for $\epsilon > 0$*

$$\left| {}^C D_{0+}^\alpha \left(\frac{\phi_P \left({}^R D_{0+}^\beta (y(t)) \right)}{f(t, y(t))} \right) - g \left(t, y(t), {}^C D_{0+}^\alpha \left(\frac{\phi_P \left({}^R D_{0+}^\beta (y(t)) \right)}{f(t, y(t))} \right) \right) \right| \leq \epsilon, t \in J. \tag{3.7}$$

Then y is also a solution satisfies the following inequality

$$|y(t) - Gy(t)| \leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)} \epsilon, t \in J. \tag{3.8}$$

Where

$$M = |\varphi| M_f + \frac{M_f (M_g + 1)}{\Gamma(\alpha+1)}.$$

Proof. Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (3.7), for $\epsilon > 0$, using the Lemma (3.1), Remark (1.1) and Lemma (1.2) related to the continuous function α where $|\alpha(t)| < \epsilon$, for all $t \in J$.

Then we have

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} (g(r, y(r), \sigma_y(r)) + \alpha(r)) dr] ds,$$

$$G(y(t)) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} g(r, y(r), \sigma_y(r)) dr] ds.$$

According to Remark (1.1), Lemma (1.2) and condition (H_1) , we obtain

$$\begin{aligned} |y(t) - G(y(t))| &= \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} (g(r, y(r), \sigma_y(r)) + \alpha(r)) dr] ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} g(r, y(r), \sigma_y(r)) dr] ds \right| \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (q-1) M^{q-2} [|\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} (g(r, y(r), \sigma_y(r)) + \alpha(r)) dr - \varphi f(s, y(s)) - \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} g(r, y(r), \sigma_y(r)) dr|] ds, \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\frac{(q-1) M^{q-2} M_f}{\Gamma(\alpha)} \times \int_0^s (s-r)^{\alpha-1} |\alpha(r)| dr \right] ds, \\ &\leq \frac{(q-1) M^{q-2} M_f}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\int_0^s (s-r)^{\alpha-1} \epsilon dr \right) ds, \\ &\leq \frac{(q-1) M^{q-2} M_f}{\Gamma(\alpha+1) \Gamma(\beta+1)} \epsilon. \end{aligned}$$

Where

$$M = |\varphi| M_f + \frac{M_f (M_g + 1)}{\Gamma(\alpha + 1)}.$$

And this satisfies the inequality (3.8). □

Theorem 3.3. *Let $1 < p \leq 2$ then $q > 2$. Assuming that conditions $(H_1) - (H_3)$ are satisfied, then the problem (P) is Ulam-Hyers stable and also generalized Ulam-Hyers stable.*

Proof. Under conditions $(H_1) - (H_3)$ and (3.6), the problem (P) has a unique solution in $C(J, \mathbb{R})$. Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (3.7). Let $x \in C(J, \mathbb{R})$ the unique solution of the inequality (3.4).

Then for all $t \in J$, we have

$$\begin{aligned} |y(t) - x(t)| &= |y(t) - G(y(t)) + Gy(t) - x(t)| \\ &\leq |y(t) - G(y(t))| + |Gy(t) - x(t)| \\ &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\epsilon + \eta \|y - x\|_\infty. \end{aligned}$$

Therefore

$$\begin{aligned} \|y - x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\epsilon + \eta \|y - x\|_\infty \\ \|y - x\|_\infty - \eta \|y - x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\epsilon \\ (1-\eta) \|y - x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\epsilon \\ \|y - x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{(1-\eta)\Gamma(\alpha+1)\Gamma(\beta+1)}\epsilon. \end{aligned}$$

Setting

$$C_q = \frac{(q-1)M^{q-2}M_f}{(1-\eta)\Gamma(\alpha+1)\Gamma(\beta+1)},$$

we obtain

$$\|y - x\|_\infty \leq C_q \epsilon.$$

Consequently, the problem (P) is Ulam-Hyers stable. If we take $\varphi(\epsilon) = C_q \epsilon$, $\varphi(0) = 0$, then we get the generalized Ulam-Hyers stability of the problem (P) . \square

Lemma 3.3. *Let $1 < p \leq 2$ then $q > 2$. Suppose that condition (H_1) is satisfied. If y is a solution of the fractional differential inequality given, for $\epsilon > 0$.*

$$\left| {}^C D_{0+}^\alpha \left(\frac{\phi_P \left({}^R D_{0+}^\beta (y(t)) \right)}{f(t, y(t))} \right) - g \left(t, y(t), {}^C D_{0+}^\alpha \left(\frac{\phi_P \left({}^R D_{0+}^\beta (y(t)) \right)}{f(t, y(t))} \right) \right) \right| \leq \phi(t) \epsilon, \quad t \in J. \quad (3.9)$$

Then y is also a solution to the following inequality

$$|y(t) - Gy(t)| \leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)} \phi(t) \epsilon, \quad t \in J. \quad (3.10)$$

Where

$$M = |\varphi| M_f + \frac{M_f (M_g + 1)}{\Gamma(\alpha + 1)}.$$

Proof. Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (3.9), for $\epsilon > 0$, using the Lemma (3.1) and Remark (1.2) related to the continuous function α where $|\alpha(t)| < \phi(t) \epsilon$, for all $t \in J$. Then we have

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} (g(r, y(r), \sigma_y(r)) + \alpha(r)) dr] ds,$$

$$G(y(t)) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} g(r, y(r), \sigma_y(r)) dr] ds.$$

According to Remark (1.2), Lemma (1.2) and condition (H_1) , we obtain

$$\begin{aligned} |y(t) - G(y(t))| &= \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} (g(r, y(r), \sigma_y(r)) + \alpha(r)) dr] ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q [\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} g(r, y(r), \sigma_y(r)) dr] ds \right| \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (q-1) M^{q-2} [|\varphi f(s, y(s)) + \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} (g(r, y(r), \sigma_y(r)) + \alpha(r)) dr - \varphi f(s, y(s)) - \frac{f(s, y(s))}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} g(r, y(r), \sigma_y(r)) dr|] ds, \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\frac{(q-1) M^{q-2} M_f}{\Gamma(\alpha)} \right. \\ &\quad \left. \times \int_0^s (s-r)^{\alpha-1} |\alpha(r)| dr \right] ds, \\ &\leq \frac{(q-1) M^{q-2} M_f}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\int_0^s (s-r)^{\alpha-1} \phi(t) \epsilon dr \right) ds, \\ &\leq \frac{(q-1) M^{q-2} M_f}{\Gamma(\alpha + 1) \Gamma(\beta + 1)} \phi(t) \epsilon. \end{aligned}$$

Where

$$M = |\varphi| M_f + \frac{M_f (M_g + 1)}{\Gamma(\alpha + 1)}.$$

□

Theorem 3.4. *Let $1 < p \leq 2$ then $q > 2$. Assuming that conditions $(H_1) - (H_3)$ and (3.6), are satisfied, then the problem (P) is Ulam-Hyers-Rassias stable and also generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (3.9). Let $x \in C(J, \mathbb{R})$ the unique solution of the inequality (3.4).

Then for all $t \in J$, we have

$$\begin{aligned} |y(t) - x(t)| &= |y(t) - Gy(t) + Gy(t) - x(t)| \\ &\leq |y(t) - Gy(t)| + |Gy(t) - x(t)| \\ &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\phi(t)\epsilon + \eta\|y-x\|_\infty. \end{aligned}$$

Therefore

$$\begin{aligned} \|y-x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\phi(t)\epsilon + \eta\|y-x\|_\infty \\ \|y-x\|_\infty - \eta\|y-x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\phi(t)\epsilon \\ (1-\eta)\|y-x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}\phi(t)\epsilon \\ \|y-x\|_\infty &\leq \frac{(q-1)M^{q-2}M_f}{(1-\eta)\Gamma(\alpha+1)\Gamma(\beta+1)}\phi(t)\epsilon. \end{aligned}$$

Setting

$$C_q = \frac{(q-1)M^{q-2}M_f}{(1-\eta)\Gamma(\alpha+1)\Gamma(\beta+1)},$$

we obtain

$$\|y-x\|_\infty \leq C_q \phi(t) \epsilon.$$

Consequently, the problem (P) is Ulam-Hyers-Rassias stable. If we take

$C_{q, \phi} = C_q \epsilon$, then we get the generalized Ulam-Hyers-Rassias stable of the problem (P). □

3.3 Example

Corresponding to the proposed problem, we provide the following example demonstrate consistency to the main theorems.

$$\left\{ \begin{array}{l} {}^C D_{0+}^{\frac{1}{2}} \left(\frac{\phi_P \left({}^R D_{0+}^{\frac{1}{2}} x(t) \right)}{1 + \frac{1}{25} e^{-t^2} \cos(x(t))} \right) = \frac{1}{5} t \cos(x(t)) + \frac{1}{5 + \left| {}^C D_{0+}^{\frac{1}{2}} \left(\frac{\phi_P \left({}^R D_{0+}^{\frac{1}{2}} x(t) \right)}{1 + \frac{1}{25} e^{-t^2} \cos(x(t))} \right) \right|}, \\ \phi_P \left({}^R D_{0+}^{\beta} (x(0)) \right) = \varphi f(0, x(0)), \\ \lim_{t \rightarrow 0} (t^{1-\beta} x(t)) = 0. \end{array} \right. \quad (L)$$

Taking $\varphi = \alpha = \beta = \frac{1}{2}$, $q = 3$.

Where f and g two continuous functions defined for all $t \in J$ as follows

$$f(t, x(t)) = 1 + \frac{1}{25} e^{-t^2} \cos(x(t)), \quad t \in J.$$

$$g \left(t, x(t), {}^C D_{0+}^{\alpha} \left(\frac{\phi_P \left({}^R D_{0+}^{\beta} (x(t)) \right)}{f(t, x(t))} \right) \right) = \frac{1}{5} t \cos(x(t)) + \frac{1}{5 + \left| {}^C D_{0+}^{\frac{1}{2}} \left(\frac{\phi_P \left({}^R D_{0+}^{\frac{1}{2}} x(t) \right)}{1 + \frac{1}{25} e^{-t^2} \cos(x(t))} \right) \right|}, \quad t \in J.$$

Let's put

$$\begin{aligned} f(t, u) &= 1 + \frac{1}{25} e^{-t^2} \cos(u), \quad (t, u) \in J \times \mathbb{R}, \\ g(t, u, v) &= \frac{1}{5} t \cos(u) + \frac{1}{5 + |v|}. \quad (t, u, v) \in J \times \mathbb{R}^2. \end{aligned}$$

For all $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$ we have

$$\begin{aligned} |f(t, u) - f(t, \bar{u})| &\leq \frac{1}{25} |u - \bar{u}|, \\ |g(t, u, v) - g(t, \bar{u}, \bar{v})| &\leq \frac{1}{5} |u - \bar{u}| + \frac{1}{25} |v - \bar{v}|, \\ |f(t, u)| &\leq \frac{26}{25}, \\ |g(t, u, v)| &\leq \frac{2}{5}, \end{aligned}$$

so, all conditions (H_1) , (H_2) and (H_3) are satisfied, for

$$\begin{aligned} k_1 &= k_3 = \frac{1}{25}, \quad k_2 = \frac{1}{5}, \quad M_f = \frac{26}{25}, \quad M_g = \frac{2}{5}, \\ \varphi &= \alpha = \beta = \frac{1}{2}, \quad q = 3. \end{aligned}$$

$$M \simeq 0.98491 > 0.$$

By theorem (3.1) the problem (L) has one solution at least further.

$$\eta \simeq 0.63086 < 1$$

Hence by theorem (3.2) the considered problem possess unique solution .

$$\nabla = \frac{(q-1)M^{q-2}M_f}{\Gamma(\alpha+1)} \simeq 2.3221 > 0.$$

Therefore, according to the theorem (3.3) , and theorem (3.4) , the problem (L) is stable in the sense Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

Conclusion

The subject of fractional order differential equations is vast, and research in this field is extensive and requires a great deal of patience. This memorandum is merely a drop in the ocean, in which I have seriously endeavored to achieve the intended goal: studying the existence and uniqueness of solutions, as well as Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability for fractional differential equations involving the generalized Mittag-Leffler derivative.

In the final chapter, I also studied the existence and uniqueness of solutions, along with Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability for some hybrid mixed problems involving first-kind fractional derivatives of Riemann and Caputo types, in the presence of the p -Laplace operator. This work relied on fixed point theorems by Banach, Krasnoselskii, and Schafer. These problems were studied within Banach spaces, and a clarifying example was provided at the end of each chapter.

I hope that I have contributed, even in a small way, to clarifying how to study such problems. I also hope that research in this field will continue through the following topics:

Studying these problems for other types of fractional derivatives and in infinite domains, as well as investigating the Mittag-Leffler fractional derivative in various problems, studying them in Banach spaces, and using numerical methods to solve some of these problems as examples.

Bibliography

- [1] Abbas, Saïd, Mouffak Benchohra, and Gaston M. N'Guérékata. Topics in fractional differential equations. Vol. 27. Springer Science & Business Media, 2012.
- [2] Abdelouaheb, Ardjouni, and Djoudi Ahcene. "Approximating solutions of non-linear hybrid Caputo fractional integro-differential equations via Dhage iteration principle." *Ural Mathematical Journal* 5.1 (8) (2019): 3-12.
- [3] Adjimi, Naas, Maamar Benbachir, and Kaddour Guerbati. "Existence results for ψ -Caputo hybrid fractional integro-differential equations." *Malaya Journal of Matematik* 9.02 (2021): 46-54.
- [4] Ahmad, Bashir, and Sotiris K. Ntouyas. "An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions." *Abstract and Applied Analysis*. Vol. 2014. No. 1. Hindawi Publishing Corporation, 2014.
- [5] Ali, Z., A. Zada, and K. Shah. "Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem. *Bound. Value Probl.* 2018, 175 (2018)."
- [6] Atangana, Abdon, and José Francisco Gómez-Aguilar. "A new derivative with normal distribution kernel: Theory, methods and applications." *Physica A: Statistical mechanics and its applications* 476 (2017): 1-14.
- [7] Atangana, Abdon, and Dumitru Baleanu. "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model." *arXiv preprint arXiv:1602.03408* (2016).
- [8] Baleanu, Dumitru, et al. *Fractional calculus: models and numerical methods*. Vol. 3. World Scientific, 2012.

-
- [9] Baleanu, Dumitru, and Arran Fernandez. "On some new properties of fractional derivatives with Mittag-Leffler kernel." *Communications in Nonlinear Science and Numerical Simulation* 59 (2018): 444-462.
- [10] Benchohra, Mouffak, and Jamal E. Lazreg. "On stability for nonlinear implicit fractional differential equations." *Le Matematiche* 70.2 (2015): 49-61.
- [11] Brezis, Haïm. "Analyse fonctionnelle." *Théorie et applications* (1983).
- [12] Burton, T. A., and Tetsuo Furumochi. "Krasnoselskii's fixed point theorem and stability." *Nonlinear Analysis: Theory, Methods & Applications* 49.4 (2002): 445-454.
- [13] Cho, Yeol Je, and Yu-Qing Chen. *Topological degree theory and applications*. Chapman and Hall/CRC, 2006.
- [14] Diethelm, Kai, and N. J. Ford. "The analysis of fractional differential equations." *Lecture notes in mathematics* 2004 (2010).
- [15] Diethelm, Kai, and Neville J. Ford. "Analysis of fractional differential equations." *Journal of Mathematical Analysis and Applications* 265.2 (2002): 229-248.
- [16] Van den Driessche, Pauline, and James Watmough. "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission." *Mathematical biosciences* 180.1-2 (2002): 29-48.
- [17] Granas, Andrzej, and James Dugundji. *Fixed point theory*. Vol. 14. New York: Springer, 2003.
- [18] Guo, Xiufeng. "Existence of unique solution to switched fractional differential equations with p -Laplacian operator." *Turkish Journal of Mathematics* 39.6 (2015): 864-871.
- [19] D. Guo and V. Lakshmikantham, *Nonlinear Problem in Abstract Cones*, Academic Press, New York, 1988.
- [20] Kreyszig, Erwin. *Introductory functional analysis with applications*. John Wiley & Sons, 1991.

- [21] Kilbas, A. A. (2006). Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204.
- [22] Kilbas, A. A., and J. J. Trujillo. "Differential equations of fractional order: methods, results and problems. II." *Applicable Analysis* 81.2 (2002): 435-493.
- [23] Kilbas, A. A., and J. J. Trujillo. "Differential equations of fractional order: methods results and problem—I." *Applicable analysis* 78.1-2 (2001): 153-192.
- [24] Koca, Ilknur. "Modelling the spread of Ebola virus with Atangana-Baleanu fractional operators." *The European Physical Journal Plus* 133 (2018): 1-11.
- [25] Liu, Zhenhai, and Liang Lu. "A class of BVPs for nonlinear fractional differential equations with p-Laplacian operator." *Electronic Journal of Qualitative Theory of Differential Equations* 2012.70 (2012): 1-16.
- [26] Liu, Xiping, Mei Jia, and Xiufen Xiang. "On the solvability of a fractional differential equation model involving the p-Laplacian operator." *Computers & Mathematics with Applications* 64.10 (2012): 3267-3275.
- [27] Shen, Tengfei, Wenbin Liu, and Ren Zhao. "Fractional boundary value problems with p (t) p(t)-Laplacian operator." *Advances in Difference Equations* 2016 (2016): 1-10.
- [28] Machado, J. Tenreiro, Virginia Kiryakova, and Francesco Mainardi. "Recent history of fractional calculus." *Communications in nonlinear science and numerical simulation* 16.3 (2011): 1140-1153.
- [29] Miller, Kenneth S., and Bertram Ross. "An introduction to the fractional calculus and fractional differential equations." (No Title) (1993).
- [30] Podlubny, Igor. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Vol. 198. elsevier, 1998.
- [31] Jung, Soon-Mo, and Themistocles M. Rassias. "Generalized Hyers-Ulam stability of Riccati differential equation." *Math. Inequal. Appl* 11.4 (2008): 777-782.

- [32] Rus, Ioan A. "Ulam stabilities of ordinary differential equations in a Banach space." *Carpathian journal of Mathematics* (2010): 103-107.
- [33] Shah, Kamal, et al. "Qualitative analysis of a mathematical model in the time of COVID-19." *BioMed Research International* 2020.1 (2020): 5098598.
- [34] Shah, Kamal, Muhammad Sher, and Thabet Abdeljawad. "Study of evolution problem under Mittag–Leffler type fractional order derivative." *Alexandria Engineering Journal* 59.5 (2020): 3945-3951.
- [35] Vu, Ho, Behzad Ghanbari, and Ngo Van Hoa. "Fuzzy fractional differential equations with the generalized Atangana-Baleanu fractional derivative." *Fuzzy Sets and Systems* 429 (2022): 1-27.
- [36] Wang, Guotao, et al. "Fuzzy fractional delay integro-differential equation with the generalized Atangana-Baleanu fractional derivative." *Demonstratio Mathematica* 57.1 (2024): 20240008.
- [37] Jafari, Hossein, et al. "Existence criterion for the solutions of fractional order p-Laplacian boundary value problems." *Boundary Value Problems* 2015 (2015): 1-10.
- [38] Jarad, Fahd, Thabet Abdeljawad, and Zakia Hammouch. "On a class of ordinary differential equations in the frame of Atangana–Baleanu fractional derivative." *Chaos, Solitons & Fractals* 117 (2018): 16-20.
- [39] Smart, D. R. "fixed point theory, Combridge Uni." Press, Combridge (1974).
- [40] Tarasov, Vasily E. *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*. Springer Science & Business Media, 2011.